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Appendix 1:
Chapterwise Solved January 2019 JEE Main Questions (All Sets) A.1–A.6

1 Relations and Functions

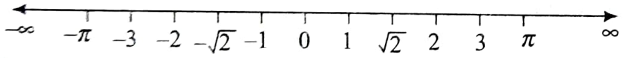
REAL NUMBER SYSTEM AND INEQUALITIES

To workout functions, we need to understand real number system and its properties. We are familiar with the following sets of different types of numbers.

N: Set of natural numbers
W: Set of whole numbers (Set of non-negative integers)
Z or I: Set of integers
Q: Set of rational numbers
R: Set of real numbers
R^+: Set of positive real numbers
R^-: Set of negative real numbers

Note:

- A rational number can be written as $\frac{a}{b}$, where a and b are integers, $b \neq 0$ and H.C.F. of a and b is 1.
- All recurring decimals are rational numbers. e.g., $0.3333... = 1/3$, $0.716716716... = 716/999$ etc.
- Those values which neither terminate nor could be expressed as recurring decimals are irrational numbers., e.g., $\sqrt{2}, \pi, -\frac{1}{\sqrt{3}}, 2+\sqrt{2}$ ... etc.
- Real number are represented on the real number line.



The set R represent set of continuous values (not discrete values).

Here '∞' is symbol (not real number) used for extremely large value. Thus symbol '∞' depicts that number line can be extended further. Symbol '∞' mean extreme right side of real number line. Symbol '−∞' means extreme left side of real number line.

INTERVALS AS SUBSET OF SET R

Now, let us define different subsets of R.

Let $a, b \in R$ and $a < b$. Then we have following different intervals.

Interval	Interval name	Plotting on real number line
$\{x \mid a < x < b\}$ or $x \in (a, b)$ or $x \in]a, b[$	Open interval	
$\{x \mid a \le x \le b\}$ or $x \in [a, b]$	Closed interval	
$\{x \mid a \le x < b\}$ or $x \in [a, b)$ or $x \in [a, b[$	Closed-Open interval	
$\{x \mid a < x \le b\}$ or $x \in (a, b]$ or $x \in]a, b]$	Open-Closed interval	

The number $(b - a)$ is called the length of any of the intervals (a, b), $[a, b]$, $[a, b)$ or $(a, b]$.

Clearly, $(a, b) \subset [a, b]$ but $[a, b] \not\subset (a, b)$.

Note

- The set $(-\infty, \infty)$ defines the set of all real numbers extending from $-\infty$ to ∞.
- The set $(0, \infty)$ defines the set of all positive real numbers denoted by R^+.
- The set $(-\infty, 0)$ defines the set of all negative real numbers denoted by R^-.
- The set $[0, \infty)$ defines the set of all non-negative real numbers $R^+ \cup \{0\}$.
- The set $(-\infty, 0]$ defines the set of all non-positive real numbers $R^- \cup \{0\}$.
- Complement of interval (a, b) is $(-\infty, a] \cup [b, \infty)$.
- Complement of interval $[a, b]$ is $(-\infty, a) \cup (b, \infty)$.

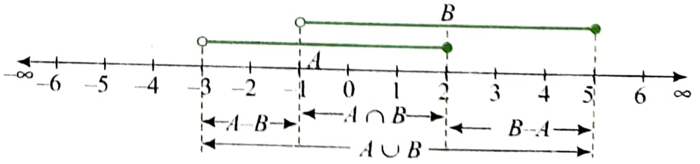
ILLUSTRATION 1.1

If sets A = (−3, 2] and B = (−1, 5], then find the following sets

(i) $A \cap B$ **(ii)** $A \cup B$ **(iii)** $A - B$ **(iv)** $B - A$

Sol. We have sets A = (−3, 2] and B = (−1, 5]

These sets are plotted on real number line as shown in the following figure.



From the figure

(i) $A \cap B = (-1, 2]$ (as $-1 \in A$ but $-1 \notin B$)

(ii) $A \cup B = (-3, 5]$

(iii) $A - B = (-3, -1]$ (as $-1 \notin$ B, so $-1 \in A - B$)

(iv) $B - A = (2, 5]$ (as $2 \in$ A, so $2 \notin B - A$)

INEQUALITIES

We have following elementary properties of inequalities.

(i) If $a \le b$ then either $a < b$ or $a = b$.

(ii) Let $a < b$. Now if both sides are increased or decreased by same quantity say k then L.H.S. is still less than the R.H.S. So, $a \pm k < b \pm k$.

(iii) If $a < b$ and $c < d$, then $a + c < b + d$ or $a - d < b - c$

(iv) If $a < b$ then $ka < kb$, if k is positive.

(v) If $a < b$ then $ka > kb$, if k is negative.
Thus, if inequality is multiplied by negative quantity on both sides, sign of inequality alters.

(vi) $a < b$ and $b < c$, then $a < b < c$. This is called transitive property of inequality.
If $a \le b$ and $b \le c$, then $a \le b \le c$.
If $a < b$ and $b > c$ then we cannot relate a and c.
Also, if $a < b < c$, then $a < c$.

(vii) $0 < a < b$, $a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$.

(viii) $a + \frac{1}{a} \ge 2$ for $a > 0$ and equality holds for $a = 1$

(ix) $a + \frac{1}{a} \le -2$ for $a < 0$ and equality holds for $a = -1$

Values of x^2 from Given Values of x

If $a < b$ then $a^2 < b^2$ is not always true.
For example,

$2 < 3$ then $4 < 9$, but $-4 < 3$ then $16 > 9$

Also, if $x > 2$ then $x^2 > 4$, but for $x < 2$ then $x^2 \ge 0$.
This is because $x < 2$ means $x < 0$ or $0 \le x < 2$.
So, $x^2 > 0$ or $0 \le x^2 < 4$
Hence $x^2 \ge 0$
If $2 < x < 4$ then $4 < x^2 < 16$
If $-2 < x < 4$ then $0 \le x^2 < 16$
If $-5 < x < 4$ then $0 \le x^2 < 25$
In fact, $a < b \Rightarrow a^2 < b^2$ follows only when distance of 'a' from '0' on real number line is less than the distance of 'b' from 0.
Also, if $x^2 \le a^2$, then $x \in [-a, a]$.
And if $x^2 \ge a^2$, then $x \in (-\infty, -a] \cup [a, \infty)$.

Values of $1/x$ from Given Values of x

Consider the following examples:

$2 < 3$ then $\frac{1}{2} > \frac{1}{3}$

$-6 < -4$, then $-\frac{1}{6} > -\frac{1}{4}$

$-3 < 2$, then $-\frac{1}{3} < \frac{1}{5}$

Thus, if values on both sides of inequality have same sign, while taking its reciprocal the sign of inequality alters.

i.e., $a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b}$ and $a < b < 0 \Rightarrow \frac{1}{a} > \frac{1}{b}$

But if values on both sides of inequality have opposite sign, then while taking reciprocal sign of inequality remains same, i.e.,

$a < 0 < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

Now, consider following examples:

(i) $2 < x < 3 \Rightarrow \frac{1}{2} > \frac{1}{x} > \frac{1}{3}$

(ii) $-6 < x < -2 \Rightarrow -\frac{1}{6} > \frac{1}{x} > -\frac{1}{2}$

(iii) $-3 < x < 4$

Now $\frac{1}{x}$ is not defined for $x = 0$

So, we consider $-3 < x < 0$ or $0 < x < 4$

$\Rightarrow \quad -\frac{1}{3} > \frac{1}{x} > -\infty$ or $\infty > \frac{1}{x} > \frac{1}{4}$

Here in $-3 < x < 0$, x is very close to zero but less than zero, i.e., 0^-.

So, $\frac{1}{0^-}$ is extremely large negative number which is denoted by '$-\infty$'.

In $0 < x < 4$, x is very close to zero but more than zero, i.e., 0^+.

So, $\frac{1}{0^+}$ is extremely large positive number which is denoted by '∞'.

Thus, $x \in [a, b]$

$$\Rightarrow \begin{cases} \frac{1}{x} \in \left[\frac{1}{b}, \frac{1}{a}\right], & \text{if } a \text{ and } b \text{ have same sign} \\ \frac{1}{x} \in \left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right), & \text{if } a \text{ and } b \text{ have opposite signs} \end{cases}$$

ILLUSTRATION 1.2

Find the values of x^2 for the given values of x.

(i) $x < 3$ **(ii)** $x > -1$ **(iii)** $x \ge 2$ **(iv)** $x < -1$

Sol.

(i) When $x < 3$, we have $x \in (-\infty, 0) \cup [0, 3)$
for $x \in [0, 3)$, $x^2 \in [0, 9)$
for $x \in (-\infty, 0)$, $x^2 \in (0, \infty)$
$\Rightarrow$ for $x < 3$, $x^2 \in [0, 9) \cup (0, \infty)$
$\Rightarrow x \in [0, \infty)$

(ii) When $x > -1$, we have $x \in (-1, 0) \cup [0, \infty)$
for $x \in (-1, 0)$, $x^2 \in (0, 1)$
for $x \in [0, \infty)$, $x^2 \in [0, \infty)$
$\Rightarrow$ for $x > -1$, $x^2 \in (0, 1) \cup [0, \infty)$
$\Rightarrow x \in [0, \infty)$

(iii) Here $x \in [2, \infty)$
$\Rightarrow x^2 \in [4, \infty)$

(iv) Here $x \in (-\infty, -1)$
$\Rightarrow x^2 \in (1, \infty)$

ILLUSTRATION 1.3

Find all possible values of the following expressions:

(i) $\sqrt{x^2 - 4}$ **(ii)** $\sqrt{9 - x^2}$ **(iii)** $\sqrt{x^2 - 2x + 10}$

Sol.

(i) $\sqrt{x^2-4}$

Least value of square root is 0 when $x^2 = 4$ or $x = \pm 2$. Also $x^2 - 4 \geq 0$

Hence, $\sqrt{x^2-4} \in [0, \infty)$.

(ii) $\sqrt{9-x^2}$

Least value of square root is 0, when $9 - x^2 = 0$ or $x = \pm 3$. Also, the greatest value of $9 - x^2$ is 9, when $x = 0$.

Hence, we have $0 \leq 9 - x^2 \leq 9 \Rightarrow \sqrt{9-x^2} \in [0, 3]$.

(iii) $\sqrt{x^2-2x+10} = \sqrt{(x-1)^2+9}$

Here, the least value of $\sqrt{(x-1)^2+9}$ is 3, when $x - 1 = 0$.

Also $(x-1)^2 + 9 \geq 9 \Rightarrow \sqrt{(x-1)^2+9} \geq 3$

Hence, $\sqrt{x^2-2x+10} \in [3, \infty)$.

ILLUSTRATION 1.4

Find the values of $1/x$ for the given values of x.

(i) $x > 3$ **(ii)** $x < -2$ **(iii)** $x \in (-1, 3) - \{0\}$

Sol.

(i) We have $3 < x < \infty$

$$\Rightarrow \frac{1}{3} > \frac{1}{x} > \frac{1}{\to \infty}$$

$$\Rightarrow 0 < \frac{1}{x} < \frac{1}{3}$$

(ii) We have $-\infty < x < -2$

$$\Rightarrow \frac{1}{\to -\infty} > \frac{1}{x} > \frac{1}{-2}$$

$$\Rightarrow 0 > \frac{1}{x} > -\frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} < \frac{1}{x} < 0$$

(iii) $x \in (-1, 3) - \{0\}$

$\Rightarrow x \in (-1, 0) \cup (0, 3)$

For $x \in (-1, 0)$, we have

$$\frac{1}{-1} > \frac{1}{x} > \frac{1}{\to 0^-}$$

$$\Rightarrow -1 > \frac{1}{x} > -\infty$$

$$\Rightarrow -\infty < \frac{1}{x} < -1 \qquad (1)$$

For $x \in (0, 3)$, we have

$$\frac{1}{\to 0^+} > \frac{1}{x} > \frac{1}{3}$$

$$\Rightarrow \infty > \frac{1}{x} > \frac{1}{3}$$

$$\Rightarrow \frac{1}{3} < \frac{1}{x} < \infty \qquad (2)$$

From (1) and (2), $\frac{1}{x} \in (-\infty, -1) \cup \left(\frac{1}{3}, \infty\right)$

ILLUSTRATION 1.5

Find all possible values of the following expressions:

(i) $\frac{1}{x^2+2}$ **(ii)** $\frac{1}{x^2-2x+3}$ **(iii)** $\frac{1}{x^2-x-1}$

Sol.

(i) We know that $x^2 \geq 0 \; \forall \; x \in R$.

$\Rightarrow x^2 + 2 \geq 2, \forall \; x \in R$.

or $2 \leq (x^2 + 2) < \infty$

or $\frac{1}{2} \geq \frac{1}{x^2+2} > 0$

$$\Rightarrow 0 < \frac{1}{x^2+2} \leq \frac{1}{2}$$

(ii) $$\frac{1}{x^2-2x+3} = \frac{1}{(x-1)^2+2}$$

Now we know that $(x-1)^2 \geq 0 \; \forall \; x \in R$. Thus,

$(x-1)^2 + 2 \geq 2 \; \forall \; x \in R$.

or $2 \leq (x-1)^2 + 2 < \infty$

or $\frac{1}{2} \geq \frac{1}{(x-1)^2+2} > 0$

or $\frac{1}{x^2-2x+3} \in \left(0, \frac{1}{2}\right]$

(iii) $$\frac{1}{x^2-x-1} = \frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}}$$

$$\left(x-\frac{1}{2}\right)^2 \geq 0, \forall \; x \in R$$

$$\Rightarrow \left(x-\frac{1}{2}\right)^2 - \frac{5}{4} \geq -\frac{5}{4}, \forall \; x \in R$$

For $\frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}}$, we must have

$$\left(x-\frac{1}{2}\right)^2 - \frac{5}{4} \in \left[-\frac{5}{4}, 0\right) \cup (0, \infty)$$

$$\Rightarrow \frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}} \in \left(-\infty, -\frac{4}{5}\right] \cup (0, \infty)$$

ILLUSTRATION 1.6

Find the values of x for which expression $\sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$ is meaningful.

Sol. $\sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$ is meaningful if

$$1-\sqrt{1-\sqrt{1-x^2}} \geq 0$$

or $\sqrt{1-\sqrt{1-x^2}} \leq 1$

or $1-\sqrt{1-x^2} \leq 1$

or $\sqrt{1-x^2} \geq 0$

or $1-x^2 \geq 0$

or $x^2 \leq 1$ or $x \in [-1, 1]$

SIGN SCHEME METHOD TO SOLVE INEQUALITIES

We know how to solve linear inequalities. Now, we shall discuss the algorithm to solve polynomial and rational inequalities.

Let $F(x) = (x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_{n-1})^{k_{n-1}}(x-a_n)^{k_n}$

where $k_1, k_2, \ldots, k_n \in Z$ and $a_1, a_2, \ldots, a_n$ are fixed real numbers satisfying the condition $a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n$.

For solving $F(x) > 0$ or $F(x) < 0$, consider the following algorithm:

Step I: Mark the numbers $a_1, a_2, \ldots, a_n$ on the number axis and put the plus sign in the interval on the right of the largest of these numbers, i.e., on the right of a_n. Clearly, when $x > a_n$, all the factors $(x - a_i)$, $i = 1, 2, 3, \ldots, n$ are positive. So, $F(x)$ is positive.

Step II: Put the plus sign in the interval on the left of a_n if k_n is an even number and the minus sign if k_n is an odd number.

In the next interval, we put a sign according to the following rule:

When passing through the point a_{n-1}, $F(x)$ changes sign if k_{n-1} is an odd number. Then we consider the next interval and put a sign in it using the same rule.

Step III: Consider all the intervals. The solution of the inequality $F(x) > 0$ is the union of all the intervals in which we have put the plus sign and the solution of the inequality $F(x) < 0$ is the union of all the intervals in which we have put the minus sign.

Frequently used inequalities:

(i) $(x-a)(x-b) < 0 \Rightarrow x \in (a, b)$, where $a < b$

(ii) $(x-a)(x-b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$, where $a < b$

(iii) $x^2 \leq a^2 \Rightarrow x \in [-a, a]$

(iv) $x^2 \geq a^2 \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

(v) If $ax^2 + bx + c < 0$, $(a > 0) \Rightarrow x \in (\alpha, \beta)$, where α and β $(\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$.

(vi) If $ax^2 + bx + c > 0$, $(a > 0) \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$, where α and β $(\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$

ILLUSTRATION 1.7

Solve $x^2 - x - 2 > 0$.

Sol. $x^2 - x - 2 > 0$

or $(x-2)(x+1) > 0$

Now $x^2 - x - 2 = 0 \Rightarrow x = -1, 2$.

On the number line (x-axis) mark $x = -1$ and $x = 2$.

Now when $x > 2$, $x + 1 > 0$ and $x - 2 > 0$

$\Rightarrow$ $(x+1)(x-2) > 0$

when $-1 < x < 2$, $x + 1 > 0$ but $x - 2 < 0$

$\Rightarrow$ $(x+1)(x-2) < 0$

when $x < -1$, $x + 1 < 0$ and $x - 2 < 0$

$\Rightarrow$ $(x+1)(x-2) > 0$

Hence, sign scheme of $x^2 - x - 2$ is

From the figure, $x^2 - x - 2 > 0$, $x \in (-\infty, -1) \cup (2, \infty)$.

ILLUSTRATION 1.8

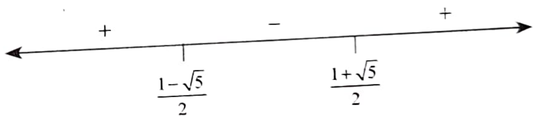
Solve $x^2 - x - 1 < 0$.

Sol. Let us first factorize $x^2 - x - 1$.

For that let $x^2 - x - 1 = 0$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Now on the number line (x-axis), mark $x = \frac{1 \pm \sqrt{5}}{2}$



From the sign scheme of $x^2 - x - 1$, which is shown in the figure, we have

$$x^2 - x - 1 < 0$$

$$\Rightarrow \quad x \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$

ILLUSTRATION 1.9

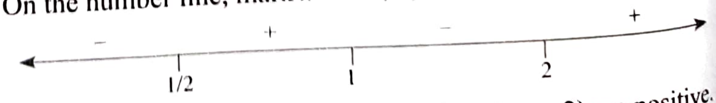
Solve $(x-1)(x-2)(1-2x) > 0$.

Sol. We have $(x-1)(x-2)(1-2x) > 0$

or $-(x-1)(x-2)(2x-1) > 0$

or $(x-1)(x-2)(2x-1) < 0$

On the number line, mark $x = 1/2, 1, 2$



When $x > 2$, all factors $(x-1)$, $(2x-1)$, and $(x-2)$ are positive.

Hence, $(x-1)(x-2)(2x-1) > 0$ for $x > 2$.

Now put positive and negative signs alternatively as shown in the figure.

Hence, solution set of $(x-1)(x-2)(1-2x) > 0$ or $(x-1)(x-2)(2x-1) < 0$ is $(-\infty, 1/2) \cup (1, 2)$.

ILLUSTRATION 1.10

Solve $\frac{2}{x} < 3$.

Sol. $\frac{2}{x} < 3$

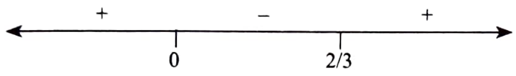
or $\frac{2}{x} - 3 < 0$ (We cannot crossmultiply with x as x can be negative or positive)

or $\frac{2-3x}{x} < 0$

or $\frac{3x-2}{x} > 0$

or $\frac{(x-2/3)}{x} > 0$

Sign scheme of $\frac{(x-2/3)}{x}$ is as follows:



$\Rightarrow \quad x \in (-\infty, 0) \cup (2/3, \infty)$

ILLUSTRATION 1.11

Solve $\frac{x-2}{x+2} > \frac{2x-3}{4x-1}$.

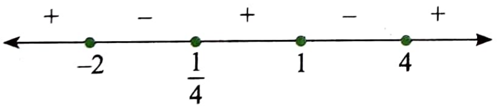
Sol. $\frac{x-2}{x+2} > \frac{2x-3}{4x-1}$

or $\frac{(x-2)(4x-1)-(x+2)(2x-3)}{(x+2)(4x-1)} > 0$

or $\frac{4x^2-9x+2-2x^2-x+6}{(x+2)(4x-1)} > 0$

or $\frac{x^2-5x+4}{(x+2)(4x-1)} > 0$

or $\frac{(x-1)(x-4)}{(x+2)(4x-1)} > 0$



$\Rightarrow \quad x \in (-\infty, -2) \cup \left(\frac{1}{4}, 1\right) 1 \cup (4, \infty)$

ILLUSTRATION 1.12

Solve $x > \sqrt{(1-x)}$.

Sol. Given inequality can be solved by squaring both sides. But sometimes squaring gives extraneous solutions which do not satisfy the original inequality. Before squaring we must restrict x for which terms in the given inequality are well defined.

$x > \sqrt{(1-x)}$. Here x must be positive.

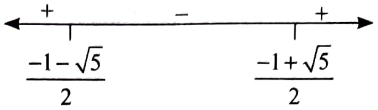
Now, $\sqrt{1-x}$ is defined only when $1-x \geq 0$ or $x \leq 1$

Thus $0 \leq x \leq 1$ (1)

Squaring given inequality both sides, $x^2 > 1-x$

$\Rightarrow \quad x^2 + x - 1 > 0$

$\Rightarrow \quad \left(x - \frac{-1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right) > 0$



$\Rightarrow \quad x < \frac{-1-\sqrt{5}}{2}$ or $x > \frac{-1+\sqrt{5}}{2}$ (2)

From (1) and (2), $x \in \left(\frac{\sqrt{5}-1}{2}, 1\right]$

ILLUSTRATION 1.13

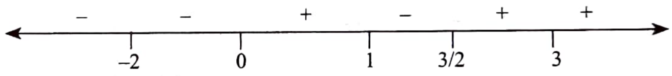
Solve $x(x+2)^2(x-1)^5(2x-3)(x-3)^4 \geq 0$.

Sol. Let $E = x(x+2)^2(x-1)^5(2x-3)(x-3)^4$.

Here for x, $(x-1)$, $(2x-3)$, exponents are odd; hence, the sign of E changes while crossing $x = 0$, 1, 3/2. Also for $(x+2)$, $(x-3)$ exponents are even; hence, the sign of E does not change while crossing $x = -2$ and $x = 3$.

Further for $x > 3$, all factors are positive; hence, the sign of the expression starts with a positive sign from the right hand side.

The sign scheme of the expression is as follows:



Hence, for $E \geq 0$, we have $x \in [0, 1] \cup [3/2, \infty)$

ILLUSTRATION 1.14

Solve $x(2^x-1)(3^x-9)(x-3) < 0$.

Sol. Let $E = x(2^x-1)(3^x-9)(x-3)$

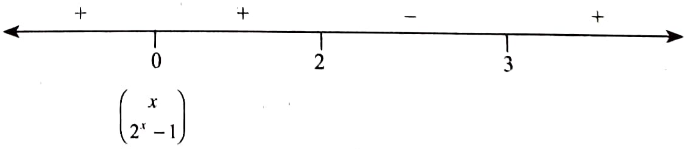
Here $2^x - 1 = 0 \Rightarrow x = 0$ and when $3^x - 9 = 0 \Rightarrow x = 2$

Now mark $x = 0$, 2, and 3 on real number line.

The sign of E starts with a positive sign from right hand side.

Also at $x = 0$, two factors vanish x and $2^x - 1$; hence, the sign of E does not change while crossing $x = 0$.

The sign scheme of E is as follows:



From the figure, we have $E < 0$ for $x \in (2, 3)$.

ILLUSTRATION 1.15

Solve $(2^x-1)(3^x-9)(\sin x - \cos x)(5^x-1) < 0$, $-\pi/2 < x < 2\pi$.

Sol. We have

$(2^x-1)(3^x-9)(\sin x - \cos x)(5^x-1) < 0, -\pi/2 < x < 2\pi$.

$2^x - 1 = 0$ if $x = 0$

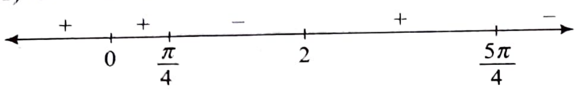
$3^x - 9 = 0$ if $x = 2$

$\sin x - \cos x = 0$ if $\tan x = 1$

or $\quad x = \frac{\pi}{4}, \frac{5\pi}{4}$

$5^x - 1 = 0$ if $x = 0$

The sign scheme of expression $(2^x - 1)(3^x - 9)(\sin x - \cos x)(5^x - 1)$ is as shown in the following figure.



When $x \in (-\pi/2, 0)$, each of the factors in the product is negative. So, expression is positive for $x \in (-\pi/2, 0)$.

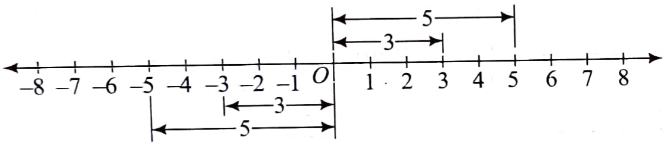
At $x = 0$, sign is not changing as two factors vanishes at $x = 0$, $2^x - 1$ and $5^x - 1$.

From the sign scheme, we have $(2^x - 1)(3^x - 9)(\sin x - \cos x)(5^x - 1) < 0$ for

$x \in (\pi/4, 2) \cup (5\pi/4, 2\pi)$

Modulus of Real Number

We represent real numbers on real number line. The interesting feature of a number on real number line is its distance from origin. Consider some numbers on a real number line as shown in the following figure:



The distance of both real numbers '–3' and '3' from '0', is 3 units.

The distances of both real numbers '–5' and '5' from '0', is 5 units.

Now, which real numbers are at distance 4 units from '0'?

The answer is '4' and '–4'.

The distance of any real number from '0' is called its absolute value or modulus of that real number.

It is denoted by $|x|$.

Thus $|x|$ = distance of 'x' from '0'.

Since $|x|$ denotes the distance, it is always non-negative i.e., $|x| \geq 0$.

Note:

- $x^2 = |x|^2$
- $|x| = |-x|$
- $|ab| = |a||b|$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Absolute Value as Square Root Function

We know that square root of real number x exists only if x is zero or positive.

Also, square root of positive real number is always positive.

Now, observe the following calculations:

$$2 = \sqrt{2^2} = \sqrt{(-2)^2} = \left[(-2)^2\right]^{\frac{1}{2}} = -2.$$

We know that this is absolutely absurd. There is some mathematical error in one of the steps.

Actually, $(-2)^2$ is positive and square root of positive is always positive.

So, we must have $\left[(-2)^2\right]^{\frac{1}{2}} = 2$ and for that we can write

$\left[(-2)^2\right]^{\frac{1}{2}} = |-2| = 2$.

Similarly, $\sqrt{4} = |4| = 4$ and $\sqrt{(-4)^2} = |-4| = -(-4) = 4$.

Thus, $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

e.g., $\sqrt{\left(1-\sqrt{2}\right)^2} = |1-\sqrt{2}| = \sqrt{2} - 1$.

Note:

$a^2 < b^2 \Rightarrow \sqrt{a^2} < \sqrt{b^2} \Rightarrow |a| < |b|$

Illustration 1.16

Find the value of x for which following expressions are defined:

(i) $\frac{1}{\sqrt{x - |x|}}$ **(ii)** $\frac{1}{\sqrt{x + |x|}}$

Sol.

(i) $x - |x| = \begin{cases} x - x = 0, \text{ if } x \geq 0 \\ x + x = 2x, \text{ if } x < 0 \end{cases}$

$\Rightarrow x - |x| \leq 0$, for all x

Thus, $\frac{1}{\sqrt{x - |x|}}$ does not take real values for any $x \in R$

So, $\frac{1}{\sqrt{x - |x|}}$ is not defined for any $x \in R$.

(ii) $x + |x| = \begin{cases} x + x = 2x, \text{ if } x \geq 0 \\ x - x = 0, \text{ if } x < 0 \end{cases}$

Thus, $\frac{1}{\sqrt{x + |x|}}$ is defined only when $x > 0$

Illustration 1.17

(i) For $2 < x < 4$, find the values of $|x|$.

(ii) For $-3 \leq x \leq -1$, find the values of $|x|$.

(iii) For $-3 \leq x < 1$, find the values of $|x|$.

(iv) For $-5 < x < 7$, find the values of $|x - 2|$.

(v) For $1 \leq x \leq 5$, find the values of $|2x - 7|$.

Sol.

(i) $2 < x < 4$

i.e., values on real number line whose distance from zero lies between 2 and 4.

Here values of x are positive $\Rightarrow |x| \in (2, 4)$

(ii) $-3 \le x \le -1$

Here values on real number line whose distance from zero lies between 1 and 3 or equal to 1 or 3.

$\Rightarrow \quad 1 \le |x| \le 3$

(iii) $-3 \le x < 1$

For $-3 \le x < 0$, $|x| \in (0, 3]$

For $0 \le x < 1$, $|x| \in [0, 1)$

So for $-3 \le x < 1$, $|x| \in [0, 1) \cup (0, 3]$ or $|x| \in [0, 3]$

(iv) $-5 < x < 7$

or $\quad -7 < x - 2 < 5$

$\Rightarrow \quad 0 \le |x-2| < 7$

(v) $1 \le x \le 5$

or $\quad 2 \le 2x \le 10$

$\Rightarrow \quad -5 \le 2x - 7 \le 3$

$\Rightarrow \quad |2x-7| \in [0, 5]$

Modulus of Any Expression

Consider two points $A(x_1, 0)$ and $B(x_2, 0)$ on the x-axis.

Now, distance $AB = \sqrt{(x_1-x_2)^2+(0-0)^2}$

$= \sqrt{(x_1-x_2)^2}$

$= |x_1 - x_2|$

Thus expression $|x_1 - x_2|$ represents the distance between two real numbers x_1 and x_2 on the real number line, just like $|x| = |x - 0|$ represents the distance between two real numbers x and 0.

Now, we shall extend the definition of absolute value.

$$|x-a| = \begin{cases} x-a, & x-a \ge 0 \\ a-x, & x-a < 0 \end{cases}$$

or

$$|x-a| = \begin{cases} x-a, & x \ge a \\ a-x, & x < a \end{cases}$$

The least value of $|x - a|$ is zero when $x = a$.

Note that $|x - a| = |a - x|$

In general, for any expression $E(x)$,

$$E(x) = \begin{cases} E(x), & E(x) \ge 0 \\ -E(x), & E(x) < 0 \end{cases}$$

ILLUSTRATION 1.18

Solve the following:

(i) $|x-2| = (x-2)$ (ii) $|x+3| = -x-3$

(iii) $|x^2-x| = x^2-x$ (iv) $|x^2-x-2| = 2+x-x^2$

Sol.

(i) $|x-2| = (x-2)$, if $x - 2 \ge 0$ or $x \ge 2$

(ii) $|x+3| = -x-3$, if $x + 3 \le 0$ or $x \le -3$

(iii) $|x^2-x| = x^2-x$, if $x^2 - x \ge 0$

or $\quad x(x-1) \ge 0$

$\Rightarrow \quad x \in (-\infty, 0] \cup [1, \infty)$

(iv) $|x^2-x-2| = 2+x-x^2$

or $\quad x^2 - x - 2 \le 0$

or $\quad (x-2)(x+1) \le 0$

or $\quad -1 \le x \le 2$

ILLUSTRATION 1.19

Prove that $\sqrt{x^2+2x+1} - \sqrt{x^2-2x+1} = \begin{cases} -2, & x < -1 \\ 2x, & -1 \le x \le 1 \\ 2, & x > 1 \end{cases}$

Sol. $\sqrt{x^2+2x+1} - \sqrt{x^2-2x+1}$

$= \sqrt{(x+1)^2} - \sqrt{(x-1)^2}$

$= |x+1| - |x-1|$

$$= \begin{cases} -x-1-(1-x), & x < -1 \\ x+1-(1-x), & -1 \le x \le 1 \\ x+1-(x-1), & x > 1 \end{cases}$$

$$= \begin{cases} -2, & x < -1 \\ 2x, & -1 \le x \le 1 \\ 2, & x > 1 \end{cases}$$

ILLUSTRATION 1.20

For $x \in R$, find all possible values of

(i) $|x-3| - 2$ (ii) $4 - |2x+3|$

Sol.

(i) We know that $|x-3| \ge 0 \ \forall \ x \in R$

$\Rightarrow \quad |x-3| - 2 \ge -2$

$\Rightarrow \quad |x-3| - 2 \in [-2, \infty)$

(ii) We know that $|2x+3| \ge 0 \ \forall \ x \in R$

$\Rightarrow \quad -|2x+3| \le 0$

$\Rightarrow \quad 4 - |2x+3| \le 4$

or $\quad 4 - |2x+3| \in (-\infty, 4]$

ILLUSTRATION 1.21

Find all possible values of

(i) $\sqrt{|x|-2}$ (ii) $\sqrt{3-|x-1|}$ (iii) $\sqrt{4-\sqrt{x^2}}$

Sol.

(i) $\sqrt{|x|-2}$

We know that square roots are defined for non-negative values only.

It implies that we must have $|x| - 2 \ge 0$. Thus,

$\sqrt{|x|-2} \ge 0$

(ii) $\sqrt{3-|x-1|}$ is defined when $3 - |x-1| \ge 0$

But the maximum value of $3 - |x-1|$ is 3, when $|x-1|$ is 0.

Hence, for $\sqrt{3-|x-1|}$ to get defined, $0 \le 3 - |x-1| \le 3$.

Thus,

$\sqrt{3-|x-1|} \in [0, \sqrt{3}]$

Alternatively, $|x-1| \ge 0$

$\Rightarrow \quad -|x-1| \le 0$

$\Rightarrow \quad 3 - |x-1| \le 3$

But for $\sqrt{3-|x-1|}$ to get defined, we must have

$0 \le 3 - |x-1| \le 3$

$\Rightarrow \quad 0 \le \sqrt{3-|x-1|} \le \sqrt{3}$

(iii) $\sqrt{4-\sqrt{x^2}} = \sqrt{4-|x|}$

$|x| \geq 0$

$\Rightarrow \quad -|x| \leq 0$

$\Rightarrow \quad 4-|x| \leq 4$

But for $\sqrt{4-|x|}$ to get defined $0 \leq 4-|x| \leq 4$

$\therefore \quad 0 \leq \sqrt{4-|x|} \leq 2$

ILLUSTRATION 1.22

Solve $|x-3| + |x-2| = 1$.

Sol. $|x-3| + |x-2| = 1$

$\Rightarrow \quad |x-3| + |x-2| = (3-x) + (x-2)$

$\Rightarrow \quad x-3 \leq 0$ and $x-2 \geq 0$

$\Rightarrow \quad x \leq 3$ and $x \geq 2$

$\Rightarrow \quad 2 \leq x \leq 3$

ILLUSTRATION 1.23

Solve $\dfrac{|x+3|+x}{x+2} > 1$.

Sol. We have $\dfrac{|x+3|+x}{x+2} > 1$

Clearly, the L.H.S. of this inequation is meaningful for $x \neq -2$.

Given $\dfrac{|x+3|+x}{x+2} > 1$

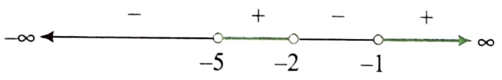
or $\dfrac{|x+3|+x}{x+2} - 1 > 0$

or $\dfrac{|x+3|+x-x-2}{x+2} > 0$

or $\dfrac{|x+3|-2}{x+2} > 0$

If $|x+3|-2=0$, then $x+3=\pm 2$ or $x=-5,-1$.

Hence, the sign scheme of the expression $\dfrac{|x+3|-2}{x+2}$ is as follows:



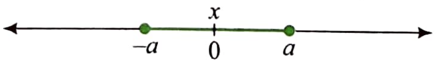
From the above sign scheme, $x \in (-5,-2) \cup (-1, \infty)$.

Inequalities Involving Modulus

We have following inequalities involving modulus

(i) $|x| \leq a$ (where $a > 0$)

The values of x satisfying above inequality are those values of x on real number line which are at distance 'a' or less than 'a'.



$\Rightarrow \quad -a \leq x \leq a$

e.g., $|x| \leq 2 \Rightarrow -2 \leq x \leq 2$

$|x| < 3 \Rightarrow -3 < x < 3$ etc.

In general, if $|f(x)| \leq a$ (where $a > 0$), then $-a \leq f(x) \leq a$.

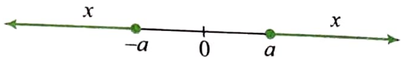
We solve $f(x) \leq a$ and $f(x) \geq -a$ further and find common values of x.

Note:

If $|x| \leq a$, then $x \in [-a, a]$, but if $0 < |x| \leq a$, then $a \in [-a, a] - \{0\}$.

(ii) $|x| \geq a$ (where $a > 0$)

The values of x satisfying above inequality are those values of x on real number line which are at distance a or more than a



$\Rightarrow \quad x \leq -a$ or $x \geq a$

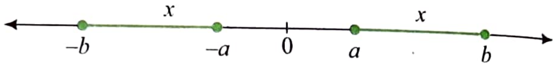
e.g., $|x| \geq 3 \Rightarrow x \leq -3$ or $x \geq 3$.

$|x| > 2 \Rightarrow x < -2$ or $x > 2$ etc.

In general, $|f(x)| \geq a \Rightarrow f(x) \leq -a$ or $f(x) \geq a$

(iii) $a \leq |x| \leq b$ (where $a, b > 0$)

The values of x satisfying above inequality are those values of x on real number line which are at distance a or b or between a and b.



$\Rightarrow \quad [-b, -a] \cup [a, b]$

e.g., $2 \leq |x| \leq 4 \Rightarrow x \in [-4,-2] \cup [2, 4]$

ILLUSTRATION 1.24

Solve $|3x-2| \leq \dfrac{1}{2}$.

Sol. $|3x-2| \leq \dfrac{1}{2}$

or $-\dfrac{1}{2} \leq 3x-2 \leq \dfrac{1}{2}$

or $\dfrac{3}{2} \leq 3x \leq \dfrac{5}{2}$

or $\dfrac{1}{2} \leq x \leq \dfrac{5}{6}$

or $x \in \left[\dfrac{1}{2}, \dfrac{5}{6}\right]$

ILLUSTRATION 1.25

Solve $||x-1|-5| \geq 2$.

Sol. $||x-1|-5| \geq 2$

i.e., $|x-1|-5 \leq -2$ or $|x-1|-5 \geq 2$

i.e., $|x-1| \leq 3$ or $|x-1| \geq 7$

i.e., $-3 \leq x-1 \leq 3$ or $x-1 \leq -7$ or $x-1 \geq 7$

i.e., $-2 \leq x \leq 4$ or $x \leq -6$ or $x \geq 8$

i.e., $x \in (-\infty, -6] \cup [-2, 4] \cup [8, \infty)$

ILLUSTRATION 1.26

Solve: $\dfrac{-1}{|x|-2} \geq 1$.

Sol. Given, $\frac{-1}{|x|-2} \geq 1$

$\Rightarrow \frac{-1}{|x|-2} - 1 \geq 0$

$\Rightarrow \frac{-1-(|x|-2)}{|x|-2} \geq 0$

$\Rightarrow \frac{1-|x|}{|x|-2} \geq 0$

$\Rightarrow \frac{|x|-1}{|x|-2} \leq 0$

$\Rightarrow \frac{y-1}{y-2} \leq 0$, where $y = |x|$

$\Rightarrow 1 \leq y < 2$

$\Rightarrow 1 \leq |x| < 2$

$\Rightarrow x \in (-2, -1] \cup [1, 2)$

Modulus of Addition and Subtraction

We know that modulus is distributive over multiplication and division. But it not distributive over addition and subtraction always.

Also, we know that $|x-y|$ = distance between two real numbers x and y on real number line.

We have following properties.

(i) $|x+y| < |x| + |y|$ if x and y have opposite sign.

(ii) $|x-y| < |x| + |y|$ if x and y have same sign.

(iii) $|x+y| = |x| + |y|$ if x and y have same sign or at least one of x and y is zero.

(iv) $|x-y| = |x| + |y|$ if x and y have opposite sign or at least one of x and y is zero.

Hence in general, $|x \pm y| \leq |x| + |y|$.

Illustration 1.27

Solve $|x-1| + |x-2| \geq 4$.

Sol. Let $f(x) = |x-1| + |x-2|$

A. x	**B.** $f(x)$	**C.** x **for which** $f(x) \geq 4$	**D.** $A \cap C$
$x < 1$	$1-x+2-x$ $= 3-2x$	$3-2x \geq 4 \Rightarrow x \leq -\frac{1}{2}$	$x \leq -1/2$
$1 \leq x \leq 2$	$x-1+2-x = 1$	$1 \geq 4$, not possible	
$x > 2$	$x-1+x-2$ $= 2x-3$	$2x-3 \geq 4 \Rightarrow x \geq 7/2$	$x \geq 7/2$

Hence, solutions is $x \in (-\infty, -1/2] \cup [7/2, \infty)$.

Alternative Method:

Consider $|x-1| + |x-2| < 4$

Now, $|(x-1)+(x-2)| \leq |x-1| + |x-2|$

$\therefore |(x-1)+(x-2)| < 4$

$\Rightarrow |2x-3| < 4$

$\Rightarrow -4 < 2x-3 < 4$

$\Rightarrow -1 < 2x < 7$

$\Rightarrow -1/2 < x < 7/2$

Therefore, for $|x-1| + |x-2| \geq 4$, we have $x \in (-\infty, -1/2] \cup [7/2, \infty)$.

Illustration 1.28

Solve $|\sin x + \cos x| = |\sin x| + |\cos x|$, $x \in [0, 2\pi]$.

Sol. The given relation holds only when $\sin x$ and $\cos x$ have the same sign or at least one of them is zero.

Hence, $x \in [0, \pi/2] \cup [\pi, 3\pi/2] \cup \{2\pi\}$.

Illustration 1.29

Solve $|-2x^2 + 1 + e^x + \sin x| = |2x^2 - 1| + e^x + |\sin x|$, $x \in [0, 2\pi]$.

Sol. $|-2x^2 + 1 + e^x + \sin x| = |2x^2 - 1| + e^x + |\sin x|$, $x \in [0, 2\pi]$

In the R.H.S., each term is positive and $e^x > 0$. So,

$1 - 2x^2 \geq 0$ and $\sin x \geq 0$

or $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ and $x \in [0, \pi]$

$\therefore x \in \left[0, \frac{1}{\sqrt{2}}\right]$

Concept Application Exercise 1.1

1. If sets A = [–4, 1) and B = [0, 3), then find the following sets:
   (a) $A \cap B$ (b) $A \cup B$ (c) $A - B$
   (d) $B - A$ (e) $(A \cup B)'$ (f) $(A \cap B)'$

2. Find the value of x^2 for the following values of x:
   (a) $[-5, -1]$ (b) $(3, 6)$ (c) $(-2, 3]$
   (d) $(-3, \infty)$ (e) $(-\infty, 4)$

3. Find the values of $1/x$ for the following values of x:
   (a) $(2, 5)$ (b) $[-5, -1)$ (c) $(3, \infty)$
   (d) $(-\infty, -2]$ (e) $[-3, 4]$

4. Find all possible values (range) of the following quadratic expressions when $x \in R$ and when $x \in [-3, 2]$.
   (a) $4x^2 + 28x + 41$ (b) $1 + 6x - x^2$

5. Find all the possible values of expression $\frac{2+x^2}{4-x^2}$.

6. Solve $\frac{x(3-4x)(x+1)}{(2x-5)} < 0$.

7. Solve $\frac{(2x+3)(4-3x)^3(x-4)}{(x-2)^2 x^5} \leq 0$.

8. Solve $\frac{5x+1}{(x+1)^2} < 1$

9. Find the number of integral values of x satisfying $\sqrt{-x^2+10x-16} < x-2$.

10. Find all possible values of $f(x) = \frac{1-x^2}{x^2+3}$.

11. Find the values of x for which the following function is defined: $f(x) = \sqrt{\frac{1}{|x-2|-(x-2)}}$

12. Solve $|4-|x-1||=3$

13. Find all values of $f(x)$ for which $f(x)=x+\sqrt{x^2}$.

14. Solve the following:

(a) $1 \le |x-2| \le 3$ (b) $0 < |x-3| \le 5$

(c) $|x-2|+|2x-3|=|x-1|$ (d) $\left|\frac{x-3}{x+1}\right| \le 1$

15. Find all possible values of expression $\sqrt{1-\sqrt{x^2-6x+9}}$.

ANSWERS

1. (a) $[0, 1)$ (b) $[-4, 3)$ (c) $[-4, 0)$ (d) $[1, 3)$ (e) $(-\infty, -4) \cup [3, \infty)$ (f) $(-\infty, 0) \cup [1, \infty)$

2. (a) $[1, 5]$ (b) $(9, 36)$ (c) $[0, 9]$ (d) $[0, \infty)$ (e) $[0, \infty)$

3. (a) $\left(\frac{1}{5}, \frac{1}{2}\right)$ (b) $\left(-1, -\frac{1}{5}\right]$ (c) $\left(0, \frac{1}{3}\right)$ (d) $\left[-\frac{1}{2}, 0\right)$ (e) $\left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{1}{4}, \infty\right)$

4. (a) $[-8, \infty)$ when $x \in R$; $[-7, 113]$ when $x \in [-3, 2]$
(b) $(-\infty, 10]$ when $x \in R$; $[-26, 9]$ when $x \in [-3, 2]$

5. $(-\infty, -1) \cup \left[\frac{1}{2}, \infty\right)$

6. $x \in (-\infty, -1) \cup (0, 3/4) \cup (5/2, \infty)$

7. $x \in (-\infty, -3/2) \cup (0, 4/3] \cup [4, \infty)$

8. $x < 0$ or $x > 3, x \ne -1$ **9.** 3

10. $y \in (-1, 1/3]$ **11.** $(-\infty, 2)$

12. $x = -6, 0, 2, 8$ **13.** $[0, \infty)$

14. (a) $[-1,1] \cup [3,5]$ (b) $[-2,8]-\{3\}$ (c) $[3/2, 2]$ (d) $[1, \infty)$

15. $[0, 1]$

RELATION

A relation from a non-empty set A to a non-empty set B is a subset of cartesian product $A \times B$.

Thus, a relation is also a set of ordered pairs. There need be no relationship between the components of the ordered pairs; any set of ordered pairs is a relation. Usually, however, we choose which ordered pairs belong to the relation so that components are related in some way, so we think of the relation as somehow representing the connection. Thus, many times subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the image of the first element. A relation on A is a subset of $A \times A$.

Consider sets $A = \{1, 4, 5\}$ and $B = \{1, 2, 7\}$

Let us define relation R from A to B as set of ordered pairs in which first element is less than the second element.

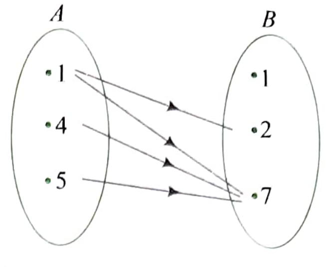
So, we have relation as set of ordered pairs $R = \{(1, 2), (1, 7), (4, 7), (5, 7)\}$. Clearly this is a subset of cartesian product $A \times B$ which has nine elements. Here relation is expressed as set a of ordered pairs. This representation of relation is called Roaster form which is the algebraic way.

One more way to represent relation is Set builder form. In this form we express relation by writing the rule between first and second element (mathematically or in words) of all ordered pairs of relation.

In above example we write,

$$R = \{(x, y) : x < y, x \in A, y \in B\}$$

A visual representation of relation is Arrow diagram as shown in the following figure.



The fact that "ordered pair (a, b) is element of relation R" is also denoted as aRb.

DOMAIN OF RELATION

The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R. Domain of relation is either equal to or subset of set of first elements. In above example domain is $\{1, 4, 5\}$.

RANGE AND CODOMAIN OF RELATION

The set of all second elements in a relation R from a set A to a set B is called the range of the relation R.

The whole set B is called the codomain of the relation R.

Range of relation is either equal to or subset of set of second elements.

Thus, Range $\subseteq$ Codomain.

In above example range is $\{2, 7\}$ and codomain is $\{1, 2, 7\}$.

NUMBER OF RELATIONS

Since any subset of cartesian product $A \times B$ is relation, the total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is 2^{pq}, which are total number of subsets of $A \times B$.

Note:

We know that all the points on coordinate plane represent ordered pairs of cartesian product $R \times R$, where R is set of real numbers. Now if we consider graph of some relation on coordinate plane then set of points (ordered pair) lying on the graph is subset of $R \times R$ (whole plane).

ILLUSTRATION 1.30

Let $A = \{1, 2, 3, 4, 6\}$. Let R be the relation on A defined by $\{(a, b): a, b \in A, a$ is exactly divisible by $b\}$. Write R in roster form. Find its domain and range.

Sol. We have $A = \{1, 2, 3, 4, 6\}$,

$R = \{(a, b): a, b \in A, a$ is exactly divisible by $b\}$

$\therefore$ $R = \{(1, 1), (2, 1), (3, 1), (4, 1), (6, 1), (2, 2), (4, 2), (6, 2), (3, 3), (6, 3), (4, 4), (6, 6)\}$

Domain of R is $\{1, 2, 3, 4, 6\}$.

Range is of $R = \{1, 2, 3, 4, 6\}$.

Here Range is same as codomain.

ILLUSTRATION 1.31

If given $R = \{(x, y) : x, y \in W, x^2 + y^2 = 25\}$. Find the domain and range of R.

Sol. We have,

$$R = \{(x, y) : x, y \in W, x^2 + y^2 = 25\}$$
$$= \{(0, 5), (3, 4), (4, 3), (5, 0)\}$$

Domain of $R \equiv$ Set of first element of ordered pairs in R

$$\equiv \{0, 3, 4, 5\}$$

Range of $R \equiv$ Set of second element of ordered pairs in R

$$\equiv \{5, 4, 3, 0\}$$

ILLUSTRATION 1.32

If $R_1 = \{(x, y) \mid y = 2x + 7$, where $x \in R$ and $-5 \le x \le 5\}$ is a relation. Then find the domain and Range of R_1.

Sol. We have,

$$R_1 = \{(x, y) \mid y = 2x + 7, \text{ where } x \in R \text{ and } -5 \le x \le 5\}$$

Domain of $R_1 \equiv \{-5 \le x \le 5, x \in R\} \equiv [-5, 5]$

$\because \quad x \in [-5, 5]$, $2x \in [-10, 10]$

$\Rightarrow \quad 2x + 7 \in [-3, 17]$

Thus range is $[-3, 17]$.

TYPES OF RELATION

Empty Relation

A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

Universal Relation

A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e., $R = A \times A$.

Both the empty relation and the universal relation are sometimes called trivial relations.

Inverse Relation

Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Domain (R) = Range (R^{-1}) and Range (R) = Domain (R^{-1})

Identity Relation

Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In order words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

Example:

The relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on set $A = \{1, 2, 3\}$. But relations $R_1 = \{(1, 1), (2, 2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R_1$ and in R_2 element 1 is related to elements 1 and 3.

Reflexive Relation

A relation R on a set A is said to be reflexive relation, if $(a, a) \in R$, for every $a \in A$.

A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Example:

Let $A = \{1, 2, 3\}$ be a set. Then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A.

But $R_1 = \{(1, 1), (3, 2), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \notin R_1$.

Note:

- The identity relation on a non-void set A is always reflexive relation on A. However, a reflexive relation on A is not necessarily the identity relation on A. For example, the relation $R = \{(a, a), (b, b), (c, c), (a, b)\}$ is a reflexive relation on set $A = \{a, b, c\}$ but is not the identity relation on A.
- The universal relation on a non-void set A is reflexive.

Example:

A relation R on N defined by $(x, y) \in R \Leftrightarrow x \ge y$ is a reflexive relation on N, because every natural number is greater than or equal to itself.

Example:

Let X be a non-void set and $P(X)$ be the power set of X. A relation R on $P(X)$ defined by $(A, B) \in R \Leftrightarrow A \subseteq B$ is a reflexive relation since every set is subset of itself.

Example:

Let L be the set of all lines in a plane. Then relation R on L defined by $(l_1, l_2) \in R \Leftrightarrow l_1$ is parallel to l_2 is reflexive, since every line is parallel to itself.

Symmetric Relation

A relation R on a set A is said to be a symmetric relation, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

Example:

Consider, set A of natural numbers. If a relation A be defined by "$x + y = 5$", then this relation is symmetric in A.

for $a + b = 5 \Rightarrow b + a = 5$

But in the set A of natural numbers if the relation R be defined as 'x is a divisor of y', then the relation R is not symmetric as $(3, 9) \in R$ does not imply $(9, 3) \in R$; for, 3 divides 9 but 9 does not divide 3.

Transitive Relation

A relation R on a set A is said to be a transitive relation, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.

Example:

In the set A of natural numbers if the relation R be defined by 'x less than y' then $a < b$ and $b < c$ imply $a < c$, that is, $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

Hence this relation is transitive.

Equivalence Relation

A relation R on a set A is said to be transitive relation if R is reflexive, symmetric and transitive.

ILLUSTRATION 1.33

Show that the relation R in the set R of real numbers, defined as $R = \{(a, b): a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

Sol. $R = \{(a, b) : a \leq b^2\}$

Here $(1, 1) \in R$ as $1 \leq 1$, but $0.5 \notin R$ as $0.5 > (0.5)^2$

Therefore, R is not reflexive.

Now, $(1, 4) \in R$ as $1 < 4^2$

But, 4 is not less than 1.

Therefore, $(4, 1) \notin R$

Therefore, R is not symmetric.

Further, $(3, 2), (2, 1.5) \in R$ (as $3 < 2^2$ and $2 < (1.5)^2$)

But, $3 > (1.5)^2 = 2.25$

Therefore, $(3, 1.5) \notin R$

Therefore, R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

ILLUSTRATION 1.34

Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{ is even}\}$, is an equivalence relation.

Sol. $A = \{1, 2, 3, 4, 5\}$

$R = \{(a, b): |a - b| \text{ is even}\}$

It is clear that for any element $a \in A$, we have $|a - a| = 0$ (which is even).

Therefore, R is reflexive.

Let $(a, b) \in R$.

$\Rightarrow$ $|a - b|$ is even,

$\Rightarrow$ $|-(a - b)| = |b - a|$ is also even

$\Rightarrow$ $(b, a) \in R$

Therefore, R is symmetric.

Now, let $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow$ $|a - b|$ is even and $|b - c|$ is even

$\Rightarrow$ $(a - b)$ is even and $(b - c)$ is even

(assuming that $a > b > c$)

$\Rightarrow$ $(a - c) = (a - b) + (b - c)$ is even

[Sum of two even integers is even]

$\Rightarrow$ $|a - c|$ is even

$\Rightarrow$ $(a, c) \in R$

Therefore, R is transitive.

Hence, R is an equivalence relation.

ILLUSTRATION 1.35

Show that the relation R in the set A of points in a plane given by $R = \{(P, Q)$: distance of the point P from the origin is same as the distance of the point Q from the origin$\}$, is an equivalence relation.

Sol. R $\{(P, Q)$: distance of point P from the origin is the same as the distance of point Q from the origin$\}$

Clearly, $(P, P) \in R$ since the distance of point P from the origin is always the same as the distance of the same point P from the origin.

Therefore, R is reflexive.

Now, let $(P, Q) \in R$.

$\Rightarrow$ The distance of point P from the origin is the same as the distance of point Q from the origin.

$\Rightarrow$ The distance of point Q from the origin is the same as the distance of point P from the origin.

$\Rightarrow$ $(Q, P) \in R$

Therefore, R is symmetric.

Now, let $(P, Q), (Q, S) \in R$.

$\Rightarrow$ The distance of points P and Q from the origin is the same and also, the distance of points Q and S from the origin is the same.

$\Rightarrow$ the distance of points P and S from the origin is the same.

$\Rightarrow$ $(P, S) \in R$

Therefore, R is transitive.

Therefore, R is an equivalence relation.

ILLUSTRATION 1.36

Show that the relation R defined in the set A of all triangles as $R = \{(T_1, T_2): T_1 \text{ is similar to } T_2\}$, is equivalence relation.

Sol. $R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$

R is reflexive since every triangle is similar to itself.

Further, if $(T_1, T_2) \in R$, then T_1 is similar to T_2.

$\Rightarrow$ T_2 is similar to T_2

$\Rightarrow$ $(T_2, T_1) \in R$.

Therefore, R is symmetric.

Now, let $(T_1, T_2), (T_2, T_3) \in R$.

$\Rightarrow$ T_1 is similar to T_2 and T_2 is similar to T_3

$\Rightarrow$ T_1 is similar to T_3

$\Rightarrow$ $(T_1, T_3) \in R$

Therefore, R is transitive.

Thus, R is an equivalence relation.

ILLUSTRATION 1.37

Given a non-empty set X, consider $P(X)$ which is the set of all subsets of X. Define the relation R in $P(X)$ as follows: For subsets A, B in $P(X)$, ARB if and only if $A \subset B$. Is R an equivalence relation on $P(X)$? Justify your answer.

Sol. Since every set is a subset of itself, ARA for all $A \in P(X)$.

Therefore, R is reflexive.

Let $ARB \Rightarrow A \subset B$.

This cannot be implied to $B \subset A$.

For instant, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A.

Therefore, R is not symmetric.

Further if ARB and BRC, then $A \subset B$ and $B \subset C$.

$\Rightarrow$ $A \subset C$

$\Rightarrow$ ARC

Therefore, R is transitive.

Hence, R is not an equivalence relation since it is not symmetric.

CONCEPT APPLICATION EXERCISE 1.2

1. **(a)** If $n(A) = 6$ and $n(A \times B) = 42$ then find $n(B)$
   (b) If some of the elements of $A \times B$ are (x, p), (p, q), (r, s). Then find the minimum value of $n(A \times B)$.
2. Let $A = \{1, 2, 3, \ldots, 14\}$. Define a relation R from A to A by $R = \{(x, y): x - 3y = 0$, where $x, y \in A\}$. Write down its domain, codomain and range.
3. Let $A = \{1, 2, 3\}$. Then find the number of equivalence relations containing $(1, 2)$
4. Let S be the set of all real numbers. Then show that the relation $R = \{(a, b) : 1 + ab > 0\}$ on S is Reflexive and symmetric but not transitive.
5. Let Z be the set of all integers and R is the relation on Z defined as $R = \{(a, b) : a, b \in Z$ and $a - b$ is divisible by $5\}$. Prove that R is an equivalence relation.

ANSWERS

1. **(a)** 7 **(b)** 9
2. $R = \{(3, 1), (6, 2), (9, 3), (12, 4)\}$
   Domain: $\{3, 6, 9, 12\}$, Range: $\{1, 2, 3, 4\}$, Codomain: A
3. 2

FUNCTION (MAPPING)

Function is special type of relation. A function or mapping from a set A to a set B is a relation f from A to B with the property that for every element $a \in A$ there exists one and only one element $b \in B$ such that $(a, b) \in f$.

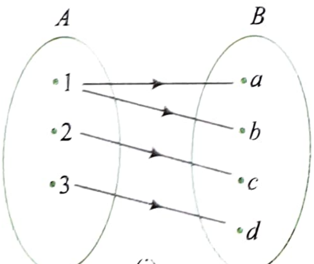
In other words, a function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

If f is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where 'b' is called the image of 'a' under f and 'a' is called the preimage of 'b' under f.

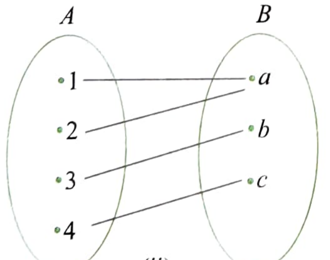
The function f from set A to set B is denoted by $f: A \to B$

Following are some relations expressed as arrow diagram.

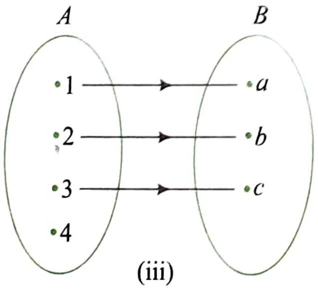
Let us check which of these represent function from set A to set B.



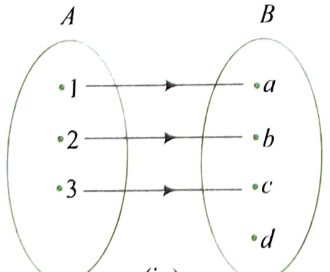
(i)



(ii)



(iii)



(iv)

Relation in arrow diagram (i) is not a function as preimage '1' has two distinct images 'a' and 'b'.

Relation in arrow diagram (ii) is a function as every preimage has unique image.

However, preimages '1' and '2' have common image. This is possible in function.

Relation in arrow diagram (iii) is not a function as preimage '4' is not associated with any image.

Relation in arrow diagram (iv) is a function as every preimage has unique image.

However, 'd' is not associated with any preimage. This is possible in function.

DOMAIN, CODOMAIN AND RANGE OF FUNCTION

In function $f : A \to B$ defined from set A to B, set A is called domain and set B is called codomain of the function.

Unlike relation where domain may be subset of set A the whole set A is domain of the function.

The set of all possible images of the preimages is called range of the function.

Range is either equal to codomain or subset of codomain.

ILLUSTRATION 1.38

Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not?

(i) $R = \{(4, 1), (5, 1), (6, 7)\}$
(ii) $R = \{(2, 3), (2, 5), (3, 3), (6, 6)\}$
(iii) $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7)\}$
(iv) $R = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)\}$

Sol.

(i) Since 4, 5, 6 are the elements of domain of R having their unique images, this relation R is a function.
(ii) Since the same first element 2 corresponds to two different images 3 and 5, this relation is not a function.
(iii) Since every element has one and only one image, this relation is a function.
(iv) Since every element has one and only one image, this relation is a function.

ILLUSTRATION 1.39

If A is set of different triangles in the plane and B is set of all positive real numbers. A relation R is defined from set A to set B such that every element of set A is associated with some number in set B which is measure of area of triangle. Is this relation as function?

Sol. Two or more different triangles may have same area.

But area of each triangle is unique.

This means each element(triangle) of set A is associated with unique value (Area) of set B.

So, given relation is function.

ILLUSTRATION 1.40

A relation R is defined from N to N as $R = \{(ab, a + b) : a, b \in N\}$. Is R a function from N to N? Justify your answer.

Sol. The relation f is defined as $f = \{(ab, a + b) : a, b \in Z\}$

We know that a relation f from a set A to a set B is said to be function if every element of set A has unique images in set B.

Let $ab = 16$. We have following possibilities.

If $\quad a = 1, b = 16$, then $a + b = 17$

If $\quad a = 2, b = 8$, then $a + b = 10$

If $\quad a = 4, b = 4$, then $a + b = 8$

Thus, first element '16' has three corresponding second elements 17, 10, 8.

So, Relation R is not a function.

ILLUSTRATION 1.41

Set A has m distinct elements and set B has n distinct elements. Then how many different mappings from set A to set B can be formed?

Sol. Since each element(preimage) in set A can corresponds to any one of the n elements (images) in set B, the number of mappings can be formed is $n \times n \times n \ldots m$ times $= n^m$.

FUNCTION AS A RELATION BETWEEN INDEPENDENT AND DEPENDENT VARIABLES

We know that in mathematics constant is the value which remains unchanged and variable is a value that may change within the scope of a given problem or set of operations.

Variables are further distinguished as being either a dependent variable or an independent variable. Independent variables are considered as inputs or arguments to a function and may take on different values freely.

Dependent variables are considered as output that change as a consequence to changes in other values in the function. When one value is completely determined by another then it is called a function of the other value or values. In this case the value of the function is a dependent variable and the other values are independent variables. The notation $f(x)$ is used for the value of the function f with x representing the independent variable.

For example, area of circle is dependent on radius r. So, we can write $y = f(r) = \pi r^2$, here we can take r as any positive real number, hence r is independent variable. But value of Area depends on value of radius r, hence area, $f(r)$ is dependent variable. We plot the graph of the function on the coordinate plane with x-axis and y-axis as axis of independent variable and dependent variable respectively.

Now consider function $f : R \to R$, where $f(x) = 3x + 2$.

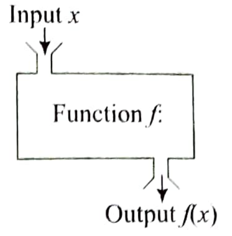
Here x is independent variable, which can take any real value. For given value of x we get specific value of $f(x)$ or y which is dependent variable. Here '$3x + 2$' can be considered as rule to relate independent and dependent variables.

For different inputs $x = 1, 2, 3 \ldots$ we get different outputs as $f(1) = 3 + 2 = 5, f(2) = 3 \times 2 + 2 = 8, f(3) = 3 \times 3 + 2 = 11 \ldots$.

Input values are preimages and output values are images.

Function can also be thought of as an input-output machine; you give the function an input, and it gives you an output.

For example, the squaring function takes the input 4 and gives the output value 16. The same squaring function takes the input -1 and gives the output value 1.



Now given function cannot process all the inputs. Like if we consider square root function, $f(x) = \sqrt{x}$, it cannot take negative real numbers as input. The input values to the function are non-negative real number only. We can say that domain of function is non-negative real numbers (zero and positive real numbers). Input outside the domain cannot be processed by the function. Thus complete (exhaustive) domain of the function is the set value real values of x for which the rule of the function is meaningful or well defined.

Consider function $f : N \to R$, where $f(x) = 2x - 3$. Here given domain of the function is N, set of natural numbers. But function $f(x)$ can process any real number to produce output. So, exhaustive or complete domain of the function is R, set of real numbers. Here domain N appears by the choice. But certainly, given domain should not contain any value which function can't process.

Depending upon rule to express the relation between dependent and independent variable, there are two types of functions.

Explicit Function

The word explicit describes something that is expressed clearly. An explicit function is a function that is expressed clearly. A function in which the dependent variable can be written explicitly in terms of the independent variable is called explicit function. e.g., functions: $y = 2x + 3, y = x^2 + x + 1, y = \log_2 x, y = e^x + \sin x$ etc.

Implicit Function

The Implicit function is a function in which the dependent variable has not been given "explicitly" in terms of the independent variable. Or it is a function in which the dependent variable is not expressed in terms of some independent variables.

Implicit functions are usually given in terms of both dependent and independent variables. It is denoted by $R(x, y) = 0$. Some examples of Implicit Functions are $x^2 + y^2 - xy = 0, x + y + \sin(xy) = 0$ etc.

Although we can convert a implicit function into explicit function it is generally not done because after conversion the new explicit function becomes very complex and sometimes also gives two different function branch. For example we can convert the implicit function $x^2 + y^2 = 16$ to a explicit function. But after conversion it gives the explicit functions $y = \pm\sqrt{4 - x^2}$. Now which branch is to be used? Because of this confusion we consider original implicit function for further processing.

Bounded and Unbounded Functions

A function f defined on some set X is called bounded, if the set of its values is bounded. In other words, if there exists a real number M such that $|f(x)| \leq M$, then function is bounded. A function that is not bounded is said to be unbounded.

If f is real-valued and $f(x) \le A$ for all x in X, then the function is said to be bounded (from) above by A. If $f(x) \ge B$ for all x in X, then the function is said to be bounded (from) below by B. A real-valued function is bounded if and only if it is bounded from above and below.

Some examples of bounded function are

Function	Range
$\sin x$, $\cos x$	$[-1, 1]$
$\frac{1}{1+x^2}$	$(0, 1]$
Dirichlet function, $\begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$	$\{0, 1\}$

Some examples of unbounded function are

Function	Range
$\sec x$, $\operatorname{cosec} x$	$(-\infty, -1] \cup [1, \infty)$
$\tan x$, $\cot x$, $\log_e x$	R
e^x	$(0, \infty)$

ILLUSTRATION 1.42

Write explicitly functions of y defined by the following equations.

(i) $x + |y| = 2y$ **(ii)** $e^y - e^{-y} = 2x$ **(iii)** $10^x + 10^y = 10$

Sol.

(i) $x + |y| = 2y$

If $y \ge 0$, we have $x + y = 2y \Rightarrow y = x$

$\Rightarrow y = x, x \ge 0$

If $y < 0$, we have $x - y = 2y \Rightarrow y = \frac{x}{3}$

$\Rightarrow y = \frac{x}{3}, x < 0$

Hence, $y = \begin{cases} \frac{x}{3}, & x < 0 \\ x, & x \ge 0 \end{cases}$

(ii) $e^y - e^{-y} = 2x$

$\Rightarrow e^{2y} - 2xe^y - 1 = 0$ (Multiplying by e^y)

$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$

$\Rightarrow e^y = x + \sqrt{x^2 + 1}$ (as $\sqrt{x^2+1} > x$, then $x - \sqrt{x^2+1} < 0$, which is not possible)

$\Rightarrow y = \log_e(x + \sqrt{x^2 + 1})$

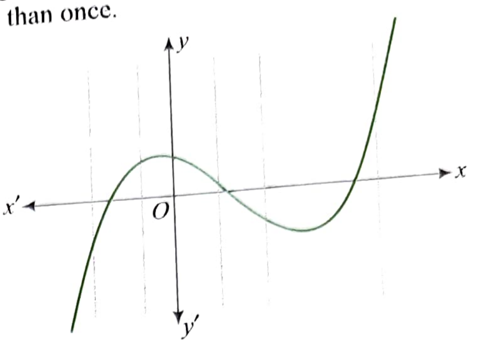
(iii) $10^x + 10^y = 10$

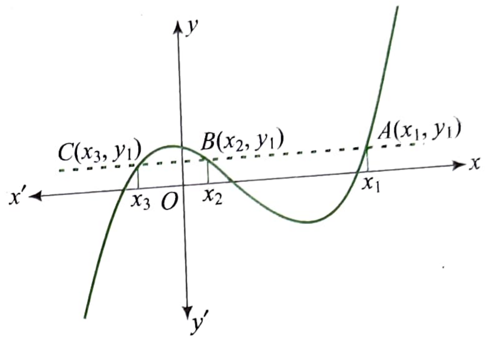
$\Rightarrow 10^y = 10 - 10^x$

$\Rightarrow y = \log_{10}(10 - 10^x)$

VERTICAL LINE TEST

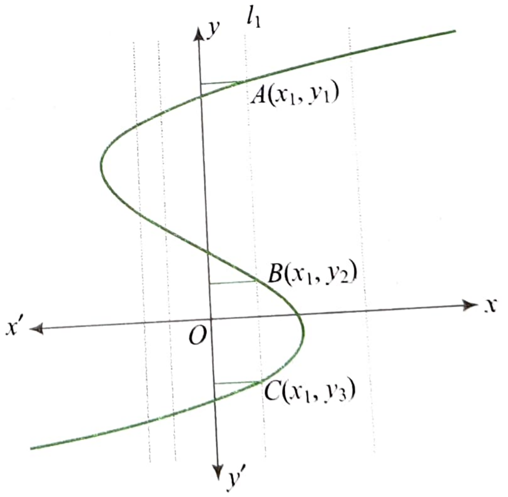
All the ordered pairs (x, y) satisfying the function $y = f(x)$ are plotted on coordinate plane and joined to get the graph of the function which is subset of $R \times R$ (R is set of real numbers). For given value of x(preimage) there is one and only one value of y (image) on the graph, i.e., there is no vertical line on the plane which meets the graph of the function in more than one point. However, there may be images which have more than one preimage. So, there may be horizontal lines intersecting the graph more than once.





Above two figures represent graph of the same relation. In first figure we observe that each of the vertical lines is intersecting the graph once. So, the relation in the graph represents function. In second figure we observe that one horizontal line is intersecting the graph at three distinct points $A(x_1, y_1)$, $B(x_2, y_1)$ and $C(x_3, y_1)$. Thus $f(x_1) = f(x_2) = f(x_3) = y_1$, i.e., three preimages x_1, x_2 and x_3 have common image y_1 or three different inputs x_1, x_2 and x_3 have same output y_1. This is possible in the function.

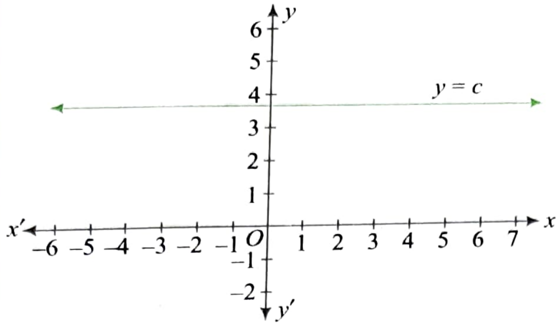
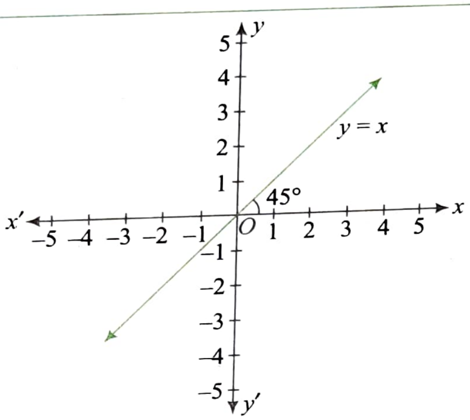
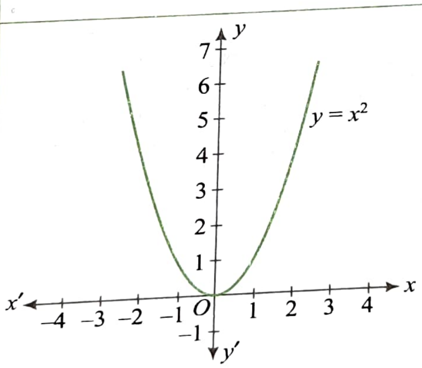
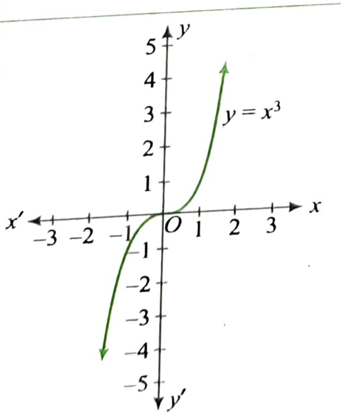
Now look at the graph of the relation in the following figure.



In the above graph there are vertical lines which intersec graph more than once. One of the vertical lines l_1 cuts the grap at points $A(x_1, y_1)$, $B(x_1, y_2)$ and $C(x_1, y_3)$. Thus, for one input there are three different outputs y_1, y_2 and y_3. This contradicts t definition of the function. So, the relation in the graph does n represent function.

This method to check whether the graph of given relati represent the function or not is called **Vertical Line Test.**

Some Elementary Functions

Function	Graph
Constant function $f: R \rightarrow R$, $y = f(x) = c$, where c is constant. Domain: R Range: $\{c\}$	
Identity function $f: R \rightarrow R$, $y = f(x) = x$. Domain: R Range: R	
Square function $f: R \rightarrow R$, $y = f(x) = x^2$ Domain: R Range: $[0, \infty)$	
Cubic function $f: R \rightarrow R$, $y = f(x) = x^3$ Domain: R Range: R	

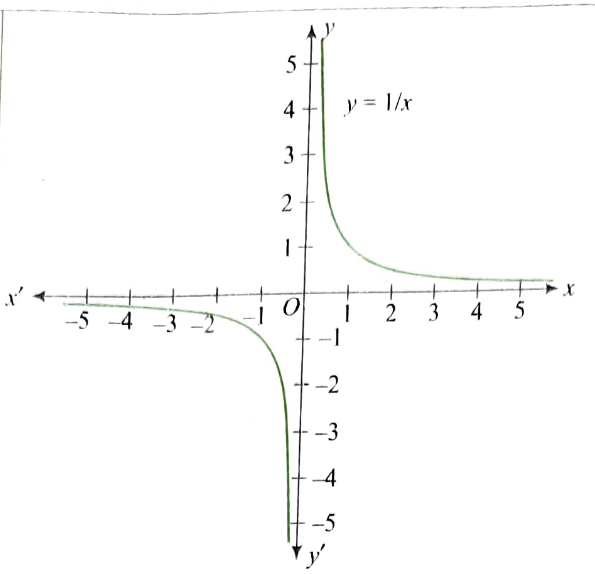
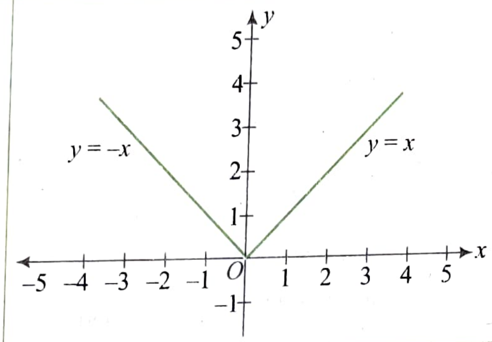
Reciprocal function $f: R \rightarrow R,\ y = f(x) = \frac{1}{x}$ Domain: $R - \{0\}$ Range: $R - \{0\}$	
Modulus function or Absolute value function $f: R \rightarrow R, y = f(x) = \lvert x \rvert$ Domain: R Range: $[0, \infty)$	

ILLUSTRATION 1.43

Find the domain and range of the following functions.

(i) $f(x) = \sqrt{2x - 3}$ (ii) $f(x) = \frac{1}{x-2}$

(iii) $f(x) = x^2 + 3$ (iv) $f(x) = \frac{1}{x^2+2}$

Sol.

(i) $f(x) = \sqrt{2x-3}$ is defined if $2x - 3 \geq 0$ or $x \geq 3/2$.

$\therefore\ x \in [2/3, \infty)$, which is the domain of the function.

For range, $2x - 3 \geq 0$

$\therefore\ \sqrt{2x-3} \geq 0$.

Thus, range is $[0, \infty)$

(ii) $f(x) = \frac{1}{x-2}$.

Clearly above function is defined for any real value of x but not for $x = 2$. For $x = 2$, denominator becomes zero and division with zero is undefined.

So, domain is $R - \{2\}$.

For range let $y = \frac{1}{x-2}$.

We have to find all possible values of y. So, let us get x in terms of y.

$$x - 2 = \frac{1}{y} \Rightarrow x = \frac{1}{y} + 2 = \frac{1+2y}{y}$$

Cleary y cannot be zero. For any value of y other than zero there exists corresponding value of x.

Thus range is $R - \{0\}$

(iii) $f(x) = x^2 + 3$

Clearly x can take any real value. So, domain is R.

For range, $x^2 \geq 0$ for real values of x.

$\therefore\ x^2 + 3 \geq 3$ or $f(x) \geq 3$.

So, range is $[3, \infty)$.

(iv) $f(x) = \frac{1}{x^2+2}$

Above function is not defined for those values of x for which $x^2 + 2 = 0$. But there are no real values of x for which $x^2 = -2$. So, we can take any real value of x. Therefore, domain is set R.

For range let $y = \frac{1}{x^2+2}$.

So, $x^2 + 2 = \frac{1}{y}$ or $x^2 = \frac{1}{y} - 2 = \frac{1-2y}{y}$.

Now $x^2 \geq 0$ for all real values of x.

$$\therefore\ \frac{1-2y}{y} \geq 0$$

$$\Rightarrow \frac{2y-1}{y} \leq 0 \quad \Rightarrow 0 < y \leq 1/2.$$

Therefore, range is $\left(0, \frac{1}{2}\right]$.

ILLUSTRATION 1.44

Find the domain and range function $f(x) = \dfrac{x^2 - 3x + 2}{x^2 - 4x + 3}$.

Sol. $y = f(x) = \dfrac{(x-1)(x-2)}{(x-3)(x-1)} = \dfrac{x-2}{x-3}, x \neq 1, 3$

Domain of function $= R - \{1, 3\}$

Now $yx - 3y = x - 2$

$\therefore \quad x = \dfrac{3y-2}{y-1}$

Clearly $y \neq 1$.

Also, when $x = 1, y = 1/2$

Therefore, range of function is $R - \left\{\dfrac{1}{2}, 1\right\}$

IDENTICAL FUNCTIONS

Two functions f and g are said to be identical if

(i) The domain of f = the domain of g, i.e., $D_f = D_g$

(ii) The range of f = the range of g

(iii) $f(x) = g(x)\ \forall\ x \in D_f$ or $x \in D_g$

e.g., $f(x) = x$ and $g(x) = \sqrt{x^2}$ are not identical functions as domain of both the functions is same but the range of former is R and latter is $[0, \infty)$.

ILLUSTRATION 1.45

Find the values of x for which the following functions are identical.

(i) $f(x) = x$ and $g(x) = \dfrac{1}{1/x}$

(ii) $f(x) = \dfrac{\sqrt{9-x^2}}{\sqrt{x-2}}$ and $g(x) = \sqrt{\dfrac{9-x^2}{x-2}}$

Sol.

(i) $f(x) = x$ is defined for all x.

but $g(x) = \dfrac{1}{1/x} = x$ is not defined for $x = 0$ as $1/x$ is not defined at $x = 0$.

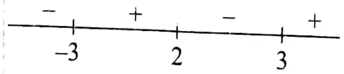
Hence both the functions are identical for $x \in R - \{0\}$.

(ii) $f(x) = \dfrac{\sqrt{9-x^2}}{\sqrt{x-2}}$ is defined if $9 - x^2 \geq 0$ and $x - 2 > 0$

$\Rightarrow \quad x \in [-3, 3]$ and $x > 2 \Rightarrow x \in (2, 3]$

$g(x) = \sqrt{\dfrac{9-x^2}{x-2}}$ is defined if $\dfrac{9-x^2}{x-2} \geq 0$

$\Rightarrow \quad \dfrac{x^2-9}{x-2} \leq 0$



From the sign scheme $x \in (-\infty, -3] \cup (2, 3]$

Hence, $f(x)$ and $g(x)$ are identical if $x \in (2, 3]$

PIECEWISE FUNCTIONS

Till now we have represented function by a single equation. In many real-life problems, however, functions are represented by a combination of equations, each corresponding to a part of the domain. Such functions are called piecewise functions. Let us consider one real life example.

We know that the parking charges in the mall are based on the length of the time. Suppose for first three hours flat charges are Rs. 50. For more than three hours but less than 5 hours flat charges are Rs. 80. For more than 5 hours charges are Rs. 80 for first five hours and Rs. 30 per hour above five hours. We can write the parking charges as a function of time t as

$$f(t) = \begin{cases} 50, & t \leq 3 \\ 80, & 3 < t \leq 5 \\ 80 + 30(t-5), & t > 5 \end{cases}$$

Thus, in piecewise function there are different rules to find out value of function in different sub-intervals.

For car parked for 2 hrs., $t = 2$. So, parking charges are $f(2) =$ Rs. 50

Similarly $f(3.5) =$ Rs. 80 and $f(8) = 80 + 30(8 - 5) =$ Rs. 170.

Also, function $f(x) = |x - 2|$ can be written in piecewise definition as

$$f(x) = \begin{cases} x - 2, & x \geq 2 \\ 2 - x, & x < 2 \end{cases}$$

Function $f(x) = |x^2 - 1|$ as

$$f(x) = \begin{cases} x^2 - 1, & x^2 - 1 \geq 0 \\ 1 - x^2, & x^2 - 1 < 0 \end{cases}$$

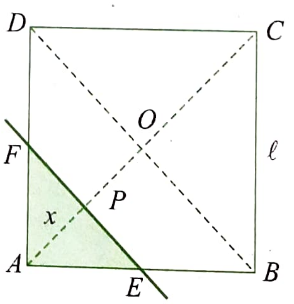
$$= \begin{cases} x^2 - 1, & x \leq -1 \text{ or } x \geq 1 \\ 1 - x^2, & -1 < x < 1 \end{cases}$$

ILLUSTRATION 1.46

$ABCD$ is a square of side ℓ. A line parallel to the diagonal BD at a distance 'x' from the vertex A cuts two adjacent sides. Express the area of the segment of the square with A at a vertex, as a function of x. Find this area at $x = 1/\sqrt{2}$ and at $x = 2$, when $\ell = 2$.

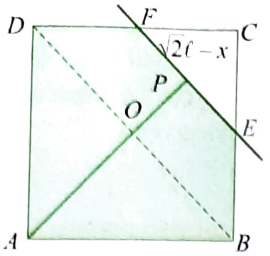
Sol. There are two cases

Case-I : when $x = AP \leq OA$, i.e., $x \leq \dfrac{\ell}{\sqrt{2}}$



$\therefore \quad ar(\Delta AEF) = \dfrac{1}{2}x.2x = x^2 \qquad (\because PE = PF = AP = x)$

Case-II : when $x = AP > OA$, i.e., $x > \frac{\ell}{\sqrt{2}}$ but $x \le \sqrt{2}\ell$



ar(ABEFDA) = ar(ABCD) – ar(ΔCFE)

$$= \ell^2 - \frac{1}{2}\left(\sqrt{2}\ell - x\right).2\left(\sqrt{2}\ell - x\right) \qquad [\because CP = \sqrt{2}\ell - x]$$

$$= \ell^2 - \left(2\ell^2 + x^2 - 2\sqrt{2}\ell x\right) = 2\sqrt{2}\ell x - x^2 - \ell^2$$

So, the required function $s(x)$ is

$$s(x) = \begin{cases} x^2, & 0 \le x \le \frac{\ell}{\sqrt{2}} \\ 2\sqrt{2}\,\ell x - x^2 - \ell^2, & \frac{\ell}{\sqrt{2}} < x \le \sqrt{2}\ell \end{cases}$$

$$\therefore \quad s(x) = \begin{cases} \frac{1}{2} & \text{at } x = \frac{1}{\sqrt{2}} \\ 8(\sqrt{2} - 1) & \text{at } x = 2 \end{cases}$$

ILLUSTRATION 1.47

The relation f is defined by $f(x) = \begin{cases} 3x + 2, & 0 \le x \le 2 \\ x^3, & 2 \le x \le 5 \end{cases}$.

The relation g is defined by $g(x) = \begin{cases} 3x + 2, & 0 \le x \le 1 \\ x^3, & 1 \le x \le 5 \end{cases}$

Show that f is a function and g is not a function

Sol. For $0 \le x < 2$, $f(x) = 3x + 2$ and for $2 \le x \le 10$, $f(x) = x^3$.

Now $x = 2$ appears in both the branches.

From the first branch $f(2) = 3(2) + 2 = 8$

From the second branch $f(2) = 2^3 = 8$.

Thus, for $x = 2$, we have a unique image.

So, given relation is function.

In relation $g(x)$, $x = 1$ appears in both the branches.

From the first branch $g(1) = 3(1) + 2 = 5$.

From the second branch $g(1) = 1^3 = 1$.

Thus, element '1' of domain of the relation g corresponds to two different images, viz., 5 and 1.

Hence, this relation is not a function.

ALGEBRA OF FUNCTIONS

Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$. Then, we define different algebraic operations between function as follows:

Operation	Function
Addition	$(f + g) : X \to R$, $(f + g)(x) = f(x) + g(x)$, for all $x \in X$.
Subtraction	$(f - g) : X \to R$, $(f - g)(x) = f(x) - g(x)$, for all $x \in X$.
Multiplication	$(f \times g) : X \to R$, $(f \times g)(x) = f(x) \times g(x)$, for all $x \in X$.
Division	$\frac{f}{g} : X \to R$, $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, for all $x \in X$, provided $g(x) \ne 0$ for $x \in X$.

ILLUSTRATION 1.48

If $f : [-3, 4] \to R$, $f(x) = 2x$, and $g : [-2, 6] \to R$, $g(x) = x^2$. Then find function $(f + g)(x)$.

Sol. $f : [-3, 4] \to R$, $f(x) = 2x$, and $g : [-2, 6] \to R$, $g(x) = x^2$

For $(f + g)(x)$ we consider common domain of $f(x)$ and $g(x)$.

Now $\quad [-3, 4] \cap [-2, 6] \equiv [-2, 4]$

So, $(f + g) : [-2, 4] \to R$, $(f + g)(x) = 2x + x^2$.

ILLUSTRATION 1.49

If $f(x) = \begin{cases} x^3, & x < 1 \\ 2x - 1, & x \ge 1 \end{cases}$ and $g(x) = \begin{cases} 3x, & x \le 2 \\ x^2, & x > 2 \end{cases}$ then find $(f - g)(x)$.

Sol. We have

$$f(x) = \begin{cases} x^3, & x < 1 \\ 2x - 1, & 1 \le x \le 2 \\ 2x - 1, & x > 2 \end{cases} \text{ and } g(x) = \begin{cases} 3x, & x < 1 \\ 3x, & 1 \le x \le 2 \\ x^2, & x > 2 \end{cases}$$

$$\Rightarrow \quad (f + g)(x) = \begin{cases} x^3 + 3x, & x < 1 \\ 2x - 1 + 3x, & 1 \le x \le 2 \\ 2x - 1 + x^2, & x > 2 \end{cases}$$

$$= \begin{cases} x^3 + 3x, & x < 1 \\ 5x - 1, & 1 \le x \le 2 \\ x^2 + 2x - 1, & x > 2 \end{cases}$$

CONCEPT APPLICATION EXERCISE 1.3

1. Find the domain of the following functions

   (a) $f(x) = \frac{1}{\sqrt{x - 2}}$ **(b)** $f(x) = \frac{1}{x^3 - x}$

   (c) $f(x) = \sqrt[3]{x^2 - 2}$

2. Find the range of the following functions.

   (a) $f(x) = 5 - 7x$ **(b)** $f(x) = 5 - x^2$

   (c) $f(x) = \frac{x^2}{x^2 + 1}$

3. Find the domain and range of $f(x) = \frac{2 - 5x}{3x - 4}$.

4. Find the domain and range of $f(x) = \sqrt{4-16x^2}$.

5. Find the range of the function $f(x) = \dfrac{x^2-x-6}{x-3}$

6. If the relation $f(x) = \begin{cases} 2x-3, & x \le 2 \\ x^3 - a, & x \ge 2 \end{cases}$ is a function, then find the value of a.

7. If the relation $f(x) = \begin{cases} 1, & x \in Q \\ 2, & x \notin Q \end{cases}$, where Q is set of rational numbers, then find the value $f(\pi) + f\left(\dfrac{22}{7}\right)$.

8. Let $f(x) = \begin{cases} x^2-4x+3, & x<3 \\ x-4, & x \ge 3 \end{cases}$ and

$g(x) = \begin{cases} x-3, & x<4 \\ x^2+2x+2, & x \ge 4 \end{cases}$.

Describe the function f/g and find its domain.

9. Which of the following functions is/are identical to $|x-2|$?

(a) $f(x) = \sqrt{x^2-4x+4}$ (b) $g(x) = |x| - |2|$

(c) $h(x) = \dfrac{|x-2|^2}{|x-2|}$ (d) $t(x) = \left|\dfrac{x^2-x-2}{x+1}\right|$

ANSWERS

1. **(a)** $(2, \infty)$ **(b)** $R - \{-1, 0, 1\}$ **(c)** R
2. **(a)** R **(b)** $(-\infty, 5]$ **(c)** $[0, 1)$
3. Domain: $R - \{4/3\}$, Range: $R - \{-5/3\}$
4. Domain: $[-1/2, 1/2]$, Range: $[0, 2]$
5. $R - \{5\}$ **6.** 7 **7.** 3
8. $\dfrac{f}{g}(x) = \begin{cases} \dfrac{x^2-4x+3}{x-3}, & x<3 \\ \dfrac{x-4}{x-3}, & 3<x<4, \text{ Domain: } R-\{3\} \\ \dfrac{x-4}{x^2+2x+2}, & x \ge 4 \end{cases}$
9. (a)

DIFFERENT TYPES OF FUNCTIONS (MAPPINGS)

ONE-ONE AND MANY-ONE FUNCTIONS

If each element in the domain of a function has a distinct image in the co-domain, the function is said to be one-one. One-one function is also called injective function.

e.g., $f: R \to R$ given by $f(x) = y = 3x + 5$ is one-one as for any value of y, we have one and only one value of x. In other words, for any image we have one and only one pre-image.

One-one function is also called **Injective function**.

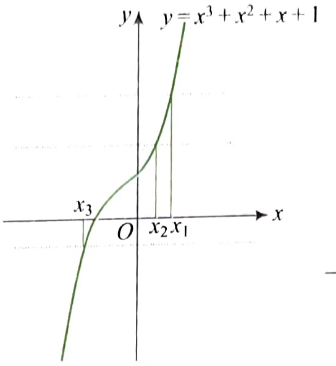
On the other hand, if there are at least two elements in the domain whose images are the same, the function is known as many-one.

e.g., $f: R \to R$ given by $f(x) = y = x^2 + 1$ is many-one as for value of y equal to 5/4 we have two values of x, viz. $\pm 1/2$.

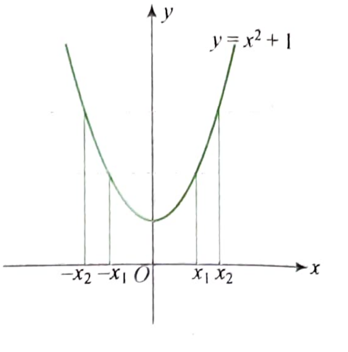
i.e., $f(1/2) = f(-1/2) = 5/4$.

Note that a function will either be one-one or many-one.

Lines drawn parallel to the x-axis from each corresponding image point should interest the graph of $y = f(x)$ at one and only one point if $f(x)$ is one-one and there will be at least one line parallel to x-axis that will cut the graph in more than one points if $f(x)$ is many-one and vice-versa.



One-one function



Many-one function

Methods to Determine One-One and Many-One Functions

Method 1: Analytical method

Let $x_1, x_2 \in$ Domain of f and if $x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$ for every x_1, x_2 in the domain, then f is one-one else many–one.

Conversely if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for every x_1, x_2 in the domain, then f is one-one else many–one.

Method 2: Graphical method

Draw the graph of the function. If there exists at least one line parallel to x-axis that cuts the graph in more than one points, then function is many-one.

Method 3: Differentiation method

If for continuous function $f(x)$, derivative $f'(x) \ge 0$ or $f'(x) \le 0$ $\forall x \in D_f$, then $f(x)$ is one-one. It should be noted that the values of x for which $f'(x) = 0$ should not form an interval.

For function which are not continuous, this method must be used cautiously.

For example, $f(x) = \dfrac{x-2}{x+1}$ is not continuous at $x = -1$.

$$f'(x) = \frac{x+1-(x-2)}{(x+1)^2} = \frac{3}{(x+1)^2} > 0 \ \forall x \in R - \{-1\}$$

If we draw the graph of the function, we find that $f(x)$ is one-one.

Now consider, $f(x) = \tan x$, which is not continuous at $(2n+1)\pi/2, n \in Z$.

We know that this function is many-one in its actual domain as $f(0) = f(\pi) = f(2\pi) = \cdots = 0$.

However, $f'(x) = \sec^2 x \ge 1, \forall x \in D_f$, where D_f is domain of $f(x)$.

ILLUSTRATION 1.50

Check the nature of the following functions.

(i) $f(x) = \sin x, x \in R$ **(ii)** $f(x) = \sin x, x \in N$

Sol.

(i) $f(x) = \sin x, x \in R$

Let $f(x_1) = f(x_2)$

$\Rightarrow \sin x_1 = \sin x_2$

$\Rightarrow x_1 = n\pi + (-1)^n x_2, n \in Z.$

There are infinite pair (x_1, x_2), $x_1 \neq x_2$ which satisfy above equation.

i.e., $f(0) = f(\pi) = 0$

So, $f(x)$ is many-one.

(ii) $f(x) = \sin x, x \in N$

Let $f(x_1) = f(x_2)$

$\Rightarrow \sin x_1 = \sin x_2$

$\Rightarrow x_1 = n\pi + (-1)^n x_2$, where $x_1, x_2 \in N$ and $n \in Z$.

This is equation is possible only for $n = 0$, for which $x_1 = x_2$.

So, $f(x)$ is one-one.

ILLUSTRATION 1.51

Check the nature of the function $f(x) = x^3 + x + 1, x \in R$ using analytical method and differentiation method.

Sol. $f(x) = x^3 + x + 1$

Let $f(x_1) = f(x_2)$

$\Rightarrow x_1^3 + x_1 + 1 = x_2^3 + x_2 + 1$

$\Rightarrow x_1^3 - x_2^3 + x_1 - x_2 = 0$

$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 1) = 0$

$\Rightarrow x_1 = x_2$ or $x_1^2 + x_1x_2 + x_2^2 + 1 = 0$

From $x_1^2 + x_1x_2 + x_2^2 + 1 = 0$

$$x_1 = \frac{-x_2 \pm \sqrt{x_2^2 - 4(x_2^2 + 1)}}{2}$$

$$= \frac{-x_2 \pm \sqrt{-3x_2^2 - 4}}{2}$$

$$= \frac{-x_2 \pm i\sqrt{3x_2^2 + 4}}{2}, \text{ where } i = \sqrt{-1}$$

Thus, $x_1 = x_2$ only.

Hence $f(x)$ is one-one.

Also, $f'(x) = 3x^2 + 1 > 0 \ \forall \in R$.

So, $f(x)$ is one-one function.

ILLUSTRATION 1.52

Let $f: R \to R$ where $f(x) = \dfrac{x^2 + 4x + 7}{x^2 + x + 1}$. Is $f(x)$ one–one?

Sol. We have

$$f(x) = \frac{x^2 + 4x + 7}{x^2 + x + 1} = 1 + \frac{3(x+2)}{x^2 + x + 1}$$

Let $f(x_1) = f(x_2)$

$$\Rightarrow 1 + \frac{3(x_1+2)}{x_1^2 + x_1 + 1} = 1 + \frac{3(x_2+2)}{x_2^2 + x_2 + 1}$$

$\Rightarrow x_1x_2^2 + x_1x_2 + x_1 + 2x_2^2 + 2x_2 + 2$
$= x_1^2x_2 + x_1x_2 + x_2 + 2x_1^2 + 2x_1 + 2$

$\Rightarrow (x_1 - x_2)(2x_1 + 2x_2 + x_1x_2 + 1) = 0$

Let us consider $2x_1 + 2x_2 + x_1x_2 + 1 = 0$

$$\Rightarrow x_2 = -\frac{1 + 2x_1}{2 + x_1}$$

This relation is satisfied by infinite number of pairs (x_1, x_2), where $x_1 \neq x_2$, e.g., $(0, -1/2)$, $(1, -1)$ etc.

Hence $f(x)$ is many-one.

ONTO AND INTO FUNCTIONS

Let $f: X \to Y$ be a function. If each element in the codomain 'Y' has at least one pre-image in the domain X that is, for every $y \in Y$ there exists at least one element $x \in X$ such that $f(x) = y$, then f is onto.

In other words, range of f is same as codomain Y, for onto function.

On the other hand, if there exists at least one element in the codomain Y which is not an image of any element in the domain X, then f is into. Thus, range of into function is subset of codomain Y.

Onto function is also called **Surjective function** and a function which is both one-one and onto is called **Bijective function**.

e.g., $f: R \to R$ where $f(x) = x^2$ is into.

$f: R \to R$ where $f(x) = ax^3 + b$ is onto where $a, b \in R, a \neq 0$.

Note that a function will either be onto or into.

Methods to Determine Onto and Into Functions

Method 1:

Find the range of function. If range is same is codomain then function is onto(surjective). If range is subset of codomain then function is into.

Method 2:

Solve $f(x) = y$ for x, say $x = g(y)$.

Now if $g(y)$ is defined for each $y \in$ codomain and $g(y) \in$ domain of f for all $y \in$ codomain, then $f(x)$ is onto.

If this requirement is not met by at least one value of y in codomain, then $f(x)$ is into.

ILLUSTRATION 1.53

Let $f: R \to R$ where $f(x) = \sin x$. Show that f is into. Also find the codomain if f is onto.

Sol. $f: R \to R, f(x) = \sin x$

Range of the function is $[-1, 1]$, which is subset of codomain R.

So, $f(x)$ is into.

To make $f(x)$ onto, we modify the codomain to $[-1, 1]$.

ILLUSTRATION 1.54

Let $f: N \to Z$ be a function defined as $f(x) = x - 1000$. Show that f is an into function.

Sol. Let $f(x) = y = x - 1000$

$\Rightarrow x = y + 1000 = g(y)$ (say)

Here $g(y)$ is defined for each $y \in Z$, but $g(y) \notin N$ for $y \leq -1000$

Hence f is into.

ILLUSTRATION 1.55

If the function $f: R \to A$ given by $f(x) = \dfrac{x^2}{x^2 + 1}$ is surjection, then find A.

Sol. $f: R \to A, f(x) = \dfrac{x^2}{x^2 + 1}$

Here domain is all real numbers.

Since $f(x)$ is onto, range must be same as codomain(A).

To find the range of the function,

$$f(x) = 1 - \frac{1}{x^2+1}$$

Now $x^2 + 1 \geq 1 \ \forall \in R$.

$\Rightarrow \quad 0 < \frac{1}{x^2+1} \leq 1$

$\Rightarrow \quad -1 \leq -\frac{1}{x^2+1} < 0$

$\Rightarrow \quad 0 \leq 1 - \frac{1}{x^2+1} < 1$

Thus, range is $[0, 1)$.

Hence codomain $A = [0, 1)$.

ILLUSTRATION 1.56

Let $A = \{x: -1 \leq x \leq 1\} = B$ be a function $f: A \rightarrow B$. Then find the nature of each of the following functions.

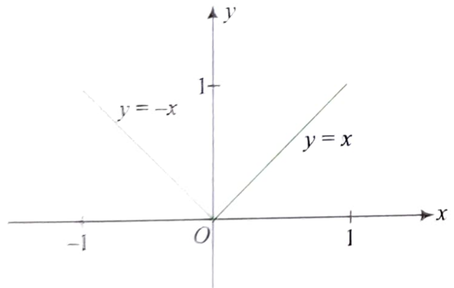
(i) $f(x) = |x|$ **(ii)** $f(x) = x|x|$

(iii) $f(x) = x^3$ **(iv)** $f(x) = \sin \frac{\pi x}{2}$

Sol.

(i) $f: [-1, 1] \rightarrow [-1, 1]$, where $f(x) = |x|$.

Graph of the function is as shown in the following figure.

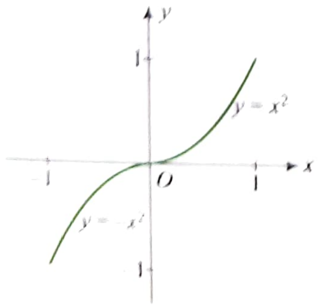


From the graph we observe that $f(x)$ is many-one.

Also, we observe that the range is $[0, 1]$. Hence $f(x)$ is into.

(ii) $f: [-1, 1] \rightarrow [-1, 1]$, where $f(x) = x|x|$

$$\text{Now } f(x) = x|x| = \begin{cases} x \times x, & x \geq 0 \\ x \times (-x), & x < 0 \end{cases} = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

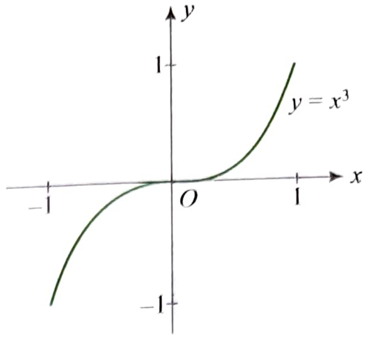


From the graph we observe that $f(x)$ is one-one as there is no horizontal line which intersect the curve more than once.

Also, we observe that the range is $[-1, 1]$. Hence $f(x)$ is onto.

Thus $f(x)$ is bijective.

(iii) $f: [-1, 1] \rightarrow [-1, 1]$, where $f(x) = x^3$



Clearly, $f(x)$ is one-one.

Also, for $x \in [-1, 1]$, $x^3 \in [-1, 1]$.

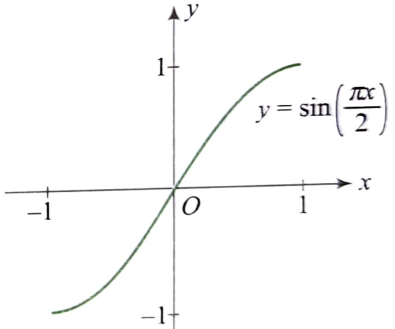
So, $f(x)$ is onto.

Thus, $f(x)$ is bijective.

(iv) $f: [-1, 1] \rightarrow [-1, 1]$, where $f(x) = \sin \frac{\pi x}{2}$

For $x \in [-1, 1]$, $\frac{\pi x}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So, graph of $f(x)$ is as shown in the following figure.



From the graph we observe that $f(x)$ is one-one.

Also, range of the $f(x)$ is $[-1, 1]$. Hence $f(x)$ is onto.

Thus, $f(x)$ is bijective.

ILLUSTRATION 1.57

If $f: R \rightarrow R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$, then check the nature of the function.

Sol. We have,

$f(x) = 2x + \sin x, x \in R$

$\Rightarrow \quad f'(x) = 2 + \cos x$

but $\quad -1 \leq \cos x \leq 1$

$\Rightarrow \quad 1 \leq 2 + \cos x \leq 3$

$\therefore \quad f'(x) > 0, \forall x \in R$

Since, $f(x) = 2x + \sin x$ continuously exists, $f(x)$ is one-one.

Also, when x approaches to infinity, $f(x)$ approaches to infinity and when x approaches to negative infinity, $f(x)$ approaches to negative infinity.

So, Range of $f(x)$ is R. Hence, $f(x)$ is onto.

Thus $f(x)$ is one-one and onto, i.e., bijective.

ILLUSTRATION 1.58

If $f: [0, \infty) \rightarrow [0, 1)$, and $f(x) = \frac{x}{1+x}$ then check the nature of the function.

Sol. Given that $f: [0, \infty) \rightarrow [0, \infty)$, $f(x) = \frac{x}{x+1}$

Let $f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1}{x_1+1} = \frac{x_2}{x_2+1}$

$\Rightarrow x_1x_2 + x_1 = x_1x_2 + x_2$

$\Rightarrow x_1 = x_2$.

Thus $f(x)$ is one-one.

Now let $y = \frac{x}{1+x}$

$\Rightarrow y + yx = x$

$\Rightarrow x = \frac{y}{1-y}$

As $x \geq 0$, $\frac{y}{1-y} \geq 0$

$\Rightarrow \frac{y}{y-1} \leq 0$

$\Rightarrow 0 \leq y < 1$ or range of $f(x)$ is $[0, 1)$.

Thus $f(x)$ is onto.

ILLUSTRATION 1.59

If the functions $f(x)$ and $g(x)$ are defined on $R \rightarrow R$ such that

$$f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases} \text{ and } g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases}$$

then find $(f - g)(x)$ and also check the nature of the function.

Sol. We have, $(f - g) : R \rightarrow R$,

$$(f - g)(x) = \begin{cases} -x \text{ if } x \in \text{rational} \\ x \text{ if } x \in \text{irrational} \end{cases}$$

Clearly $(f - g)(x)$ is one-one and onto.

ILLUSTRATION 1.60

Show $f : R \rightarrow R$ defined by $f(x) = (x - 1)(x - 2)(x - 3)$ is surjective but not injective.

Sol. We have

$f : R \rightarrow R$, where $f(x) = (x - 1)(x - 2)(x - 3)$

Clearly $f(1) = f(2) = f(3) = 0$.

So, $f(x)$ is many-one.

Also, when x approaches to infinity, $f(x)$ approaches to infinity and when x approaches to negative infinity, $f(x)$ approaches to negative infinity.

Since, polynomial function continuously exists for all real x, given function $f(x)$ takes all real values.

Hence, range of the $f(x)$ is R.

Therefore, $f(x)$ is onto.

ILLUSTRATION 1.61

If $f: R \rightarrow R$, $f(x) = \begin{cases} x|x| - 4, & x \in Q \\ x|x| - \sqrt{3}, & x \notin Q \end{cases}$, then identify the type of function.

Sol. $f(2) = f(3^{1/4}) = 0$. $\Rightarrow$ many-to-one function

Also, $f(x) \neq \sqrt{3}\ \forall x \in R \Rightarrow$ into function

ILLUSTRATION 1.62

If $f : R \rightarrow R$ be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$, then discuss the nature of the function.

Sol. $f(x) = x^3 + x^2 + 3x + \sin x$

$\therefore \quad f'(x) = 3x^2 + 2x + 3 + \cos x$

$$= 3\left[\left(x + \frac{1}{3}\right)^2 + \frac{8}{9}\right] - (-\cos x)$$

Now $3\left[\left(x + \frac{1}{3}\right)^2 + \frac{8}{9}\right]_{\min} = \frac{8}{3}$ and $(-\cos x)$ has maximum value '1'.

Thus $f'(x) > 0$.

Hence, $f(x)$ is one-one.

Also, when x approaches to infinity, $f(x)$ approaches to infinity and when x approaches to negative infinity, $f(x)$ approaches to negative infinity.

Also, $f(x)$ continuously exists for all real x. So, $f(x)$ takes all real values.

Hence, range of the $f(x)$ is R.

Therefore, $f(x)$ is onto.

ILLUSTRATION 1.63

If function $f(x)$ is defined from set A to B, such that $n(A) = 3$ and $n(B) = 5$. Then find the number of one-one functions and number of onto functions that can be formed.

Sol. For one-one function each preimage in set A must have distinct image in set B.

So, number of one-one functions from set A to B

= number of permutations of five objects (images) taken three (preimages) at a time

$= {}^5P_3 = 5 \times 4 \times 3 = 60$.

For onto function each image must have pre-image.

Now consider three elements of set A as three distinct boxes and five elements of set B as five distinct objects.

Number of onto functions from set A to B

= number of ways five distinct objects can be distributed into three boxes so that no box remains empty

$= 3^5 - {}^3C_1(3 - 1)^5 + {}^3C_2(3 - 2)^5$

$= 243 - 96 + 3$

$= 150$

CONCEPT APPLICATION EXERCISE 1.4

1. Which of the following functions from Z to itself are bijections?
   (a) $f(x) = x^3$ (b) $f(x) = x + 2$
   (c) $f(x) = 2x + 1$ (d) $f(x) = x^2 + x$

2. If $f: N \to Z$ $f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$ identify the type of the function.

3. If $f: R \to R$ is given by $f(x) = \frac{x^2-4}{x^2+1}$, identify the type of function.

4. If $f: R \to S$, defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$, is onto, then find the set S.

5. Let g: $R \to \left(0, \frac{\pi}{3}\right]$ be defined by $g(x) = \cos^{-1}\left(\frac{x^2-k}{1+x^2}\right)$.
   Then find the possible values of k for which g is a surjective function.

6. Identify the type of the function $f: R \to R$,
   $f(x) = e^{x^2} + \cos x$.

7. Let a function $f: R \to R$ be defined by $f(x) = 2x + \cos x + \sin x$ for $x \in R$. Then find the nature of $f(x)$.

8. If $f: R \to R$ given by $f(x) = x^3 + px^2 + qx + r$, is, then find the condition for which $f(x)$ is one-one.

ANSWERS

1. (b) 2. bijective 3. many-one, into
4. [–1, 3] 5. (–1, –1/2] 6. many-one, into
7. bijective 8. $p^2 \le 3q$

ALGEBRAIC FUNCTIONS

We have following algebraic functions:

Polynomial Function: $y = f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$, where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \ne 0$. Here $f(x)$ is called a polynomial function of degree n. If $n = 1, 2, 3$, then polynomial is called linear, quadratic and cubic, respectively.

Polynomial of odd degree has range R.

Polynomial of even degree does not take all real values.

Rational Function: The function which can be written as the quotient of two polynomial function is said to be a rational function. If $P(x)$ and $Q(x)$ are two polynomial functions, then a function $f(x)$ defined by $f(x) = \frac{P(x)}{Q(x)}$ is a rational function of x.

Note: A function which is not algebraic is called transcendental function. Exponential function, the logarithm, and the trigonometric functions are transcendental functions.

QUADRATIC FUNCTIONS

The most common polynomial function used is quadratic function.

Let $f(x) = ax^2 + bx + c$, where $a, b, c, \in R$ and $a \ne 0$.

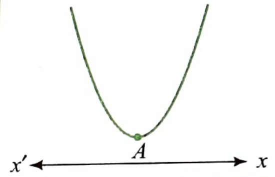
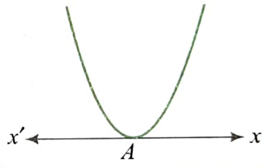
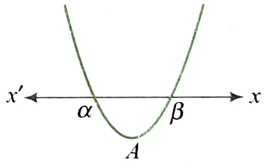
We have $y = f(x) = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$

$= a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$, where $D = b^2 - 4ac$

$\Rightarrow \left(y+\frac{D}{4a}\right) = a\left(x+\frac{b}{2a}\right)^2$

Thus $y = f(x)$ represent a parabola whose axis is parallel to y-axis and vertex at $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$. For $a > 0$, the parabola opens upwards and for $a < 0$, the parabola opens downwards, this gives the following cases:

Case I: $a > 0$

$D < 0$, $f(x) > 0 \ \forall \ x \in R$		
$D = 0$, $f(x) \ge 0 \ \forall \ x \in R$		Range: $\left[-\frac{D}{4a}, \infty\right)$ If $f(x) \ge 0$, $\forall \ x \in R$ then $a > 0$ and $D \le 0$
$D > 0, f(x) > 0$, $\forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0 \ \forall \ x \in (\alpha, \beta)$		

Case II: $a < 0$

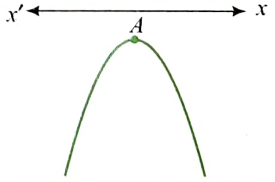
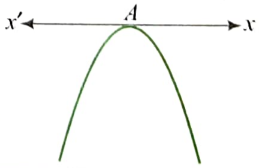
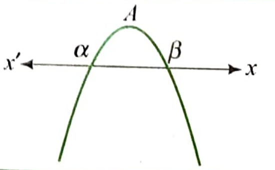
$D < 0$, $f(x) < 0 \ \forall \ x \in R$		
$D = 0$, $f(x) \le 0 \ \forall \ x \in R$		Range: $\left(-\infty, -\frac{D}{4a}\right]$ If $f(x) \le 0$, $\forall \ x \in R$ then $a < 0$ and $D \le 0$
$D > 0, f(x) < 0$, $\forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0$ $\forall \ x \in (\alpha, \beta)$		

ILLUSTRATION 1.64

Find the Range of $f(x) = x^2 - x - 3$

Sol. $f(x) = x^2 - x - 3$

$$= \left(x-\frac{1}{2}\right)^2 - \frac{1}{4} - 3 = \left(x-\frac{1}{2}\right)^2 - \frac{13}{4}$$

Now $\left(x-\frac{1}{2}\right)^2 \ge 0\ \forall x \in R$

$$\Rightarrow \left(x-\frac{1}{2}\right)^2 - \frac{13}{4} \ge \frac{-13}{4}\ \forall x \in R$$

Hence range is $\left[-\frac{13}{4}, \infty\right)$

Alternatively, let $y = x^2 - x - 3$

$\therefore\ x^2 - x - 3 - y = 0$

Since x is real, $D \ge 0$

$\therefore\ (-1)^2 - 4(-3-y) \ge 0$

$\Rightarrow\ 13 + 4y \ge 0$

$\Rightarrow\ y \ge -13/4$

Therefore, range is $\left[-\frac{13}{4}, \infty\right)$

ILLUSTRATION 1.65

Find the domain and range of the following

(i) $f(x) = \sqrt{x^2 - 3x + 2}$ **(ii)** $f(x) = \sqrt{x^2 - 4x + 6}$

Sol.

(i) Clearly $f(x)$ is defined if $x^2 - 3x + 2 \ge 0$

$\Rightarrow (x-1)(x-2) \ge 0$

$\Rightarrow x \in (-\infty, 1] \cup [2, \infty)$

Now, $f(x) = \sqrt{x^2 - 3x + 2}$

$$= \sqrt{\left(x-\frac{3}{2}\right)^2 + 2 - \frac{9}{4}}$$

$$= \sqrt{\left(x-\frac{3}{2}\right)^2 - \frac{1}{4}}$$

Clearly, $\left(x-\frac{3}{2}\right)^2 - \frac{1}{4} \ge 0$

$$\therefore \sqrt{\left(x-\frac{3}{2}\right)^2 - \frac{1}{4}} \ge 0$$

Therefore, range is $[0, \infty)$.

Here least value of $f(x)$ occurs when $x - \frac{3}{2} = \pm\frac{1}{2}$, i.e., $x = 1$ or $x = 2$.

(ii) $f(x) = \sqrt{x^2 - 4x + 6}$

$$= \sqrt{(x-2)^2 + 2}$$

Clearly $(x-2)^2 + 2 \ge 2,\ \forall x \in R$.

So, domain of $f(x)$ is R.

Also, $\sqrt{(x-2)^2 + 2} \ge \sqrt{2}$

Hence, range is $[\sqrt{2}, \infty)$.

ILLUSTRATION 1.66

Find the range of $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1}$.

Sol. Let $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$

or $(1-y)x^2 - (1+y)x + 1 - y = 0$

Now, x is real. Then

$D \ge 0$

or $(1+y)^2 - 4(1-y)^2 \ge 0$

or $(1 + y - 2 + 2y)(1 + y + 2 - 2y) \ge 0$

or $(3y-1)(3-y) \ge 0$

or $3\left(y - \frac{1}{3}\right)(y-3) \le 0$

or $\frac{1}{3} \le y \le 3$

Hence, the range is $\left[\frac{1}{3}, 3\right]$.

ILLUSTRATION 1.67

Find the complete set of values of a such that $\dfrac{x^2 - x}{1 - ax}$ attains all real values.

Sol. $y = \dfrac{x^2 - x}{1 - ax}$

or $x^2 - x = y - axy$

or $x^2 + x(ay - 1) - y = 0$

Since x is real we get

$(ay-1)^2 + 4y \ge 0$

or $a^2y^2 + 2y(2-a) + 1 \ge 0\ \forall y \in R$

So, as $a^2 > 0$,

$4(2-a)^2 - 4a^2 \le 0$

or $4 - 4a \le 0$

or $a \in [1, \infty)$

CONCEPT APPLICATION EXERCISE 1.5

1. The entire graphs of the equation $y = x^2 + kx - x + 9$ is strictly above the x-axis then find the values of k.
2. Find the range of $f(x) = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$.
3. Find the range of $f(x) = \sqrt{x-1} + \sqrt{5-x}$.
4. If $f(x) = \sqrt{x^2 + ax + 4}$ is defined for all x, then find the values of a.
5. Find the domain and range of $f(x) = \sqrt{3 - 2x - x^2}$.

ANSWERS

1. $-5 < k < 7$ **2.** $(-\infty, 5] \cup [9, \infty)$

3. $[2, 2\sqrt{2}]$ **4.** $[-4, 4]$

5. Domain: $[-3, 1]$, Range: $[0, 2]$

TRIGONOMETRIC FUNCTIONS

We have six trigonometric function. Some important aspects of these functions are as follows.

I. $f(x) = \sin x$

Domain: R

Range: $[-1, 1]$

$\sin^2 x$, $|\sin x| \in [0, 1]$

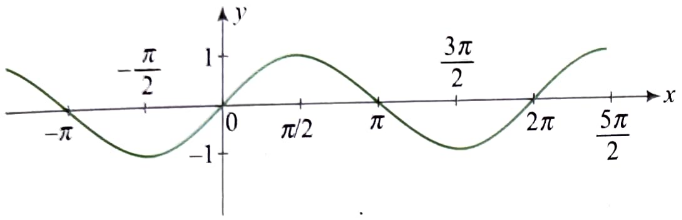
$\sin x = 0 \Rightarrow x = n\pi,\ n \in Z$

$\sin x = 1 \Rightarrow x = (4n + 1)\pi/2,\ n \in Z$

$\sin x = -1 \Rightarrow x = (4n - 1)\pi/2,\ n \in Z$

$\sin x = \sin \alpha \Rightarrow x = n\pi + (-1)^n \alpha,\ n \in Z$

$\sin x \geq 0 \Rightarrow x \in \bigcup_{n\in Z} [2n\pi, (2n+1)\pi]$



From the graph we observe that sin x is periodic function having period 2π, i.e., graph of the function repeats after length of 2π units. Also, graph is symmetrical about origin. Such function is called odd function. Clearly function is many-one.

II. $f(x) = \cos x$

Domain: R

Range: $[-1, 1]$

$\cos^2 x$, $|\cos x| \in [0, 1]$

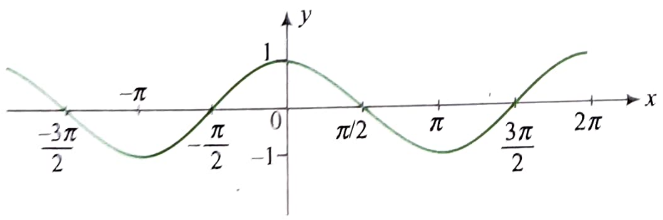
$\cos x = 0 \Rightarrow x = (2n + 1)\pi/2,\ n \in Z$

$\cos x = 1 \Rightarrow x = 2n\pi,\ n \in Z$

$\cos x = -1 \Rightarrow x = (2n + 1)\pi,\ n \in Z$

$\cos x = \cos\alpha \Rightarrow x = 2n\pi \pm \alpha,\ n \in Z$

$\cos x \geq 0 \Rightarrow x \in \bigcup_{n\in Z} \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$



From the graph we observe that cos x is periodic function having period 2π. Also, graph is symmetrical about y-axis. Such function is called an even function. Clearly function is many-one.

III. $f(x) = \tan x$

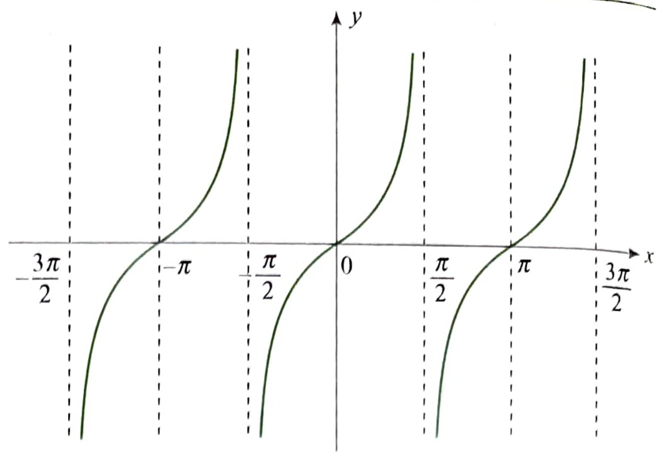
Domain: $R - \{(2n + 1)\pi/2,\ n \in Z\}$

Range: $(-\infty, \infty)$

$\tan^2 x$, $|\tan x| \in [0, \infty)$

$\tan x = 0 \Rightarrow x = n\pi,\ n \in Z$

$\tan x = \tan\alpha \Rightarrow x = n\pi + \alpha,\ n \in Z$



Nature of the function: periodic with period π, odd, many-one.

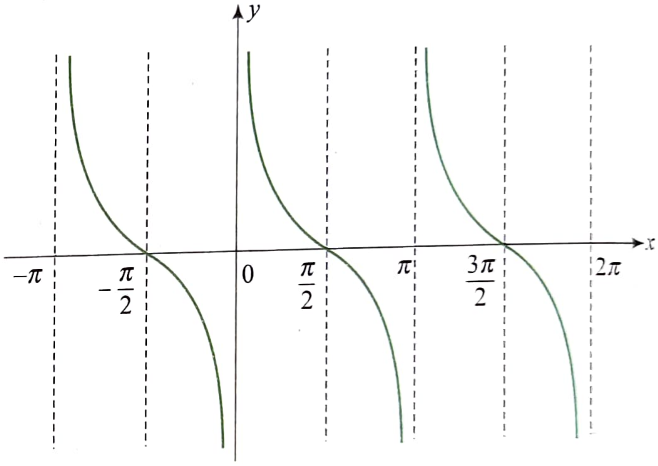
IV. $f(x) = \cot x$

Domain: $R - \{n\pi,\ n \in Z\}$

Range: $(-\infty, \infty)$

$\cot^2 x$, $|\cot x| \in [0, \infty)$

$\cot x = 0 \Rightarrow x = (2n + 1)\ \pi/2,\ n \in Z$



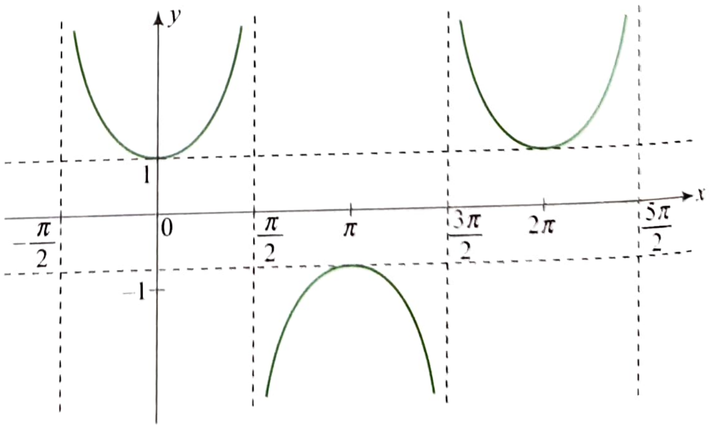
Nature: Odd, many-one, period π

V. $f(x) = \sec x$

Domain: $R - \{(2n + 1)\pi/2,\ n \in Z\}$

Range: $(-\infty, -1] \cup [1, \infty)$

$\sec^2 x$, $|\sec x| \in [1, \infty)$



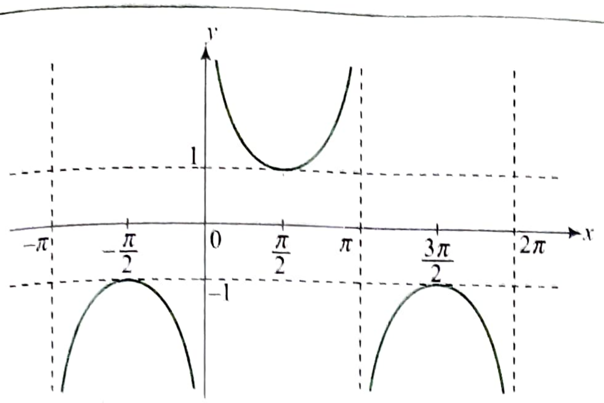
Nature: Even, many-one, period 2π

VI. $f(x) = \operatorname{cosec} x$

Domain: $R - \{n\pi,\ n \in Z\}$

Range: $(-\infty, -1] \cup [1, \infty)$

$\operatorname{cosec}^2 x$, $|\operatorname{cosec} x| \in [1, \infty)$.



Nature: Odd, many-one, period 2π

Important Result

Consider function $f(x) = a\cos x + b\sin x$

Many times, it is required to write the function entirely in the form of sine or cosine.

$$f(x) = a\cos x + b\sin x$$

$$= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x\right)$$

$$= \sqrt{a^2+b^2}\,(\sin\alpha\cos x + \cos\alpha\sin x)$$

$$= \sqrt{a^2+b^2}\,\sin(x+\alpha), \text{ where } \tan\alpha = \frac{a}{b}$$

Also. $f(x) = a\cos x + b\sin x$

$$= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x\right)$$

$$= \sqrt{a^2+b^2}\,(\cos\alpha\cos x + \sin\alpha\sin x)$$

$$= \sqrt{a^2+b^2}\,\cos(x-\alpha), \text{ where } \tan\alpha = \frac{b}{a}$$

Now $-1 \le \sin(x+a) \le 1$

$$\Rightarrow \quad -\sqrt{a^2+b^2} \le \sqrt{a^2+b^2}\,\sin(x+\alpha) \le \sqrt{a^2+b^2}$$

Thus, range of $f(x) = a\cos x + b\sin x$ is $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$

ILLUSTRATION 1.68

Find the domain of the function $f(x) = \dfrac{1}{1+2\sin x}$

Sol. To define $f(x)$, we must have $1 + 2\sin x \ne 0$

or $\quad \sin x \ne -\dfrac{1}{2}$ or $x \ne n\pi + (-1)^n \dfrac{7\pi}{6}, n \in Z$

Hence, the domain of the function is

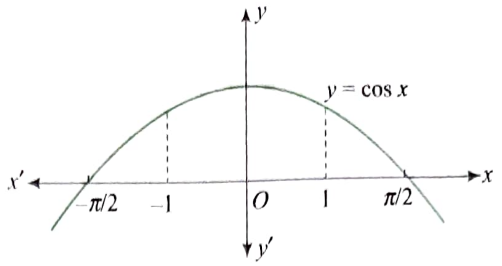
$$R - \left\{n\pi + (-1)^n\frac{7\pi}{6}, n \in Z\right\}$$

ILLUSTRATION 1.69

Find the domain of $f(x) = \sqrt{\cos(\sin x)}$.

Sol. $f(x) = \sqrt{\cos(\sin x)}$ is defined if

$\cos(\sin x) \ge 0 \qquad (1)$



As we know, $-1 \le \sin x \le 1$ for all x.

So, $\quad \cos\theta \ge 0$

(Here, $\theta = \sin x$ lies in the first and fourth quadrants)

i.e., $\cos(\sin x) \ge 0$ for all x

i.e., $x \in R$

Thus, the domain $f(x)$ is R.

ILLUSTRATION 1.70

Find the range of $f(x) = \sin^2 x - \sin x + 1$.

Sol. $f(x) = \sin^2 x - \sin x + 1 = \left(\sin x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$

Now, $-1 \le \sin x \le 1$ or $\quad -\dfrac{3}{2} \le \sin x - \dfrac{1}{2} \le \dfrac{1}{2}$

or $\quad 0 \le \left(\sin x - \dfrac{1}{2}\right)^2 \le \dfrac{9}{4}$ or $\dfrac{3}{4} \le \left(\sin x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4} \le 3$

Hence, the range is $\left[\dfrac{3}{4}, 3\right]$.

ILLUSTRATION 1.71

Find the range of $f(x) = \dfrac{1}{2\cos x - 1}$.

Sol. $-1 \le \cos x \le 1$

or $\quad -2 \le 2\cos x \le 2$

or $\quad -3 \le 2\cos x - 1 \le 1$

For $\quad \dfrac{1}{2\cos x - 1}$,

$-3 \le 2\cos x - 1 < 0$ or $0 < 2\cos x - 1 \le 1$

i.e., $\quad -\infty < \dfrac{1}{2\cos x - 1} \le \dfrac{-1}{3}$ or $1 \le \dfrac{1}{2\cos x - 1} < \infty$

Hence, the range is $\left(-\infty, -\dfrac{1}{3}\right] \cup [1, \infty)$.

ILLUSTRATION 1.72

Find the values of x for which the following functions are identical.

$$f(x) = \cos x \text{ and } g(x) = \frac{1}{\sqrt{1+\tan^2 x}}$$

Sol. $f(x) = \cos x$ has domain R and range $[-1, 1]$

$$g(x) = \frac{1}{\sqrt{1+\tan^2 x}} = \frac{1}{\sqrt{\sec^2 x}} = |\cos x|,$$

has domain $R - \{(2n+1)\pi/2, n \in Z$ as $\tan x$ is not defined for $x = (2n+1)\pi/2, n \in Z$

Also range of $g(x) = |\cos x|$ is $[0, 1]$

Hence $f(x)$ and $g(x)$ are identical if x lies in first and fourth quadrant.

$$\Rightarrow \quad x \in \bigcup_{n \in Z} \left(-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right)$$

ILLUSTRATION 1.73

Find the range of the function $f(x) = 3\sin\left(\sqrt{\frac{\pi^2}{16} - x^2}\right)$.

Sol. We must have $\frac{\pi^2}{16} - x^2 \geq 0$

Also, $\frac{\pi^2}{16} - x^2 \leq \frac{\pi^2}{16}$

$$\Rightarrow \quad 0 \leq \frac{\pi^2}{16} - x^2 \leq \frac{\pi^2}{16}$$

$$\Rightarrow \quad 0 \leq \sqrt{\frac{\pi^2}{16} - x^2} \leq \frac{\pi}{4}$$

$$\Rightarrow \quad 0 \leq \sin\sqrt{\frac{\pi^2}{16} - x^2} \leq \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad 0 \leq 3\sin\sqrt{\frac{\pi^2}{16} - x^2} \leq \frac{3}{\sqrt{2}}$$

Hence, range is $\left[0, \frac{3}{\sqrt{2}}\right]$

ILLUSTRATION 1.74

If $f(x) = \frac{\sin x}{\sqrt{1+\tan^2 x}} - \frac{\cos x}{\sqrt{1+\cot^2 x}}$, then find the range of $f(x)$.

Sol. $f(x) = \frac{\sin x}{|\sec x|} - \frac{\cos x}{|\operatorname{cosec} x|} = \sin x\,|\cos x| - \cos x\,|\sin x|$

Clearly, the domain of $f(x)$ is $R \sim \left\{\frac{n\pi}{2}, n \in Z\right\}$ and the period of $f(x)$ is 2π.

$$f(x) = \begin{cases} 0, & x \in (0, \pi/2) \\ -\sin 2x, & x \in (\pi/2, \pi) \\ 0, & x \in (\pi, 3\pi/2) \\ \sin 2x, & x \in (3\pi/2, 2\pi) \end{cases}$$

The range of $f(x)$ is $[-1, 1]$.

ILLUSTRATION 1.75

Find the range of $f(x) = |\sin x| + |\cos x|, x \in R$.

Sol. $f(x) = |\sin x| + |\cos x| \;\forall x \in R$.

Clearly, $f(x) > 0$.

Also, $f^2(x) = \sin^2 x + \cos^2 x + |2\sin x\cos x| = 1 + |\sin 2x|$

Now, $0 \leq |\sin zx| \leq 1$

$\therefore \quad 1 \leq 1 + |\sin zx| \leq 2$

$\therefore \quad 1 \leq f^2(x) \leq 2$

or $\quad 1 \leq f(x) \leq \sqrt{2}$

ILLUSTRATION 1.76

Find the range of $f(\theta) = 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$

Sol. $f(\theta) = 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$

$$= 5\cos\theta + \frac{3}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

$$= \sqrt{\left(\frac{169}{4} + \frac{27}{4}\right)}\sin(\alpha - \theta) + 3$$

$$= 7\sin(\alpha - \theta) + 3$$

Thus, the range of $f(\theta)$ is $[-4, 10]$.

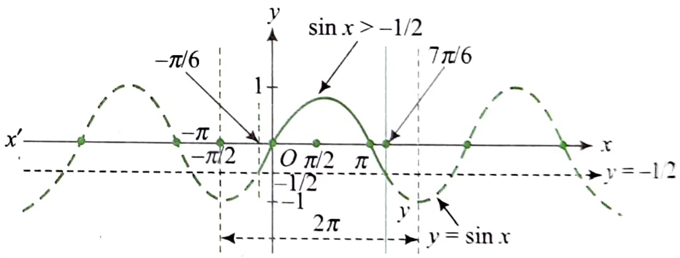
ILLUSTRATION 1.77

Solve $\sin x > -\frac{1}{2}$ or find the domain of $f(x) = \frac{1}{\sqrt{1+2\sin x}}$.

Sol. To define $f(x)$, we must have $1 + 2\sin x > 0$ or $\sin x > -\frac{1}{2}$.

The function $\sin x$ has the least positive period 2π. That is why it is sufficient to solve the inequalities of the form $\sin x > a$, $\sin x \geq a$, $\sin x < a$, and $\sin x \leq a$ first on the interval of length 2π, and then get the solution set by adding numbers of the form $2\pi n, n \in Z$, to each of the solutions obtained on that interval.

Thus, let us solve this inequality on the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.



From the figure,

$$\sin x > -\frac{1}{2} \text{ when } -\frac{\pi}{6} < x < \frac{7\pi}{6}$$

Thus, on generalizing the above solution, we get

$$x \in \bigcup_{n \in z} \left(\frac{-\pi}{6} + 2n\pi, \frac{7\pi}{6} + 2n\pi\right)$$

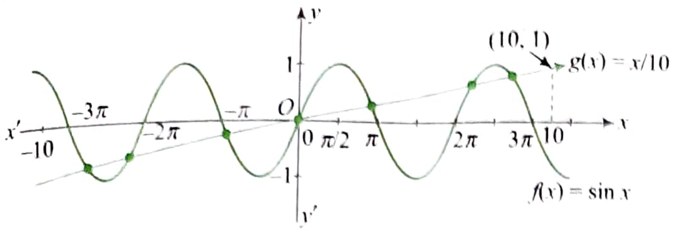
ILLUSTRATION 1.78

Find the number of solutions of $\sin x = \frac{x}{10}$.

Sol. Here, let $f(x) = \sin x$ and $g(x) = \frac{x}{10}$.

Also, we know that $-1 \leq \sin x \leq 1$ or $-1 \leq \frac{x}{10} \leq 1$ or $-10 \leq x \leq 10$

Now, sketch both the curves when $x \in [-10, 10]$.



From the figure, $f(x) = \sin x$ and $g(x) = \frac{x}{10}$ intersect at 7 points. So, number of solutions is 7.

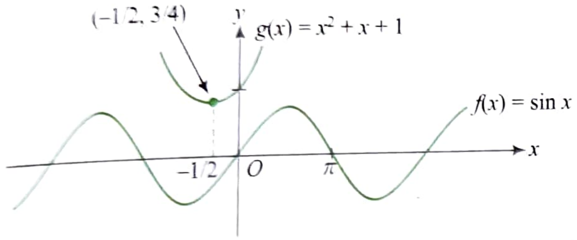
ILLUSTRATION 1.79

Find the number of solutions of the equation $\sin x = x^2 + x + 1$.

Sol. Let $f(x) = \sin x$ and

$$g(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

As shown in the figure, graphs of $y = f(x)$ and $y = g(x)$ do not intersect at any point. Therefore, there is no solution.



CONCEPT APPLICATION EXERCISE 1.6

1. Find the domain of $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$.
2. Solve **(a)** $\tan x < 2$, **(b)** $\cos x \le -\frac{1}{2}$.
3. Prove that the least positive value of x, satisfying $\tan x = x + 1$, lies in the interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.
4. Find the range of $f(x) = \sec\left(\frac{\pi}{4}\cos^2 x\right)$, where $-\infty < x < \infty$.
5. If $x \in [1, 2]$, then find the range of $f(x) = \tan x$.
6. Find the range of $f(x) = \dfrac{1}{1 - 3\sqrt{1 - \sin^2 x}}$.
7. Find the range of $f(x) = \dfrac{2\sin^2 x + 2\sin x + 3}{\sin^2 x + \sin x + 1}$
8. Draw the graph of $y = (\sin 2x)\sqrt{1 + \tan^2 x}$. Find its domain and range.

ANSWERS

1. $[-4, -\pi] \cup [0, \pi]$
2. (a) $\bigcup\limits_{n \in Z}\left(n\pi - \frac{\pi}{2}, n\pi + \tan^{-1} 2\right)$

   (b) $\bigcup\limits_{n \in Z}\left[2n\pi + \frac{2\pi}{3}, 2n\pi + \frac{4\pi}{3}\right]$
4. $[1, \sqrt{2}]$
5. $(-\infty, \tan 2] \cup [\tan 1, \infty)$
6. $(-\infty, -1/2] \cup [1, \infty)$
7. $[7/3, 10/3]$
8. Domain: $R - \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$

   Range: $(-2, 2)$

INVERSE TRIGONOMETRIC FUNCTIONS

Inverse or reverse function of any function is defined if function is bijective (one–one and onto both). We will be studying inverse functions in detail later in this chapter. Right now, let us define inverse trigonometric functions. Since each trigonometric function is many–one in its actual domain, we restrict the domain of the function so that its inverse is defined.

For given function $y = f(x)$ its inverse function is written as $x = f^{-1}(y)$ or $y = f^{-1}(x)$. The domain and range of function $y = f(x)$ is respectively range and domain of its inverse function $y = f^{-1}(x)$.

$f(x) = \sin^{-1} x$

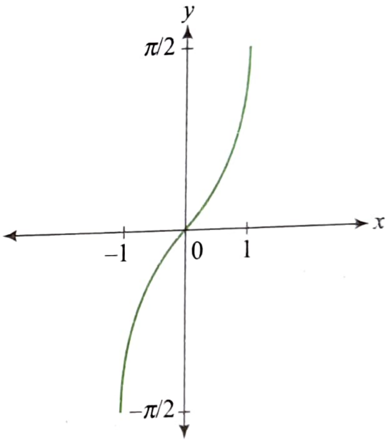
Domain: $[-1, 1]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{-1}(\sin x) = x$, for all $x \in [-\pi/2, \pi/2]$

$\sin(\sin^{-1} x) = x$, for all $x \in [-1, 1]$

$\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$



$f(x) = \cos^{-1} x$

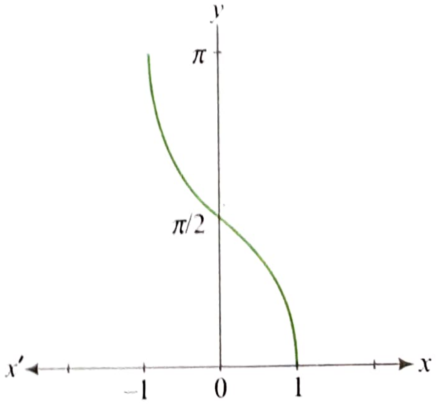
Domain: $[-1, 1]$

Range: $[0, \pi]$

$\cos^{-1}(\cos x) = x$, for all $x \in [0, \pi]$

$\cos(\cos^{-1} x) = x$, for all $x \in [-1, 1]$

$\cos^{-1}(-x) = \pi - \cos^{-1} x$, for all $x \in [-1, 1]$

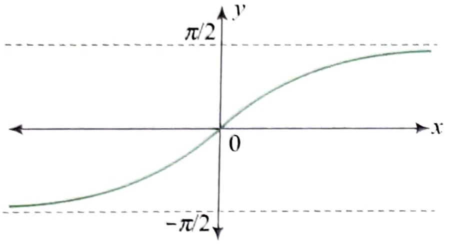


$f(x) = \tan^{-1}x$

Domain: R

Range: $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\tan^{-1}(\tan x) = x,$	for all $x \in (-\pi/2, \pi/2)$
$\tan(\tan^{-1} x) = x,$	for all $x \in R$
$\tan^{-1}(-x) = -\tan^{-1} x,$	for all $x \in R$

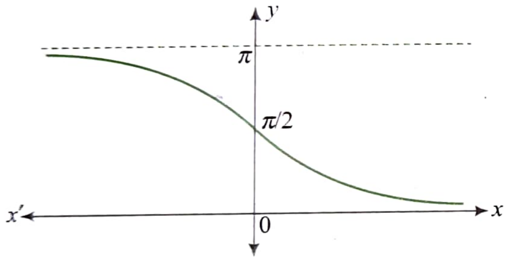


$f(x) = \cot^{-1}x$

Domain: R

Range: $(0, \pi)$

$\cot^{-1}(\cot x) = x,$	for all $x \in (0, \pi)$
$\cot(\cot^{-1} x) = x,$	for all $x \in R$
$\cot^{-1}(-x) = \pi - \cot^{-1} x,$	for all $x \in R$

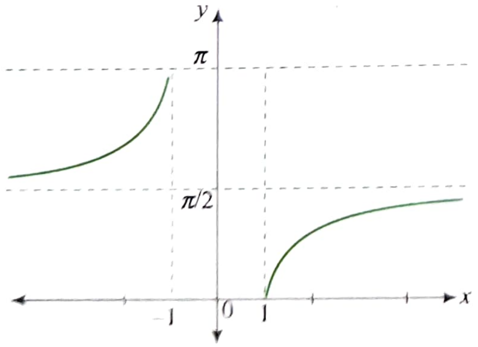


$f(x) = \sec^{-1}x$

Domain: $(-\infty, -1] \cup [1, \infty)$

Range: $[0, \pi] - \{\pi/2\}$

$\sec^{-1}(\sec x) = x,$	for all $x \in [0, \pi] - \{\pi/2\}$
$\sec(\sec^{-1} x) = x,$	for all $x \in (-\infty, -1] \cup [1, \infty)$
$\sec^{-1}(-x) = \pi - \sec^{-1} x,$	for all $x \in (-\infty, -1] \cup [1, \infty)$

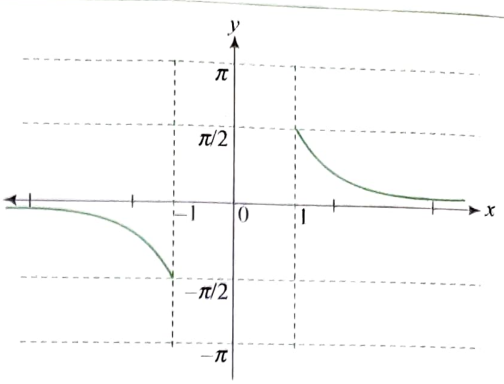


$f(x) = \text{cosec}^{-1}x$

Domain: $(-\infty, -1] \cup [1, \infty)$

Range: $[-\pi/2, \pi/2] - \{0\}$

$\text{cosec}^{-1}(\text{cosec } x) = x,$	for all $x \in [-\pi/2, \pi/2] - \{0\}$
$\text{cosec}(\text{cosec}^{-1} x) = x,$	for all $x \in (-\infty, -1] \cup [1, \infty)$
$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x,$	for all $x \in (-\infty, -1] \cup [1, \infty)$



Some of the useful properties of inverse trigonometric functions are as follows:

- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, |x| \le 1$
- $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}, |x| \ge 1$
- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in R$
- $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$
- $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$
- $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

Note:

- Domain of $\sin^{-1}(\sin x)$ is R as $\sin x \in [-1, 1]$ for all $x \in R$. Similarly, domain of $\cos^{-1}(\cos x)$ is R.
- Domain of $\sec^{-1}(\sec x)$, $\tan^{-1}(\tan x)$ is $R - \{(2n + 1)\pi/2, n \in Z\}$.
- Domain of $\text{cosec}^{-1}(\text{cosec } x)$, $\cot^{-1}(\cot x)$ is $R - \{n\pi, n \in Z\}$.

ILLUSTRATION 1.80

Find the domain and range of the function $f(x) = \sin^{-1} \frac{x^2}{2}$

Sol. $f(x) = \sin^{-1} \frac{x^2}{2}$

We must have $0 \le \frac{x^2}{2} \le 1$

$\Rightarrow \quad 0 \le x^2 \le 2$

$\Rightarrow \quad -\sqrt{2} \le x \le \sqrt{2}$

So, domain is $[-\sqrt{2}, \sqrt{2}]$

Now, $0 \le \frac{x^2}{2} \le 1$

$\Rightarrow \quad \sin^{-1} 0 \le \sin^{-1} \frac{x^2}{2} \le \sin^{-1} 1$

$\Rightarrow \quad 0 \le \sin^{-1} \frac{x^2}{2} \le \pi/2$

Hence range is $[0, \pi/2]$.

ILLUSTRATION 1.81

Find the domain of the function $f(x) = \sin^{-1}(1 + 3x + 2x^2)$.

Sol. $f(x) = \sin^{-1}(1 + 3x + 2x^2)$

We must have $-1 \le 1 + 3x + 2x^2 \le 1$

Now $2x^2 + 3x + 1 \ge -1$

$\Rightarrow$ $2x^2 + 3x + 2 \ge 0$, which is always true as discriminant < 0.

Also, $2x^2 + 3x + 1 \le 1$

$\Rightarrow$ $2x^2 + 3x \le 0$

$\Rightarrow$ $x(2x + 3) \le 0$

$\Rightarrow$ $\frac{-3}{2} \le x \le 0$

$\Rightarrow$ $x \in \left[-\frac{3}{2}, 0\right]$

ILLUSTRATION 1.82

Find the values of x for which the following pair of functions are identical.

(i) $f(x) = \tan^{-1}x + \cot^{-1}x$ and $g(x) = \sin^{-1}x + \cos^{-1}x$

(ii) $f(x) = \cos(\cos^{-1}x)$ and $g(x) = \cos^{-1}(\cos x)$

Sol.

(i) $f(x) = \tan^{-1}x + \cot^{-1}x, x \in R$.

and $g(x) = \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for $x \in [-1, 1]$

Hence functions are identical if $x \in [-1, 1]$.

(ii) $f(x) = \cos(\cos^{-1}x)$ is defined if $\cos^{-1}x$ is defined for which $x \in [-1, 1]$.

$g(x) = \cos^{-1}(\cos x)$ is defined if $\cos x \in [-1, 1]$, which is true $\forall x \in R$. So, domain is R.

Thus, common domain of two functions is $[-1, 1]$.

$\therefore$ $f(x) = x$ and $g(x) = x$ for $[-1, 1]$.

ILLUSTRATION 1.83

Find the domain and range of the function $f(x) = \sin^{-1}((1 + e^x)^{-1})$.

Sol. $f(x) = \sin^{-1}((1 + e^x)^{-1})$

We know that $e^x > 0$ for all real x.

$\therefore$ $e^x + 1 > 1$

$\Rightarrow$ $0 < \frac{1}{1+e^x} < 1$

So, $f(x)$ is defined for all real values of x.

Hence domain is R.

Also $0 < \frac{1}{1+e^x} < 1$

$\Rightarrow$ $\sin^{-1}0 < \sin^{-1}\frac{1}{1+e^x} < \sin^{-1}1$

$\Rightarrow$ $0 < \sin^{-1}\frac{1}{1+e^x} < \frac{\pi}{2}$

Therefore, range of the function is $\left(0, \frac{\pi}{2}\right)$

ILLUSTRATION 1.84

Find the domain for $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$.

Sol. $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is defined for

$$-1 \le \frac{1+x^2}{2x} \le 1 \text{ or } \left|\frac{1+x^2}{2x}\right| \le 1$$

or $|1 + x^2| \le |2x|$, for all x

or $1 + x^2 \le |2x|$, for all x (As $1 + x^2 > 0$)

or $x^2 - 2|x| + 1 \le 0$ (As $x^2 = |x^2|$)

or $|x|^2 - 2|x| + 1 \le 0$

or $(|x| - 1)^2 \le 0$

But $(|x| - 1)^2$ is always either positive or zero.

Thus, $(|x| - 1)^2 = 0$ or $|x| = 1$ or $x = \pm 1$

Thus, the domain for $f(x)$ is $\{-1, 1\}$.

ILLUSTRATION 1.85

Find the range of $f(x) = \sin^{-1}x + \tan^{-1}x + \cos^{-1}x$.

Sol. Clearly, the domain of the function is $[-1, 1]$. Also

$$\tan^{-1}x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \text{ for } x \in [-1, 1]$$

Now, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for $x \in [-1, 1]$

Thus, $f(x) = \tan^{-1}x + \frac{\pi}{2}$, where $x \in [-1, 1]$

Hence, the range is

$$\left[-\frac{\pi}{4} + \frac{\pi}{2}, \frac{\pi}{4} + \frac{\pi}{2}\right] \equiv \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

ILLUSTRATION 1.86

Find the domain of $f(x) = \sqrt{\cos^{-1}x - \sin^{-1}x}$.

Sol. We must have $\cos^{-1}x \ge \sin^{-1}x$

or $\frac{\pi}{2} - \sin^{-1}x \ge \sin^{-1}x$

or $\frac{\pi}{2} \ge 2\sin^{-1}x$

or $\sin^{-1}x \le \frac{\pi}{4}$, but $-\frac{\pi}{2} \le \sin^{-1}x$

or $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{4}$

or $\sin\left(-\frac{\pi}{2}\right) \le x \le \sin\frac{\pi}{4}$

$\left(\because \sin x \text{ is increasing function in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

or $x \in \left[-1, \frac{1}{\sqrt{2}}\right]$

ILLUSTRATION 1.87

Find the range of $\tan^{-1}\left(\dfrac{2x}{1+x^2}\right)$.

Sol. First, we must get the range of

$$\frac{2x}{1+x^2}=y$$

We have $yx^2-2x+y=0$

Since x is real, $D \geq 0$, i.e., $4-4y^2 \geq 0$ or $-1 \leq y \leq 1$. So,

$$\tan^{-1}(y) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \quad \text{(As } \tan x \text{ is an increasing function)}$$

ILLUSTRATION 1.88

Find the range of $f(x) = \cot^{-1}(2x-x^2)$.

Sol. $f(x) = \cot^{-1}(2x-x^2)$

$= \cot^{-1}(1-(1-2x+x^2))$

$= \cot^{-1}(1-(1-x)^2)$

Now $-\infty < 1-(1-x)^2 \leq 1$

$\Rightarrow \cot^{-1}(-\infty) > \cot^{-1}(1-(1-x)^2) \geq \cot^{-1}1$

$\Rightarrow \pi > \cot^{-1}(1-(1-x)^2) \geq \pi/4$

$\Rightarrow f(x) \in \left[\dfrac{\pi}{4}, \pi\right)$

Thus, range is $\left[\dfrac{\pi}{4}, \pi\right)$

ILLUSTRATION 1.89

Find the range of $f(x) = \cos^{-1}\left(\dfrac{\sqrt{1+2x^2}}{1+x^2}\right)$

Sol. $f(x) = \cos^{-1}\left(\dfrac{\sqrt{1+2x^2}}{1+x^2}\right)$

$= \sin^{-1}\left(\sqrt{1-\dfrac{1+2x^2}{(1+x^2)^2}}\right) \quad (\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2})$

$= \sin^{-1}\left(\sqrt{\dfrac{x^4}{(1+x^2)^2}}\right)$

$= \sin^{-1}\left(\dfrac{x^2}{1+x^2}\right)$

$= \sin^{-1}\left(1-\dfrac{1}{1+x^2}\right)$

Now, $\left(1-\dfrac{1}{1+x^2}\right) \in [0, 1)$

$\therefore \quad f(x) \in \left[0, \dfrac{\pi}{2}\right)$

CONCEPT APPLICATION EXERCISE 1.7

1. Find the domain of the following functions:

 (a) $f(x) = \dfrac{\sin^{-1}x}{x}$

 (b) $f(x) = \sin^{-1}(|x-1|-2)$

 (c) $f(x) = \cos^{-1}(1+3x+2x^2)$

 (d) $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$

 (e) $f(x) = \cos^{-1}\left(\dfrac{6-3x}{4}\right) + \operatorname{cosec}^{-1}\left(\dfrac{x-1}{2}\right)$

 (f) $f(x) = \sqrt{\sec^{-1}\left(\dfrac{2-|x|}{4}\right)}$

2. Find the range of $f(x) = \tan^{-1}\left(\sqrt{x^2-2x+2}\right)$.

3. Find the range of the function $f(x) = \cot^{-1}\log_{0.5}(x^4-2x^2+3)$.

4. Find the domain of definition of the function $f(x) = \sqrt{\sin^{-1}(2x)+\dfrac{\pi}{6}}$.

5. Find the domain and range of $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$.

ANSWERS

1. (a) $[-1, 0) \cup (0, 1]$ (b) $[-2, 0] \cup [2, 4]$
 (c) $[-3/2, 0]$ (d) $[2, 3)$
 (e) $[3, 10/3]$ (f) $(-\infty, -6] \cup [6, \infty)$
2. $[\pi/4, \pi/2)$
3. $[3\pi/4, \pi)$
4. $[-1/4, 1/2]$
5. Domain:
 Range: $\{\pi/4, 3\pi/4\}$

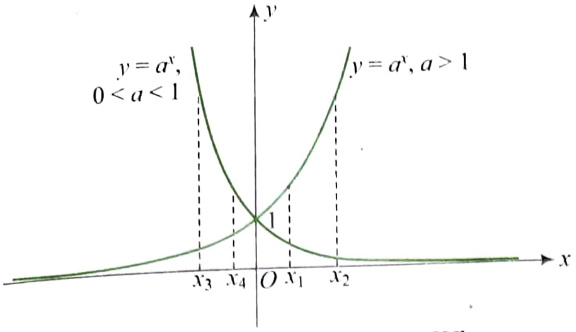
EXPONENTIAL AND LOGARITHMIC FUNCTIONS

EXPONENTIAL FUNCTION

If a is positive real number other than '1' then exponential function is defined as $f(x) = a^x$. Here 'a' is called the base and x can take any real value e.g., functions $f(x) = 2^x$, $g(x) = (4/7)^x$ are exponential functions. There is a big difference between an exponential function and a polynomial. The function $g(x) = x^3$ is a polynomial in which exponent is constant. The function $f(x) = 3^x$ is an exponential function in which exponent is variable. Here, the restriction on the base a is that it is not 0 or 1 as $f(x) = 0^x = 0$ and $f(x) = 1^x = 1$ and hence, these are constant functions and won't have many of the same properties that general exponential functions have. Also, we avoid negative numbers as base so that we don't get any complex values out of the function evaluation. For instance, if we allowed $a = -4$ the function would be, $f(x) = (-4)^x$, then we have $f(0.5) = (-4)^{0.5}$, which is a complex number. We only want real numbers to arise from function evaluation and so to make sure of this we require that a is not a negative number.

Graph of Exponential Function

Graphs of exponential function $y = a^x$ for $a > 1$ and for $0 < a < 1$ is as shown in the following figure.



Clearly if $a > 1$, a^x increases as x increases. When x approaches to infinity, a^x approaches to infinity and when x approaches to negative infinity, a^x approaches to zero.

Also, if $x_2 > x_1$ then $a^{x_2} > a^{x_1}$

If $0 < a < 1$, a^x decreases as x increases. When x approaches to infinity, a^x approaches to zero and when x approaches to negative infinity, a^x approaches to infinity.

Also, if $x_4 > x_3$ then $a^{x_4} < a^{x_3}$

ILLUSTRATION 1.90

Find the domain of $f(x) = \sqrt{\left(\dfrac{1-5^x}{7^{-x}-7}\right)}$.

Sol. We must have

$$g(x) = \left(\frac{1-5^x}{7^{-x}-7}\right) \geq 0 \quad \text{or} \quad \frac{5^x-1}{7^{-x}-7} \leq 0$$

Now, $5^x - 1 = 0$ or $x = 0$ and $7^{-x} - 7 = 0$ or $x = -1$.

The sign scheme of $g(x)$ is as follows:

$$\begin{array}{ccccc} - & & + & & - \\ \hline & -1 & & 0 & \end{array}$$

Hence, from the sign scheme of $g(x)$, $x \in (-\infty, -1) \cup [0, \infty)$.

ILLUSTRATION 1.91

Find the domain of $f(x) = \sqrt{(0.625)^{4-3x} - (1.6)^{x(x+8)}}$

Sol. Clearly, $(0.625)^{4-3x} \geq (1.6)^{x(x+8)}$

or $\left(\dfrac{5}{8}\right)^{4-3x} \geq \left(\dfrac{8}{5}\right)^{x(x+8)}$

or $\left(\dfrac{8}{5}\right)^{3x-4} \geq \left(\dfrac{8}{5}\right)^{x(x+8)}$

or $3x - 4 \geq x^2 + 8x$ or $x^2 + 5x + 4 \leq 0$

or $-4 \leq x \leq -1$

Hence, the domain of function $f(x)$ is $x \in [-4, -1]$.

LOGARITHMIC FUNCTION

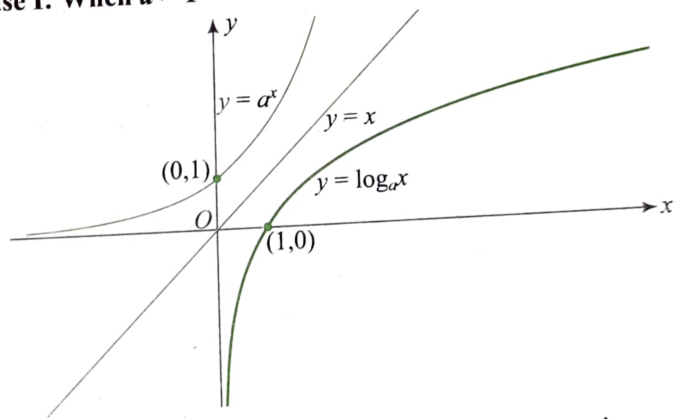
The **logarithm** of a number to a given base is the exponent to which the base must be raised in order to produce that number. For example, the logarithm of 1000 to base 10 is 3, because 10 to the power of 3 is 1000, i.e., $10^3 = 1000$. We write $\log_{10}1000 = 3$.

Here '10' is called the base of the logarithm. Similarly $\log_2 64$ is the value to which '2' must be raised to get 64. Since $2^6 = 64$, $\log_2 64 = 6$. Thus, if $a^x = y$, then we have $\log_a y = x$. In exponential function $y = a^x$, x can take any real value, so $\log_a y$ can also take any real value. Also, $y = a^x > 0$, so $\log_a y$ is defined only if $y > 0$. Thus, we can define logarithmic function as $f(x) = \log_a x$, $a > 0$, $a \neq 1$, having domain $(0, \infty)$ and range R. Thus, logarithmic function is actually inverse of exponential function. So, domain and range of exponential function are range and domain of logarithmic function, respectively.

Graph of the Logarithmic Function

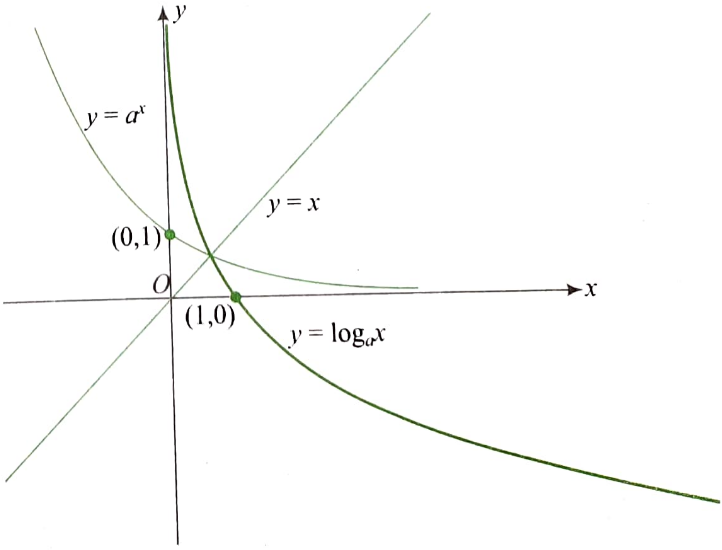
If we consider any point (p, q) on the graph of $y = a^x$, we have $q = a^p$. So, we have $p = \log_a q$, or there is a point (q, p) on the graph of $y = \log_a x$. Thus, for any point (p, q) on the graph of $y = a^x$, we have a point (q, p) on the graph of $y = \log_a x$. Since point (q, p) is the mirror image of the point (p, q) in the line $y = x$, we can draw the graph of $y = \log_a x$ by taking the mirror image of the graph of $y = a^x$ in line $y = x$. Thus, graphs of $y = a^x$ and $y = \log_a x$ are symmetrical about the line $y = x$ as shown in the following figure.

Case I: When $a > 1$



From the graph, we can see that $y = \log_a x$ is an increasing function in its domain. The value of $\log_a x$ approaches to infinity as x approaches to infinity. The value of $\log_a x$ approaches to negative infinity as x approaches to 0. Also, for $x_2 > x_1$, we have $\log_a x_2 > \log_a x_1$.

Case II: When $0 < a < 1$



From the graph, we can see that $y = \log_a x$ is decreasing function in its domain. The value of $\log_a x$ approaches to negative infinity as x approaches to infinity. The value of $\log_a x$ approaches to positive infinity as x approaches to 0. Also, for $x_2 > x_1$, we have $\log_a x_2 < \log_a x_1$.

Properties of Logarithmic Function

For $m, n > 0$ and $a > 0, a \neq 1$

- $\log_a m + \log_a n = \log_a mn$
- $\log_a m - \log_a n = \log_a \dfrac{m}{n}$
- $\log_a(m^p) = p\log_a m$
- $\log_{(a^q)} m = \dfrac{1}{q}\log_a m$
- $\log_a p = \dfrac{1}{\log_p a}$
- $\log_a p = \dfrac{\log_b p}{\log_b a}$
- $a^{\log_a p} = p$
- $\log_a p > \log_a q$ then $p > q$ if $a > 1$
- $\log_a p > \log_a q$ then $p < q$ if $0 < a < 1$
- $\log_a p > m$ then $p > a^m$ if $a > 1$
- $\log_a p > m$ then $0 < p < a^m$ if $0 < a < 1$

ILLUSTRATION 1.92

Is the pair of the functions $e^{\sqrt{\log_e x}}$ and $\sqrt{x}$ identical?

Sol. $e^{\sqrt{\log_e x}} = \left(e^{(\log_e x)}\right)^{\frac{1}{2}} = \sqrt{x} = \sqrt{x}$, where $x \geq 1$

For $\sqrt{x}$, $x \geq 0$.

Thus, functions are not identical as domain is not same.

ILLUSTRATION 1.93

Find the domain and range of following functions

(i) $f(x) = \log_e(\sin x)$

(ii) $f(x) = \log_3(5 - 4x - x^2)$

Sol.

(i) $f(x) = \log_e \sin x$ is defined if $\sin x \in (0, 1]$

$\therefore \quad x \in \cdots (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi) \ldots$

Also, $0 < \sin x \leq 1$

$\therefore \quad -\infty < \log_e(\sin x) \leq 0$

Thus, range is $(-\infty, 0]$

(ii) $f(x) = \log_3(5 - 4x - x^2)$

$= \log_3(9 - (x + 2)^2)$

$f(x)$ is defined if $5 - 4x - x^2 > 0$

or $x^2 + 4x - 5 < 0$

$\Rightarrow \quad (x - 1)(x + 5) < 0$

$\Rightarrow \quad -5 < x < 1$

So, domain is $(-5, 1)$.

Also $9 - (x + 2)^2 \leq 9$

Thus, $0 < 9 - (x + 2)^2 \leq 9$

$\Rightarrow \quad -\infty < \log_3(9 - (x + 2)^2) \leq \log_3 9$

Hence, range of $f(x)$ is $(-\infty, 2]$

ILLUSTRATION 1.94

Find the domain and range of function

$$f(x) = \log_2\left(\frac{\pi + 2\sin^{-1}\left(\frac{3-x}{7}\right)}{\pi}\right).$$

Sol. $f(x) = \log_2\left(\dfrac{\pi + 2\sin^{-1}\left(\frac{3-x}{7}\right)}{\pi}\right)$

Least value of $2\sin^{-1}\left(\dfrac{3-x}{7}\right)$ is '−2' for which

$\dfrac{\pi + 2\sin^{-1}\left(\frac{3-x}{7}\right)}{\pi}$ is positive.

So, for domain we must have $-1 \leq \dfrac{3-x}{7} \leq 1$

$\therefore \quad -7 \leq 3 - x \leq 7$

$\Rightarrow \quad -10 \leq -x \leq 4$

$\Rightarrow \quad -4 \leq x \leq 10$

Thus, domain of $f(x)$ is $[-4, 10]$.

Also, $\quad -\dfrac{\pi}{2} \leq \sin^{-1}\left(\dfrac{3-x}{7}\right) \leq \dfrac{\pi}{2}$

$\Rightarrow \quad -\pi \leq 2\sin^{-1}\left(\dfrac{3-x}{7}\right) \leq \pi$

$\Rightarrow \quad 0 \leq \pi + 2\sin^{-1}\left(\dfrac{3-x}{7}\right) \leq 2\pi$

$\Rightarrow \quad 0 < \dfrac{\pi + 2\sin^{-1}\left(\frac{3-x}{7}\right)}{\pi} \leq 2$ (for logarithm to get defined)

$\Rightarrow \quad -\infty < \log_2\left(\dfrac{\pi + 2\sin^{-1}\left(\frac{3-x}{7}\right)}{\pi}\right) \leq 1$

Hence range is $(-\infty, 1]$

ILLUSTRATION 1.95

Find the domain of $f(x) = \log_{10} \log_2 \log_{2/\pi} (\tan^{-1} x)^{-1}$.

Sol. We must have

$\log_2 \log_{2/\pi} (\tan^{-1} x)^{-1} > 0$

or $\quad \log_{2/\pi} (\tan^{-1} x)^{-1} > 1$

or $\quad 0 < (\tan^{-1} x)^{-1} < \dfrac{2}{\pi}$

or $\frac{\pi}{2} < \tan^{-1} x < \infty$, which is not possible. Hence, the domain is ϕ.

ILLUSTRATION 1.96

Find the domain and range of $f(x) = \sqrt{\log_3\{\cos(\sin x)\}}$

Sol. $f(x) = \sqrt{\log_3\{\cos(\sin x)\}}$

$f(x)$ is defined only if $\log_3\{\cos(\sin x)\} \geq 0$

or $\cos(\sin x) \geq 1$

or $\cos(\sin x) = 1$ as $-1 \leq \cos\theta \leq 1$

or $\sin x = 0$ or $x = n\pi, n \in I$

Hence, Domain = $\{n\pi; n \in I\}$

Also, the range is $\{0\}$.

ILLUSTRATION 1.97

Find the domain of $f(x) = \sin^{-1}\{\log_9(x^2/4)\}$.

Sol. We have

$$f(x) = \sin^{-1}\left\{\log_9\left(\frac{x^2}{4}\right)\right\}$$

Since the domain of $\sin^{-1} x$ is $[-1, 1]$. Therefore,

$$f(x) = \sin^{-1}\left\{\log_9\left(\frac{x^2}{4}\right)\right\}$$

is defined if

$$-1 \leq \log_9\left(\frac{x^2}{4}\right) \leq 1$$

or $9^{-1} \leq \frac{x^2}{4} \leq 9^1$

or $\frac{4}{9} \leq x^2 \leq 36$

or $\frac{2}{3} \leq |x| \leq 6$

or $x \in \left[-6, -\frac{2}{3}\right] \cup \left[\frac{2}{3}, 6\right]$

$(\because a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b])$

Hence, the domain of $f(x)$ is $\left[-6, -\frac{2}{3}\right] \cup \left[\frac{2}{3}, 6\right]$.

ILLUSTRATION 1.98

Find the domain of function $f(x) = \log_4[\log_5\{\log_3(18x - x^2 - 77)\}]$.

Sol. $f(x)$ is defined if

$\log_5\{\log_3(18x - x^2 - 77)\} > 0$ and $18x - x^2 - 77 > 0$

or $\log_3(18x - x^2 - 77) > 5^0$ and $x^2 - 18x + 77 < 0$

or $\log_3(18x - x^2 - 77) > 1$ and $(x - 11)(x - 7) < 0$

or $18x - x^2 - 77 > 3^1$ and $7 < x < 11$

or $18x - x^2 - 80 > 0$ and $7 < x < 11$

or $x^2 - 18x + 80 < 0$ and $7 < x < 11$

or $(x - 10)(x - 8) < 0$ and $7 < x < 11$

or $8 < x < 10$ and $7 < x < 11$

or $8 < x < 10$

or $x \in (8, 10)$

Hence, the domain of $f(x)$ is $(8, 10)$.

ILLUSTRATION 1.99

Let $x \in \left(0, \frac{\pi}{2}\right)$. Then find the domain of the function

$$f(x) = \frac{1}{\sqrt{-\log_{\sin x} \tan x}}.$$

Sol. Here, $x \in \left(0, \frac{\pi}{2}\right)$

or $0 < \sin x < 1$ (1)

and we know $\begin{cases} \log_a x < b \Rightarrow x > a^b, & \text{if } 0 < a < 1 \\ x < a^b, & \text{if } a > 1 \end{cases}$ (2)

Thus, $f(x) = \frac{1}{\sqrt{-\log_{\sin x} \tan x}}$ exists if $-\log_{\sin x}(\tan x) > 0$

or $\log_{\sin x} \tan x < 0$

[As inequality changes sign on multiplying by –ve]

or $\tan x > (\sin x)^0$ [Using (1) and (2)]

or $\tan x > 1$

or $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ $\left[\text{As } x \in \left(0, \frac{\pi}{2}\right)\right]$

ILLUSTRATION 1.100

Find the domain of $f(x) = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}$.

Sol. $f(x) = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}$

exists if

$$\log_{0.4}\left(\frac{x-1}{x+5}\right) \geq 0 \text{ and } \left(\frac{x-1}{x+5}\right) > 0$$

or $\frac{x-1}{x+5} \leq (0.4)^0$ and $\frac{x-1}{x+5} > 0$

or $\frac{x-1}{x+5} \leq 1$ and $\frac{x-1}{x+5} > 0$

or $\frac{x-1}{x+5} - 1 \leq 0$ and $\frac{x-1}{x+5} > 0$

or $\frac{-6}{x+5} \leq 0$ and $\frac{x-1}{x+5} > 0$

or $x + 5 > 0$ and $\frac{x-1}{x+5} > 0$

or $x > -5$ and $x - 1 > 0$ (As $x + 5 > 0$)

or $x > -5$ and $x > 1$

Thus, the domain $f(x)$ is $(1, \infty)$.

ILLUSTRATION 1.101

Find the range of $f(x) = \log_e x - \dfrac{(\log_e x)^2}{|\log_e x|}$.

Sol. $f(x) = \log_e x - \dfrac{(\log_e x)^2}{|\log_e x|}$

$$= \begin{cases} \log_e x - \dfrac{(\log_e x)^2}{(\log_e x)}, & \log_e x > 0 \\ \log_e x - \dfrac{(\log_e x)^2}{(-\log_e x)}, & \log_e x < 0 \end{cases}$$

$$= \begin{cases} 0, & x > 1 \\ 2\log_e x, & 0 < x < 1 \end{cases}$$

Therefore, range is $(-\infty, 0]$.

ILLUSTRATION 1.102

Find the domain and range of function

$$f(x) = \sin\left(\log_e\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right)$$

Sol. We have $f(x) = \sin\left(\log_e\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)\right)$

We must have $4 - x^2 > 0$ and $1 - x > 0$

$\Rightarrow$ $x^2 < 4$ and $x < 1$

$\Rightarrow$ $-2 < x < 2$ and $x < 1$

$\Rightarrow$ $-2 < x < 1$

Thus, domain of $f(x)$ is $(-2, 1)$

When x approaches to 1 from its left-hand side $\dfrac{\sqrt{4-x^2}}{1-x}$ approaches to infinity.

When x approaches to -2 from its right-hand side $\dfrac{\sqrt{4-x^2}}{1-x}$ approaches to zero.

Also, $\dfrac{\sqrt{4-x^2}}{1-x}$ exists continuously for $x \in (-2, 1)$.

Thus $0 < \dfrac{\sqrt{4-x^2}}{1-x} < \infty$

$\Rightarrow$ $-\infty < \log_e \dfrac{\sqrt{4-x^2}}{1-x} < \infty$

$\Rightarrow$ $\sin\left(\log_e \dfrac{\sqrt{4-x^2}}{1-x}\right) \in [-1, 1]$

CONCEPT APPLICATION EXERCISE 1.8

Find the domain of the following functions (1 to 7):

1. $f(x) = \sqrt{4^x + 8^{(2/3)(x-2)} - 13 - 2^{2(x-1)}}$
2. $f(x) = \sin^{-1}(\log_2 x)$
3. $f(x) = \log_{(x-4)}(x^2 - 11x + 24)$
4. $f(x) = \dfrac{3}{4-x^2} + \log_{10}(x^3 - x)$
5. $f(x) = \sqrt{\dfrac{\log_{0.3}|x-2|}{|x|}}$
6. $f(x) = \sqrt{\log_{10}\left\{\dfrac{\log_{10} x}{2(3 - \log_{10} x)}\right\}}$
7. $f(x) = \dfrac{1}{\sqrt{\log_{1/2}(x^2 - 7x + 13)}}$
8. Find the range of $f(x) = \log_2\left(\dfrac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$.
9. Find the values of x in $[-\pi, \pi]$ for which $f(x) = \sqrt{\log_2(4\sin^2 x - 2\sqrt{3}\sin x - 2\sin x + \sqrt{3} + 1)}$ is defined.

ANSWERS

1. $[2, \infty)$ 2. $[1/2, 2]$ 3. $(8, \infty)$
4. $(-1, 0) \cup (1, 2) \cup (2, \infty)$ 5. $[1, 2) \cup (2, 3]$
6. $[100, 1000)$ 7. $(3, 4)$ 8. $[1, 2]$
9. $x \in \left[-\pi, \dfrac{\pi}{6}\right] \cup \left[\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right] \cup \left[\dfrac{5\pi}{6}, \pi\right]$

SOME MORE FUNCTIONS

GREATEST INTEGER FUNCTION

Consider number $1.23 = 1 + 0.23$. This number has two parts, integral part '1' and fractional part 0.23.

For number 2.87, integral part is 2 and fractional part is 0.87.

Now which numbers have integral part 3? There will be innumerable numbers starting with 3 but less than 4, i.e., numbers 3, 3.001, 3.02, 3.1, 3.56, 3.72, 3.99 all have integral part 3. Thus, if integral part of real number x is '3' then $x \in [3, 4)$. Integral part of real any number x is represented as $[x]$.

So, we have $[1.23] = 1$, $[2.87] = 2$. From this we understand that to get the integral part of real number x, we look to left of the real number x on real number line and pick the first integer to left of x. Since this integer is the greatest among all the integers which are less than x, $[x]$ is called greatest integer less than or equal to x. Obviously, integral part of real number x will be same as x if x is integer. Thus $[x] \le x$.

For negative real numbers also, we follow the same rule. For example, $[-1.76] = -2$, $[-4.56] = -5$, $[-0.81] = -1$ etc.

In fact, fractional part of any real number is always positive. So, $-1.76 = -2 + 0.34$, which return integral part of -1.76 as -2.

We define greatest integer function as $f(x) = [x]$

Clearly domain of the function is R.

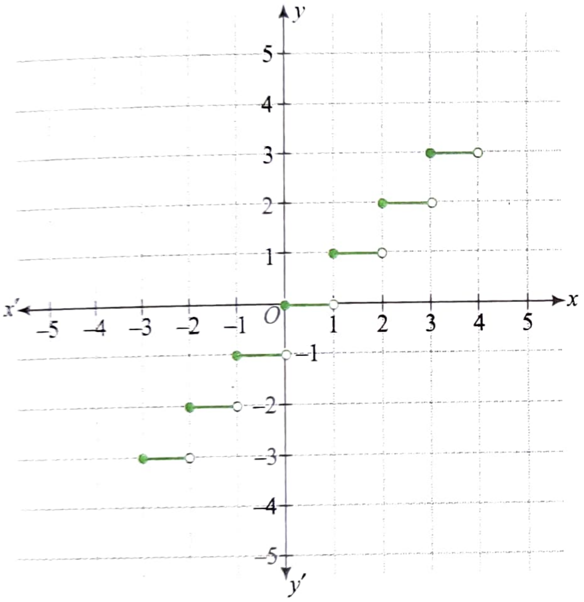
For any input value to this function we get output as integer.

So, range is Z, set of integers.

In piecewise, definition of the function is as follows.

$$f(x) = \begin{cases} .... \\ .... \\ -2,\ x \in [-2, -1) \\ -1,\ x \in [-1, 0) \\ 0,\ x \in [0, 1) \\ 1,\ x \in [1, 2) \\ 2,\ x \in [2, 3) \\ 3,\ x \in [3, 4) \\ .... \\ .... \end{cases}$$

Graph of the function is as shown in the following figure.



Properties of Greatest Integer Function

- $[x] = n\ (n \in Z) \Rightarrow x \in [n, n+1)$
- $x - 1 < [x] \le x$
- $[x + n] = [x] + n$, where $n \in Z$
- $[-x] + [x] = 0$, if $x \in Z$
- $[-x] + [x] = -1$, if $x \notin Z$
- $[x] \ge n \Rightarrow x \ge n,\ n \in Z$
- $[x] \le n \Rightarrow x < n + 1,\ n \in Z$
- $[x] > n \Rightarrow x \ge n + 1,\ n \in Z$

  e.g., $[x] \ge 2 \Rightarrow x \in [2, \infty)$

  $[x] > 3 \Rightarrow [x] \ge 4 \Rightarrow x \in [4, \infty)$

  $[x] \le 3 \Rightarrow x \in (-\infty, 4)$
- $\left[\frac{x}{n}\right] + \left[\frac{x+1}{n}\right] + \left[\frac{x+2}{n}\right] + \ldots + \left[\frac{x+n-1}{n}\right] = [x],\ n \in N$

  or $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx]$

ILLUSTRATION 1.103

Find the domain of the function $f(x) = \dfrac{1}{[x]^2 - 7[x] - 8}$, where [.] represents the greatest integer function.

Sol. We have $f(x) = \dfrac{1}{[x]^2 - 7[x] - 8}$

Clearly $f(x)$ is not defined if $[x]^2 - 7[x] - 8 = 0$

$\Rightarrow \quad ([x] - 8)([x] + 1) = 0$

$\Rightarrow \quad [x] = -1$ or 8

$\Rightarrow \quad x \in [-1, 0)$ or $x \in [8, 9)$

So, domain of $f(x)$ is $(-\infty, -1) \cup [0, 8) \cup [9, \infty)$

ILLUSTRATION 1.104

Find the domain of $f(x) = \sqrt{[x]-1} + \sqrt{4-[x]}$, where [.] represents greatest integer function.

Sol. We have $f(x) = \sqrt{[x]-1} + \sqrt{4-[x]}$

Clearly $f(x)$ is defined if $[x] - 1 \ge 0$ and $4 - [x] \ge 0$

$\therefore \quad 1 \le [x] \le 4$

$\Rightarrow \quad 1 \le x < 5$

Hence domain of $f(x)$ is $[1, 5)$.

ILLUSTRATION 1.105

Find the Domain and range of $f(x) = \sin^{-1}[x]$, where [.] represents greatest integer function.

Sol. $f(x) = \sin^{-1}[x]$ is defined if $-1 \le [x] \le 1$

$\Rightarrow \quad [x] = -1, 0, 1$

$\Rightarrow \quad x \in [-1, 2)$

Also, range is $\{\sin^{-1}(-1), \sin^{-1} 0, \sin^{-1} 1\} \equiv \{-\pi/2, 0, \pi/2\}$

ILLUSTRATION 1.106

Solve If $[\cos^{-1}x] + [\cot^{-1}x] = 0$, where [.] denotes the greatest integer function.

Sol. We have $[\cos^{-1} x] \ge 0\ \forall\ x \in [-1, 1]$

and $\quad [\cot^{-1} x] \ge 0\ \forall\ x \in R$

Hence, $[\cos^{-1} x] + [\cot^{-1} x] = 0$

only if $[\cos^{-1} x] = [\cot^{-1} x] = 0$

$[\cos^{-1} x] = 0 \Rightarrow x \in (\cos 1, 1]$

$[\cot^{-1} x] = 0 \Rightarrow x \in (\cot 1, \infty)$

$\therefore \quad x \in (\cot 1, 1]$

ILLUSTRATION 1.107

Write the piecewise definition of the following functions.

(i) $f(x) = [\sqrt{x}]$ **(ii)** $f(x) = [\tan^{-1} x]$ **(iii)** $f(x) = [\log_e x]$

In each case [.] denotes the greatest integer function.

Sol.

(i) $f(x) = [\sqrt{x}],\ x \ge 0.$

$[\sqrt{x}] = 0$ if $\sqrt{x} \in [0, 1)$ or $x \in [0, 1)$

$[\sqrt{x}] = 1$ if $\sqrt{x} \in [1, 2)$ or $x \in [1, 4)$

$[\sqrt{x}] = 2$ if $\sqrt{x} \in [2, 3)$ or $x \in [4, 9)$

and so on.

$$\therefore \quad f(x) = \begin{cases} 0, & x \in [0, 1) \\ 1, & x \in [1, 4) \\ 2, & x \in [4, 9) \\ 3, & x \in [9, 16) \\ \ldots \\ \ldots \end{cases}$$

(ii) $f(x) = [\tan^{-1} x]$

We know that $\tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore$ possible values of $[\tan^{-1} x]$ are $-2, -1, 0, 1$.

If $[\tan^{-1} x] = -2$, then $\tan^{-1} x \in \left(-\frac{\pi}{2}, -1\right)$ or $x \in (-\infty, -\tan 1)$

If $[\tan^{-1} x] = -1$, then $\tan^{-1} x \in [-1, 0)$ or $x \in [-\tan 1, 0)$

If $[\tan^{-1} x] = 0$, then $\tan^{-1} x \in [0, 1)$ or $x \in [0, \tan 1)$

If $[\tan^{-1} x] = 1$, then $\tan^{-1} x \in \left[1, \frac{\pi}{2}\right)$ or $x \in [\tan 1, \infty)$

(iii) $f(x) = [\log_e x]$

We know that $\log_e x \in (-\infty, \infty)$.

$\therefore$ $[\log_e x] \in Z$, i.e., $[\log_e x]$ takes all integral values.

If $[\log_e x] = n, n \in Z$ then $\log_e x \in [n, n+1)$ or $x \in [e^n, e^{n+1})$.

$$\text{Therefore, } [\log_e x] = \begin{cases} \ldots \\ \ldots \\ -1, & x \in [e^{-1}, 1) \\ 0, & x \in [1, e) \\ 1, & x \in [e, e^2) \\ 2, & x \in [e^2, e^3) \\ 3, & x \in [e^3, e^4) \\ \ldots \\ \ldots \end{cases}$$

ILLUSTRATION 1.108

Find the range of $f(x) = [\sin x + [\cos x + [\tan x + [\sec x]]]]$, $x \in (0, \pi/4)$, where [.] denotes the greatest integer function less than or equal to x.

Sol. Given

$$f(x) = [\sin x + [\cos x + [\tan x + [\sec x]]]]$$
$$= [\sin + p], \text{ where } p = [\cos x + [\tan x + [\sec x]]]$$
$$= [\sin x] + p, \text{ (as } p \text{ is an integer)}$$
$$= [\sin x] + [\cos x + [\tan x + [\sec x]]]$$
$$= [\sin x] + [\cos x] + [\tan x] + [\sec x]$$

Now, for $x \in (0, \pi/4)$, $\sin x \in \left(0, \frac{1}{\sqrt{2}}\right)$, $\cos x \in \left(\frac{1}{\sqrt{2}}, 1\right)$,

$\tan x \in (0, 1)$, $\sec x \in (1, \sqrt{2})$

or $[\sin x] = 0$, $[\cos x] = 0$, $[\tan x] = 0$, and $[\sec x] = 1$

Therefore, the range of $f(x)$ is 1.

ILLUSTRATION 1.109

Find the range of $f(x) =$

$$[1 + \sin x] + \left[2 + \sin\frac{x}{2}\right] + \left[3 + \sin\frac{x}{3}\right] + \ldots + \left[n + \sin\frac{x}{n}\right]$$

$\forall x \in [0, \pi]$, $n \in N$ ([.] denotes greatest integer function).

Sol. $$f(x) = [1 + \sin x] + \left[2 + \sin\frac{x}{2}\right] + \left[3 + \sin\frac{x}{3}\right] + \ldots + \left[n + \sin\frac{x}{n}\right]$$

for, $x \in (0, \pi)$

$$\sin x, \sin\frac{x}{2}, \sin\frac{x}{3}, \sin\frac{x}{4}, \sin\frac{x}{5}, \ldots \sin\frac{x}{n} \in (0, 1)$$

$\therefore$ for $x \in (0, \pi)$

$$f(x) = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

For $x = 0$, $f(x) = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

for $x = \pi$, $f(x) = 1 + [2 + 1] + 3 + 4 + \ldots + n$

$$= (1 + 2 + 3 + \ldots + n) + 1$$
$$= \frac{n(n+1)}{2} + 1 = \frac{n^2 + n + 2}{2}$$

$\therefore$ Range is $\left\{\dfrac{n(n+1)}{2}, \dfrac{n^2 + n + 2}{2}\right\}$

ILLUSTRATION 1.110

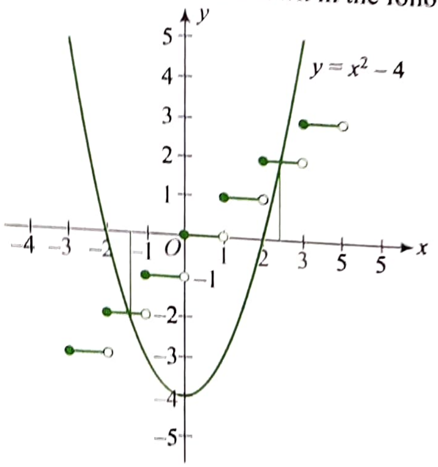
Solve: $x^2 - 4 - [x] = 0$, where [.] denotes the greatest integer unction.

Sol. We have $x^2 - 4 - [x] = 0$

or $\quad x^2 - 4 = [x]$

To, solve the equation we first locate the roots, for which we draw the graphs of $y = x^2 - 4$ and $y = [x]$ and find the points of intersection.

The graphs of the functions are as shown in the following figure.



We observe that graphs intersect when $x^2 - 4 = 2$ and $x^2 - 4 = -2$

$\Rightarrow \quad x^2 = 6$ or $x^2 = 2$

$\Rightarrow \quad x = \sqrt{6}$ or $-\sqrt{2}$

Fractional Part Function

As we discussed earlier any real number has two parts, integral and fractional.

If we subtract integral part of number from the number, we get its fractional part.

i.e., for 1.23, [1.23] = 1. So, fractional part of 1.23 denoted by {1.23} is 1.23 – [1.23] = 1.23 – 1 = 0.23

It is not surprising that the fractional part of the number lies in [0, 1).

i.e., for –2.74, fractional part is

$$\{-2.74\} = -2.74 - [-2.74] = -2.74 - (-3) = 0.26.$$

In general, fractional part function is defined as

$$f(x) = x - [x] = \{x\}.$$

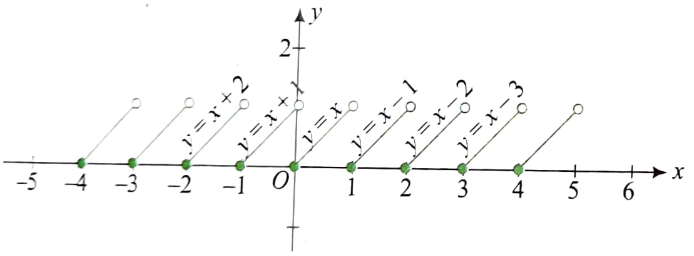
The domain of the function is R and range of the function is [0, 1).

Clearly, $\{x\} = 0$ if x is integer. For non–integral number, $\{x\} \in (0, 1)$.

Piecewise representation of the function is

$$f(x) = \{x\} = x - [x] = \begin{cases} \dots \\ \dots \\ x+1, \ x \in [-1, 0) \\ x, \ x \in [0, 1) \\ x-1, \ x \in [1, 2) \\ x-2, \ x \in [2, 3) \\ \dots \\ \dots \end{cases}$$

The graph of the function is as shown in the figure.



We observe that graph of the function repeats after interval of length '1' unit. Thus, function is periodic with period '1'.

Properties of Fractional Part Function

- $\{x\} = \{x + n\}$, where $n \in Z$
- If $\{x\} = p$ then $x = n + p$, where $n \in Z$
- $\{-x\} + \{x\} = 0$, if $x \in Z$
- $\{-x\} + \{x\} = 1$, if $x \notin Z$
- $[x + y] = [x] + [y]$, if $0 \le \{x\} + \{y\} < 1$
- $[x + y] = [x] + [y] + 1$, $1 \le \{x\} + \{y\} < 2$

Illustration 1.111

Find the domain and range of $f(x) = \log \{x\}$, where { } represents the fractional part function).

Sol. We know that $0 \le \{x\} < 1 \ \forall x \in R$.

Now, when $\{x\} = 0$, $\log \{x\}$ is not defined. So, x cannot be an integer. Hence, the domain is $R - I$.

Now, for $0 < \{x\} < 1$, $-\infty < \log \{x\} < 0$. So, range is $(-\infty, 0)$.

Illustration 1.112

Find the domain and range of $f(x) = \sin^{-1}(x - [x])$, where [.] represents the greatest integer function.

Sol. We have $f(x) = \sin^{-1}(x - [x]) = \sin^{-1}\{x\}$

We know that $\{x\} \in [0, 1)$

For these values of $\{x\}$, $\sin^{-1}\{x\}$ is well defined.

So, domain of the function is R.

Now $\quad 0 \le \{x\} < 1$

$\Rightarrow \quad \sin^{-1}0 \le \sin^{-1}\{x\} < \sin^{-1}1$

$\Rightarrow \quad 0 \le \sin^{-1}\{x\} < \pi/2$

Hence range of the function is $[0, \pi/2)$

Illustration 1.113

Write the function $f(x) = \{\sin x\}$ (where {.} denotes the fractional part function) in piecewise definition.

Sol. We have $f(x) = \{\sin x\}$

Clearly, $\{\sin x\} = 0$ if $\sin x$ is integer.

$\therefore \quad \{\sin x\} = 0$ when $x = \dfrac{n\pi}{2}$

Now when $\sin x \in (0, 1)$

$$\{\sin x\} = \sin x - [\sin x] = \sin x - 0 = \sin x$$

When $\sin x \in (-1, 0)$,

$$\{\sin x\} = \sin x - [\sin x] = \sin x - (-1) = \sin x + 1$$

Thus, $$f(x) = \begin{cases} 0, \ \sin x \in \{-1, 0, 1\} \\ \sin x, \ \sin x \in (0, 1) \\ \sin x + 1, \ \sin x \in (-1, 0) \end{cases}$$

$$f(x) = \begin{cases} 0, & x = \dfrac{n\pi}{2}, n \in Z \\ \sin x, & x \in \bigcup\limits_{n \in Z} (2n\pi, (2n+1)\pi) \\ \sin x + 1, & x \in \bigcup\limits_{n \in Z} ((2n+1)\pi, (2n+2)\pi) \end{cases}$$

Illustration 1.114

Solve $2[x] = x + \{x\}$, where [.] and {.} denote the greatest integer function and the fractional part function, respectively.

Sol. Given $2[x] = x + \{x\}$

or $\quad 2[x] = [x] + 2\{x\}$

or $\quad \{x\} = \dfrac{[x]}{2}$

or $\quad 0 \le \dfrac{[x]}{2} < 1$

or $\quad 0 \le [x] < 2$

or $\quad [x] = 0, 1$

For $\quad [x] = 0$, we get $\{x\} = 0$ or $x = 0$.

For $\quad [x] = 1$, we get $\{x\} = \dfrac{1}{2}$ or $x = \dfrac{3}{2}$.

ILLUSTRATION 1.115

Find the range of $f(x) = \dfrac{x-[x]}{1-[x]+x}$, where $[\cdot]$ represents the greatest integer function.

Sol. $f(x) = \dfrac{x-[x]}{1-[x]+x} = \dfrac{\{x\}}{1+\{x\}} = 1 - \dfrac{1}{1+\{x\}}$

Now, $0 \le \{x\} < 1$

or $1 \le \{x\}+1 < 2$

or $\dfrac{1}{2} < \dfrac{1}{1+\{x\}} \le 1$

or $-1 \le -\dfrac{1}{1+\{x\}} < -\dfrac{1}{2}$

or $0 \le 1-\dfrac{1}{1+\{x\}} < \dfrac{1}{2}$

ILLUSTRATION 1.116

Find the domain of function $f(x) = \dfrac{1}{\sqrt{\{\sin x\}+\{\sin(\pi+x)\}}}$, where $\{.\}$ denotes the fractional part function.

Sol. $f(x) = \dfrac{1}{\sqrt{\{\sin x\}+\{\sin(\pi+x)\}}} = \dfrac{1}{\sqrt{\{\sin x\}+\{-\sin x\}}}$

Now, $\{\sin x\} + \{-\sin x\} = \begin{cases} 0, & \sin x \text{ is an integer} \\ 1, & \sin x \text{ is not an integer} \end{cases}$

For $f(x)$ to get defined,

$\{\sin x\} + \{-\sin x\} \ne 0$

or $\sin x \ne$ integer

or $\sin x \ne \pm 1, 0$

or $x \ne \dfrac{n\pi}{2}, n \in I$

ILLUSTRATION 1.117

Solve: $[x]^2 = x + 2\{x\}$, where [.] and {.} denote the greatest integer and the fractional part functions, respectively.

Sol. $[x]^2 = x + 2\{x\}$

or $[x]^2 = [x] + 3\{x\}$

or $\{x\} = \dfrac{[x]^2-[x]}{3}$

or $0 \le \dfrac{[x]^2-[x]}{3} < 1$

or $0 \le [x]^2 - [x] < 3$

or $[x] \in \left(\dfrac{1-\sqrt{13}}{2}, 0\right] \cup \left[1, \dfrac{1+\sqrt{13}}{2}\right)$

or $[x] = -1, 0, 1, 2$

or $\{x\} = \dfrac{2}{3}, 0, 0, \dfrac{2}{3}$ (respectively)

or $x = -\dfrac{1}{3}, 0, 1, \dfrac{8}{3}$

ILLUSTRATION 1.118

Solve the system of equations in x, y, and z satisfying the following equations:

$x + [y] + \{z\} = 3.1$

$\{x\} + y + [z] = 4.3$

$[x] + \{y\} + z = 5.4$

(where [.] denotes the greatest integer function and {.} denotes the fractional part function.)

Sol. Adding all the three equations, we get

$2(x+y+z) = 12.8$ or $x+y+z = 6.4$ (1)

Adding the first two equations, we get

$x+y+z+[y]+\{x\} = 7.4$ (2)

Adding the second and third equations, we get

$x+y+z+[z]+\{y\} = 9.7$ (3)

Adding the first and third equations, we get

$x+y+z+[x]+\{z\} = 8.5$ (4)

From (1) and (2), $[y] + \{x\} = 1$.

From (1) and (3), $[z] + \{y\} = 3.3$.

From (1) and (4), $[x] + \{z\} = 2.1$. So,

$[x] = 2, [y] = 1, [z] = 3,$

$\{x\} = 0, \{y\} = 0.3$, and $\{z\} = 0.1$

$\therefore$ $x = 2, y = 1.3, z = 3.1$

ILLUSTRATION 1.119

If $f(x) = \begin{cases} [x], & 0 \le \{x\} < 0.5 \\ [x]+1, & 0.5 < \{x\} < 1 \end{cases}$, then prove that $f(x) = -f(-x)$ (where [.] and $\{\cdot\}$ represent the greatest integer function and the fractional part function, respectively).

Sol. $f(-x) = \begin{cases} [-x], & 0 \le \{-x\} < 0.5 \\ [-x]+1, & 0.5 < \{-x\} < 1 \end{cases}$

$= \begin{cases} [-x], & \{-x\} = 0 \\ [-x], & 0 < \{-x\} < 0.5 \\ [-x]+1, & 0.5 < \{-x\} < 1 \end{cases}$

$= \begin{cases} -[x], & \{x\} = 0 \\ -1-[x], & 0 < 1-\{x\} < 0.5 \\ -1-[x]+1, & 0.5 < 1-\{x\} < 1 \end{cases}$

$= \begin{cases} -[x], & \{x\} = 0 \\ -1-[x], & 0.5 < \{x\} < 1 \\ -[x], & 0 < \{x\} < 0.5 \end{cases}$

$= \begin{cases} -[x], & 0 \le \{x\} < 0.5 \\ -1-[x], & 0.5 < \{x\} < 1 \end{cases}$

$= -f(x)$

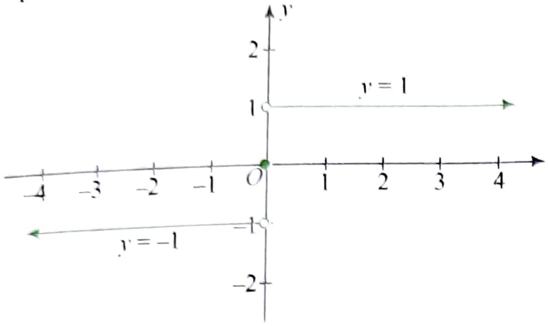
SIGNUM FUNCTION

Signum function is denoted by sgn(x) and defined as

$$f(x) = \text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Clearly domain of the function is R and range is $\{-1, 0, 1\}$.

The graph of the function is as shown in the following figure.



We can also define sgn($f(x)$) as

$$\text{sgn}(f(x)) = \begin{cases} 1, & f(x) > 0 \\ 0, & f(x) = 0 \\ -1, & f(x) < 0 \end{cases} = \begin{cases} \dfrac{|f(x)|}{f(x)}, & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$$

ILLUSTRATION 1.120

Verify that

(i) x sgn $x = |x|$ **(ii)** $|x|$ sgn $x = x$

(iii) x (sgn x) (sgn x) $= x$

Sol.

(i) $x \text{ sgn } x = \begin{cases} x \cdot 1, & x > 0 \\ 0, & x = 0 \\ x \cdot (-1), & x < 0 \end{cases} = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases} = |x|$

(ii) $|x| \text{ sgn } x = \begin{cases} x \cdot 1, & x > 0 \\ 0, & x = 0 \\ (-x) \cdot (-1), & x < 0 \end{cases} = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ x, & x < 0 \end{cases} = x$

(iii) x (sgn x) (sgn x) $= |x|$ sgn (x) $= x$

ILLUSTRATION 1.121

For the following functions write the piecewise definition and draw the graph

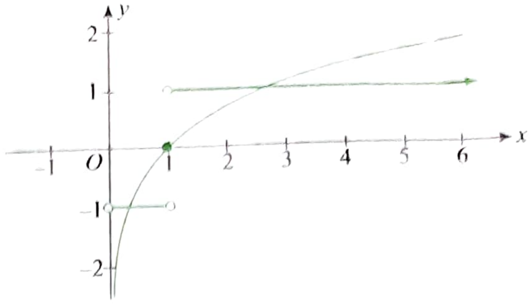
(i) $f(x) = \text{sgn}(\log_e x)$ **(ii)** $f(x) = \text{sgn}(\sin x)$

Sol.

(i) $f(x) = \text{sgn}(\log_e x)$

$$= \begin{cases} -1, & \log_e x < 0 \\ 0, & \log_e x = 0 \\ 1, & \log_e x > 0 \end{cases}$$

$$= \begin{cases} -1, & x \in (0, 1) \\ 0, & x = 1 \\ 1, & x \in (1, \infty) \end{cases}$$



(ii) $f(x) = \text{sgn}(\sin x) = \begin{cases} -1, & \sin x < 0 \\ 0, & \sin x = 0 \\ 1, & \sin x > 0 \end{cases}$

$$= \begin{cases} -1, & x \in ((2n+1)\pi, (2n+2)\pi) \\ 0, & x = n\pi, n \in Z \\ 1, & x \in (2n\pi, (2n+1)\pi) \end{cases}$$

Graphs of $y = \sin x$ and $y = \text{sgn}(\sin x)$ are as shown in the following figure.

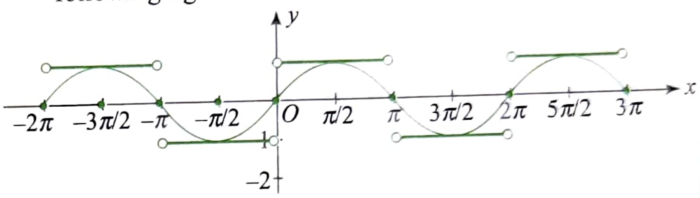


ILLUSTRATION 1.122

Find the range of the following

(i) $f(x) = \text{sgn}(x^2)$ **(ii)** $f(x) = \text{sgn}(x^2 - 2x + 3)$

Sol.

(i) We know that $x^2 \geq 0 \ \forall \ x \in R$

$$\therefore \quad \text{sgn}(x^2) = \begin{cases} 0, & x^2 = 0 \\ 1, & x^2 \neq 0 \end{cases} = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$$

Hence range of $f(x)$ is $\{0, 1\}$

(ii) $x^2 - 2x + 3 = (x-1)^2 + 2 > 0 \ \forall \ x \in R$.

So, $f(x) = \text{sgn}(x^2 - 2x + 3) = 1$

Hence, range of $f(x)$ is $\{1\}$

FUNCTIONS OF THE FORM $f(x) = \max.\{g_1(x), g_2(x), ..., g_n(x)\}$ or $f(x) = \min.\{g_1(x), g_2(x), ..., g_n(x)\}$

Let us consider the function $f(x) = \max. \{x, x^2\}$

Here, $f(2) = \max.\{2, 2^2\} = 2^2 = 4$

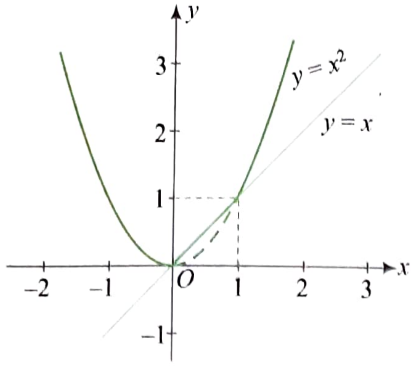
$f(0.5) = \max. \{0.5, 0.5^2\} = 0.5$

$f(1) = \max.\{1, 1^2\} = 1$

To write the equivalent definition of the function first we draw th graphs of $y = x$ and $y = x^2$.

We observe that for $x \in (-\infty, 0)$, the graph of $y = x^2$ lies above t graph of $y = x$, or $x^2 > x$.

for $x \in (0, 1)$, the graph of $y = x$ lies above the graph of $y = x^2$ $x > x^2$



for $x \in (1, \infty)$, the graph of $y = x^2$ lies above the graph of $y = x$ or $x^2 > x$

Therefore, $f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \le x \le 1 \\ x^2, & x > 1 \end{cases}$

For $\quad g(x) = \min.\{x, x^2\}$, we have

$$g(x) = \begin{cases} x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ x, & x > 1 \end{cases}$$

ILLUSTRATION 1.123

If $f: R \to R$ and $g: R \to R$ are two given functions, then prove that $2\min.\{f(x) - g(x), 0\} = f(x) - g(x) - |g(x) - f(x)|$

Sol. $h(x) = 2\min.\{f(x) - g(x), 0\}$

$$= \begin{cases} 0, & f(x) > g(x) \\ 2\{f(x) - g(x)\}, & f(x) \le g(x) \end{cases}$$

$$= \begin{cases} f(x) - g(x) - |f(x) - g(x)|, & f(x) > g(x) \\ f(x) - g(x) - |f(x) - g(x)|, & f(x) \le g(x) \end{cases}$$

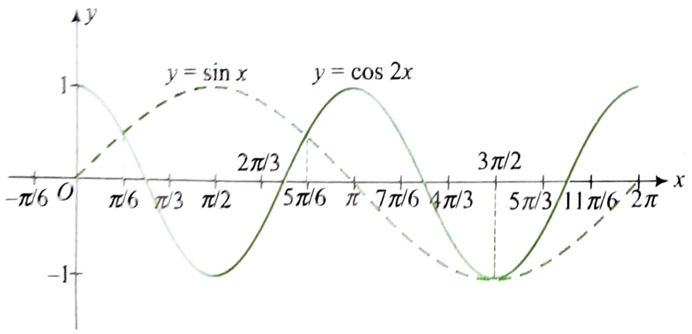
$\therefore \quad h(x) = f(x) - g(x) - |g(x) - f(x)|$

ILLUSTRATION 1.124

Draw the graph of the function $f(x) = \max.\{\sin x, \cos 2x\}$, $x \in [0, 2\pi]$. Write the piecewise definition of $f(x)$ and find the range of the function.

Sol. We have $f(x) = \max.\{\sin x, \cos 2x\}$

We first draw the graphs of $y = \sin x$ and $y = \cos 2x$



To find the points of intersection,

$\sin x = \cos 2x$

$\Rightarrow \quad \sin x = 1 - 2\sin^2 x$

$\Rightarrow \quad 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow \quad (2\sin x - 1)(\sin x + 1) = 0$

$\Rightarrow \quad \sin x = 1/2$ or $\sin x = -1$

$\Rightarrow \quad x = \pi/6, 5\pi/6$ or $x = 3\pi/2$

From the graph,

$$f(x) = \begin{cases} \cos 2x, & 0 \le x < \dfrac{\pi}{6} \\ \sin x, & \dfrac{\pi}{6} \le x < \dfrac{5\pi}{6} \\ \cos 2x, & \dfrac{5\pi}{6} \le x \le 2\pi \end{cases}$$

Clearly, range of the function is $[-1, 1]$.

CONCEPT APPLICATION EXERCISE 1.9

In the questions 1 to 10, $[x]$ and $\{x\}$ represent the greatest integer function and the fractional part function, respectively.

1. Solve $[x]^2 - 5[x] + 6 = 0$.
2. If $y = 3[x] + 1 = 4[x - 1] - 10$, then find the value of $[x + 2y]$.
3. Find the domain of

   (a) $f(x) = \dfrac{1}{\sqrt{x - [x]}}$ **(b)** $f(x) = \dfrac{1}{\log[x]}$ **(c)** $f(x) = \log\{x\}$
4. Find the domain of $f(x) = \dfrac{1}{\sqrt{|[|x|-1]|-5}}$.
5. Find the domain of $f(x) = \dfrac{\sqrt{(1-\sin x)}}{\log_5(1-4x^2)} + \cos^{-1}(1 - \{x\})$.
6. Find the range of $f(x) = \cos(\log_e\{x\})$.
7. Find the domain and range of $f(x) = \cos^{-1}\sqrt{\log_{[x]}\left(\dfrac{|x|}{x}\right)}$.
8. Find the range of $f(x) = \log_{[x-1]} \sin x$.
9. Solve $(x - 2)[x] = \{x\} - 1$.
10. Find the domain of the function $f(x) = \dfrac{1}{8\{x\}^2 - 6\{x\} + 1}$.
11. Write the equivalent definition and draw the graphs of the following functions.

    (a) $f(x) = \text{sgn}(\log_e|x|)$

    (b) $f(x) = \text{sgn}(x^3 - x)$
12. Consider the function $f(x) = \max.\{1, |x - 1|, \min\{4, |3x - 1|\}\}\ \forall x \in R$ Then find the value of $f(3)$.
13. Find the equivalent definition of $f(x) = \max.\{x^2, (1 - x)^2, 2x(1 - x)\}$ where $0 \le x \le 1$

ANSWERS

1. $[2, 4)$ **2.** 107
3. **(a)** $R - Z$ **(b)** $[2, \infty)$ **(c)** $R - Z$
4. $(-\infty, -7] \cup [7, \infty)$ **5.** $(-1/2, 1/2) - \{0\}$
6. $[-1, 1]$
7. Domain: $[2, \infty)$, Range: $\{\pi/2\}$ **8.** $(-\infty, 0]$
9. $[1, 2)$ **10.** $R - \{x : x = n + 1/4, n + 1/2, n \in Z\}$

11. (a) $f(x)=\begin{cases}1, & |x|>1\\ 0, & |x|=1\\ -1, & 0<|x|<1\end{cases}$

(b) $f(x)=\begin{cases}-1, & x<-1,\ 0<x<1\\ 1, & -1<x<0,\ x>1\\ 0, & x=-1,\ 0,\ 1\end{cases}$

12. 4

13. $f(x)=\begin{cases}(1-x)^2, & 0\le x<1/3\\ 2x(1-x), & 1/3\le x<2/3\\ x^2, & 2/3\le x\le 1\end{cases}$

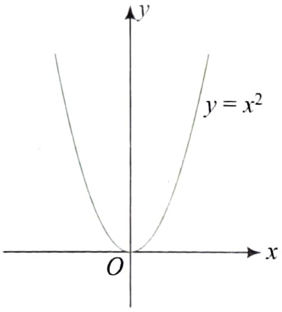
EVEN AND ODD FUNCTIONS

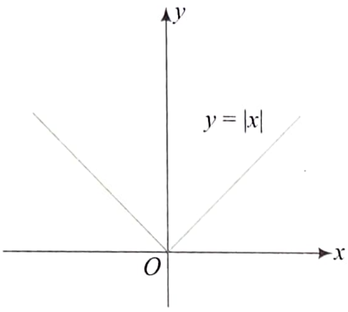
EVEN FUNCTION

A function $y = f(x)$ is said to be an even function if $f(-x) = f(x)\ \forall\ x \in D_f$, where D_f is domain of the function.

Graph of an even function $y = f(x)$ is symmetrical about the y-axis, i.e., if point (x, y) lies on the graph then $(-x, y)$ also lies on the graph.

Some examples of even function are x^2, $|x|$, $\cos x$.





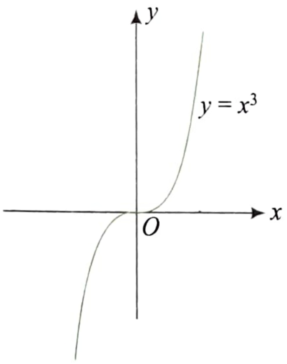
ODD FUNCTION

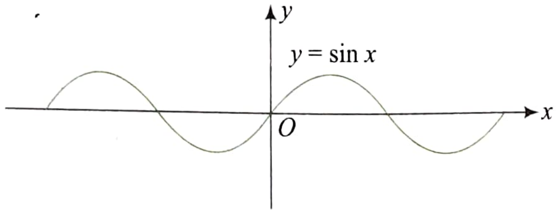
A function $y = f(x)$ is said to be an odd function if

$f(-x) = -f(x)\ \forall\ x \in D_f$.

Graph of an odd function $y = f(x)$ is symmetrical in opposite quadrants, i.e., if point (x, y) lies on the graph then $(-x, -y)$ also lies on the graph.

Some examples of odd function are x, x^3, $\sin x$.





Note:

- A function can be even or odd or neither.
- $f(x) = 0$ is both odd and even function.
- It is possible to express a function (not necessarily odd or even) as a sum of even and odd function.

  i.e., $f(x)=\underbrace{\left(\dfrac{f(x)+f(-x)}{2}\right)}_{even}+\underbrace{\left(\dfrac{f(x)-f(-x)}{2}\right)}_{odd}$.
- The first derivative of an even function is an odd function and vice versa.
- If odd function $f(x)$ is continuous at $x = 0$ then $f(0) = 0$.
- If even function $f(x)$ is continuous and differentiable at $x = 0$ then $\left(\dfrac{df(x)}{dx}\right)_{x=0}=0$.
- Every even function is many-one.
- Algebraic operations and composition of odd and even functions.

$f(x)$	Even	Even	Odd	Odd
$g(x)$	Even	Odd	Even	Odd
$f(x) \pm g(x)$	Even	Neither odd nor even	Neither odd nor even	Odd
$f(x) \times g(x)$	Even	Odd	Odd	Even
$f(x)/g(x)$	Even	Odd	Odd	Even
$f(g(x))$	Even	Even	Even	Odd

ILLUSTRATION 1.125

Which of the following functions is (are) even, odd, or neither?

(i) $f(x) = x^2 \sin x$ **(ii)** $f(x)=\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$

(iii) $f(x)=\log\left(\dfrac{1-x}{1+x}\right)$ **(iv)** $f(x)=\log\left(x+\sqrt{1+x^2}\right)$

(v) $f(x) = \sin x - \cos x$ **(vi)** $f(x)=\dfrac{e^x+e^{-x}}{2}$

Sol.

(i) $f(-x) = (-x)^2 \sin(-x) = -x^2 \sin x = -f(x)$

Hence, $f(x)$ is odd.

(ii) $f(-x)=\sqrt{1+(-x)+(-x)^2}-\sqrt{1-(-x)+(-x)^2}$

$=\sqrt{1-x+x^2}-\sqrt{1+x+x^2}$

$=-f(x)$

Hence, $f(x)$ is odd.

(iii) $f(-x)=\log\left\{\dfrac{1-(-x)}{1+(-x)}\right\}$

$=\log\left(\dfrac{1+x}{1-x}\right)$

$=-f(x)$.

Hence, $f(x)$ is odd.

(iv) $f(-x) = \log\left(-x+\sqrt{1+(-x)^2}\right)$

$$= \log\left\{\frac{(-x+\sqrt{1+x^2})(x+\sqrt{1+x^2})}{(x+\sqrt{1+x^2})}\right\}$$

$$= \log\left(\frac{1}{x+\sqrt{1+x^2}}\right) = -f(x)$$

Hence, $f(x)$ is odd.

(v) $f(-x) = \sin(-x) - \cos(-x) = -\sin x - \cos x$

Clearly, $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$.

Hence, $f(x)$ is neither even nor odd.

(vi) $f(-x) = \dfrac{e^{-x}+e^{-(-x)}}{2} = \dfrac{e^{-x}+e^{x}}{2} = f(x)$

Hence, $f(x)$ is even.

ILLUSTRATION 1.126

If $f(x) = (h_1(x) - h_1(-x))\,(h_2(x) - h_2(-x))\cdots(h_{2n+1}(x) - h_{2n+1}(-x))$ and $f(200) = 0$, then prove that $f(x)$ is a many-one function.

Sol. $f(-x) = (h_1(-x) - h_1(x))\,(h_2(-x) - h_2(x))\cdots(h_{2n+1}(-x) - h_{2n+1}(x))$

$\therefore$ $f(-x) = (-1)^{2n+1} f(x) = -f(x)$

or $f(x) + f(-x) = 0$

So. $f(x)$ is odd. Therefore,

$f(-200) = -f(200) = 0$

So. $f(x)$ is many-one.

ILLUSTRATION 1.127

Check whether the function $f(x) = \dfrac{x(\sin x + \tan x)}{\left[\dfrac{x+\pi}{\pi}\right] - \dfrac{1}{2}}$, where [.] denotes greatest integer function is even or odd function.

Sol. We have

$$f(x) = \frac{x(\sin x+\tan x)}{\left[\frac{x+\pi}{\pi}\right]-\frac{1}{2}} = \frac{x(\sin x+\tan x)}{\left[\frac{x}{\pi}\right]+1-\frac{1}{2}} = \frac{x(\sin x+\tan x)}{\left[\frac{x}{\pi}\right]+0.5}$$

$$\Rightarrow \quad f(-x) = \frac{-x(\sin(-x)+\tan(-x))}{\left[-\frac{x}{\pi}\right]+0.5}$$

$$= \begin{cases} \dfrac{x(\sin x+\tan x)}{-1-\left[\frac{x}{\pi}\right]+0.5}, & x \neq n\pi,\ n \in Z \\ 0, & x = n\pi,\ n \in Z \end{cases}$$

$$= \begin{cases} -\dfrac{x(\sin x+\tan x)}{\left[\frac{x}{\pi}\right]+0.5}, & x \neq n\pi \\ 0, & x = n\pi \end{cases}$$

Thus $f(-x) = -f(x)$

Hence, $f(x)$ is an odd function.

EVEN AND ODD EXTENSION OF FUNCTION

Let a function be defined on certain domain which is entirely nonnegative (or nonpositive). The domain of $f(x)$ can be extended to the set $X = \{-x : x \in D_f\}$, where D_f is domain of function in two ways.

Even extension:

If a function $f(x)$ is defined only on the interval $(0, L)$ then its even extension is defined as

$$f_{even} = \begin{cases} f(x), & x \in (0, L) \\ f(-x), & x \in (-L, 0) \end{cases}$$

Odd extension:

If a function $f(x)$ is defined only on the interval $(0, L)$ then its odd extension is defined as

$$f_{odd} = \begin{cases} f(x), & x \in (0, L) \\ -f(-x), & x \in (-L, 0) \end{cases}$$

ILLUSTRATION 1.128

If $f(x) = \begin{cases} x^3 + x^2, & 0 \le x \le 2 \\ x+2, & 2 < x \le 4 \end{cases}$. Then find the even and odd extension of $f(x)$.

Sol. For even extension $f(x) = f(-x)$

$$\therefore \quad f(x) = f(-x) = \begin{cases} (-x)^3 + (-x)^2, & 0 \le -x \le 2 \\ -x+2, & 2 < -x \le 4 \end{cases}$$

$$= \begin{cases} -x+2, & -4 \le x < -2 \\ -x^3 + x^2, & -2 \le x \le 0 \end{cases}$$

For odd extension, $f(x) = -f(-x)$

$$\therefore \quad f(x) = -f(-x) = \begin{cases} x-2, & -4 \le x < -2 \\ x^3 - x^2, & -2 \le x \le 0 \end{cases}.$$

CONCEPT APPLICATION EXERCISE 1.10

Identify the following functions whether odd or even or neither:

1. $f(x) = \{g(x) - g(-x)\}^3$
2. $f(x) = \log\left(\dfrac{x^4+x^2+1}{x^2+x+1}\right)$
3. $f(x) = xg(x)g(-x) + \tan(\sin x)$
4. $f(x) = \cos|x| + \left[\left|\dfrac{\sin x}{2}\right|\right]$

   where [.] denotes the greatest integer function.
5. $f(x) = \begin{cases} x|x|, & x \le -1 \\ [x+1]+[1-x], & -1 < x < 1 \\ -x|x|, & x \ge 1 \end{cases}$

   where [ ] represents the greatest integer function.
6. Let the function $f(x) = x^2 + x + \sin x - \cos x + \log(1+|x|)$ be defined on the interval [0, 1]. Define functions $g(x)$ and $h(x)$ in [–1, 0] satisfying $g(-x) = -f(x)$ and $h(-x) = f(x)\ \forall\ x \in [0, 1]$.

ANSWERS

1. odd 2. even 3. odd 4. even
5. even 6. $g(x) = -x^2 + x + \sin x + \cos x - \log(1+|x|)$

   $h(x) = x^2 - x - \sin x - \cos x + \log(1+|x|)$

PERIODIC FUNCTIONS

A function $f: X \to Y$ is said to be a periodic function if there exists a positive real number T such that $f(x+T) = f(x)$, for all $x \in X$. The least of all such positive numbers T is called the principal period or simply period of f. All periodic functions can be analysed over an interval of one period within the domain as the same pattern shall be repetitive over the entire domain.

In other words, a function is said to be periodic if its each value is repeated after a definite interval. Thus, the graph of the periodic function repeats after an interval of the length equal to its period.

Clearly $f(x) = f(x+T) = f(x+2T) = f(x+3T) = \ldots$

PERIOD OF SOME ELEMENTARY FUNCTIONS

Function	Period
$\sin x$, $\cos x$, $\sec x$, $\text{cosec } x$	2π
$\tan x$, $\cot x$	π
$\{x\}$, where $\{.\}$ represents fractional part.	1
$\sin^n x$, $\cos^n x$, $\sec^n x$, $\text{cosec}^n x$, $n \in N$	π if n is even 2π if n is odd
$\tan^n x$, $\cot^n x$, $n \in N$	π for all $n \in N$
$\|\sin x\|$, $\|\cos x\|$, $\|\sec x\|$, $\|\text{cosec } x\|$, $\|\tan x\|$, $\|\cot x\|$	π
Constant function	Function is periodic but has no fixed period.

ILLUSTRATION 1.129

Prove that period of function $f(x) = \sin x$, $x \in R$ is 2π.

Sol. Let period of $f(x) = \sin x$ be T

$\therefore$ $f(x+T) = f(x)$ for all real x.

or $\sin(x+T) = \sin x$ for all real x.

$\therefore$ $\sin(0+T) = \sin 0$ (putting $x = 0$)

$\therefore$ $T = n\pi$, $n \in Z$.

The least value of T is π.

But for $T = \pi$, $\sin(x+\pi) = -\sin x \neq f(x)$

So, let $T = 2\pi$ for which $\sin(x+2\pi) = \sin x$.

Thus, period of $\sin x$ is 2π.

ILLUSTRATION 1.130

Verify that the period of function $f(x) = \sin^{10}x$ is π.

Sol. We have $f(x) = \sin^{10}x$

Now $f(x+\pi) = \sin^{10}(x+\pi)$

$= (\sin(x+\pi))^{10}$

$= (-\sin x)^{10}$

$= \sin^{10}x$

$= f(x)$

Thus, we have $f(x+\pi) = f(x)$ for all $x \in R$.

So, period of $f(x)$ is π.

ILLUSTRATION 1.131

Prove that function $f(x) = \cos\sqrt{x}$ is non-periodic.

Sol. We have $f(x) = \cos\sqrt{x}$

Let $f(x)$ be periodic with period T, where $T > 0$.

$\therefore$ $f(x+T) = f(x)$

$\Rightarrow$ $\cos\sqrt{x+T} = \cos\sqrt{x}$ for $x \geq 0$.

In particular choosing $x = 0$, we have

$\cos\sqrt{T} = \cos\sqrt{0} = 1$ ...(1)

For $x = T$, we have

$\cos\sqrt{T+T} = \cos\sqrt{T} = 1$

or $\cos\sqrt{2T} = 1$ ...(2)

From (1), $\sqrt{T} = 2m\pi$, $m \in Z$

From (2), $\sqrt{2T} = 2n\pi$, $n \in Z$

$\therefore$ $\dfrac{\sqrt{2T}}{\sqrt{T}} = \dfrac{2n\pi}{2m\pi}$

or $\sqrt{2} = \dfrac{n}{m}$, which is not true.

So, $\cos\sqrt{x}$ is not periodic.

PERIOD OF TRANSFORMED FUNCTIONS

- If $f(x)$ is periodic with period T, then transformed functions $af(x)$, $f(x) \pm a$, $f(x \pm a)$, $a \in R$ are also periodic with the same period T.

  Proof:

  Let $g(x) = af(x)$, where $f(x)$ has period T.

  $\therefore$ $g(x+T) = af(x+T) = af(x) = g(x)$ ($\because$ $f(x+T) = f(x)$)

  Thus $g(x)$ is periodic with period T.

  Similarly, we can prove that functions $f(x) \pm a$, $f(x \pm a)$ have same period T.

  For example, period of $3\sin x$ is 2π, period of $\cot(x+4)$ is π, period of $\{x+2.3\}$ is 1. (where $\{\cdot\}$ represents fractional part function)

- If $f(x)$ is periodic with period T, then function $f(ax)$, where $a \in R - \{0\}$ is also periodic with period $\dfrac{T}{|a|}$.

  Proof:

  Consider $a > 0$

  We have $f(x+T) = f(x)$.

  Let period of $f(ax)$ be T'. Then

  $f(a(x+T')) = f(ax)$

  $\Rightarrow$ $f(ax + aT') = f(ax)$

  $\Rightarrow$ $f(y + aT') = f(y) = f(y+T)$

  $\Rightarrow$ $T = aT'$

  $\Rightarrow$ $T' = T/a$ ($\because$ period is always positive)

  Similarly, if $a < 0$ then $T' = -T/a$

  In general if period of $f(x)$ is T then period of $f(ax+b) + c$ is $T/|a|$.

  For example, period of $\sin 2x$ is $\dfrac{2\pi}{2} = \pi$

  Period of $\tan(3x+5)$ is $\dfrac{\pi}{3}$

  Period of $\{x/2\}$, where $\{.\}$ represents fractional part function is $\dfrac{1}{1/2} = 2$.

ILLUSTRATION 1.132

Find the period of the following functions

(i) $f(x) = |\sin 3x|$

(ii) $f(x) = 2\operatorname{cosec}(5x - 6) + 7$

(iii) $f(x) = x - [x - 2.6]$, where [.] represents the greatest integer function.

Sol.

(i) Period of $|\sin x|$ is π.

So, the period of $|\sin 3x|$ is $\frac{\pi}{3}$

(ii) Period of $2\operatorname{cosec}(5x - 6) + 7$ is $\frac{2\pi}{5}$

(iii) $f(x) = x - [x - 2.6]$

$= (x - 2.6) - [x - 2.6] + 2.6$

$= \{x - 2.6\} + 2.6$, where {.} represents fractional part function.

$\therefore$ Period of $f(x)$ is '1'.

ILLUSTRATION 1.133

Find the period of the function

$$f(x) = 4\cos^4\left(\frac{x-\pi}{4\pi^2}\right) - 2\cos\left(\frac{x-\pi}{2\pi^2}\right).$$

Sol. $f(x) = 4\cos^4\left(\frac{x-\pi}{4\pi^2}\right) - 2\cos\left(\frac{x-\pi}{2\pi^2}\right)$

$$= \left(2\cos^2\left(\frac{x-\pi}{4\pi^2}\right)\right)^2 - 2\cos\left(\frac{x-\pi}{2\pi^2}\right)$$

$$= \left(1 + \cos\left(\frac{x-\pi}{2\pi^2}\right)\right)^2 - 2\cos\left(\frac{x-\pi}{2\pi^2}\right)$$

(using $2\cos^2\theta = 1 + \cos 2\theta$)

$$= \cos^2\left(\frac{x-\pi}{2\pi^2}\right) + 1$$

$$= \frac{1 + \cos\left(\frac{x-\pi}{\pi^2}\right)}{2} + 1$$

$$= \frac{\cos\left(\frac{x}{\pi^2} - \frac{1}{\pi}\right) + 3}{2}$$

$\therefore$ Period of $f(x)$ is $\frac{2\pi}{\frac{1}{\pi^2}} = 2\pi^3$

PERIOD OF COMPOSITE FUNCTIONS

If $f(x)$ has period T then period of composite function $g(f(x))$ is T if there does not exist any positive number k which is less than T for which $g(f(x + k)) = g(f(x))$ holds for all x in the domain.

i.e., period of $f(x) = \sin(\cos x)$ is 2π as

$f(x + 2\pi) = \sin(\cos(x + 2\pi))$

$= \sin(\cos x) = f(x)$.

But period of $f(x) = \cos(\cos x)$ is π and not 2π as

$f(x + \pi) = \cos(\cos(x + \pi))$

$= \cos(-\cos x) = \cos(\cos x)) = f(x)$.

Note:

- Period of each of the functions $\sin^{-1}(\sin x)$, $\cos^{-1}(\cos x)$, $\sec^{-1}(\sec x)$, $\operatorname{cosec}^{-1}(\operatorname{cosec} x)$ is 2π.
- Period of each of the functions $\tan^{-1}(\tan x)$, $\cot^{-1}(\cot x)$ is π

ILLUSTRATION 1.134

Find the period of the following.

(i) $f(x) = \frac{2^x}{2^{[x]}}$, where [.] represents the greatest integer function.

(ii) $f(x) = e^{\sin x}$

(iii) $f(x) = \sin^{-1}(\sin 3x)$

(iv) $f(x) = \sqrt{\sin x}$

(v) $f(x) = \tan\left(\frac{\pi}{2}[x]\right)$, where [.] represents greatest integer function.

Sol.

(i) $f(x) = \frac{2^x}{2^{[x]}} = 2^{x-[x]} = 2^{\{x\}}$, where {.} is fractional part function.

Period of $\{x\}$ is '1'. So, period of $2^{\{x\}}$ is 1 as $2^{\{x+1\}} = 2^{\{x\}}$ for all real x.

(ii) $f(x) = e^{\sin x}$.

Period of $\sin x$ is 2π.

So, period of $e^{\sin x}$ is also 2π as $e^{\sin(x+2\pi)} = e^{\sin x}$.

(iii) $f(x) = \sin^{-1}(\sin 3x)$

We know that period of $\sin^{-1}(\sin x)$ is 2π.

So, period of $\sin^{-1}(\sin 3x)$ is $2\pi/3$.

(iv) $f(x) = \sqrt{\sin x}$

Period of sin x is 2π.

So, period of $\sqrt{\sin x}$ is also 2π as $\sqrt{\sin(x + 2\pi)} = \sqrt{\sin x}$.

(v) $f(x) = \tan\left(\frac{\pi}{2}[x]\right)$, where [.] represents greatest integer function.

Let the period be T.

$\therefore\ f(x + T) = f(x)$

$\Rightarrow \tan\left(\frac{\pi}{2}[x+T]\right) = \tan\left(\frac{\pi}{2}[x]\right)$

Putting $x = 0$,

$$\tan\left(\frac{\pi}{2}[T]\right) = 0$$

$\therefore\ \frac{\pi}{2}[T] = n\pi,\ n \in Z.$

$\Rightarrow [T] = 2n,\ n \in Z$

$\Rightarrow T = 2$ (least positive value)

Also, $\tan\left(\frac{\pi}{2}[x+2]\right) = \tan\left(\frac{\pi}{2}(2+[x])\right)$

$= \tan\left(\pi + \frac{\pi}{2}[x]\right)$

$= \tan\left(\frac{\pi}{2}[x]\right)$

ILLUSTRATION 1.135

Find the period of $f(x) = \sin(x + \cos x)$.

Sol. $f(x) = \sin(x + \cos x)$

Let period be T.

$\therefore$ $f(x + T) = f(x)$ for all real x.

Putting $x = 0$, we have

$f(T) = f(0)$

or $\sin(T + \cos T) = \sin 1$

By intelligent guessing, $T = 2\pi$, which is the least positive value.

Also, $f(x + 2\pi) = \sin(x + 2\pi + \cos(x + 2\pi))$

$= \sin(2\pi + (x + \cos x))$

$= \sin(x + \cos x) = f(x)$

PERIOD OF FUNCTIONS RESULTING FROM ALGEBRAIC OPERATIONS OF FUNCTIONS

Let $f(x)$ have period $p = m/n$ ($m, n \in N$ and coprime) and $g(x)$ has period $q = r/s$ ($r, s \in N$ and coprime) and let t be the L.C.M. of p and q,

$$t = \frac{\text{L.C.M. of } (m, r)}{\text{H.C.F. of } (n, s)}$$

Then t shall be the period of $f + g$ provided there does not exist a positive number k ($< t$) for which $f(x + k) + g(x + k) = f(x) + g(x)$, else k will be the period. The same rule is applicable for function resulting from any other algebraic operation.

It should be noted that L.C.M. of p and q exist if p and q are rational quantities. If p and q are irrational then L.C.M. of p and q does not exist unless they have same irrational surd. L.C.M. of rational and irrational does not exist. Thus, addition of two periodic functions is not necessarily periodic. But addition of periodic and non-periodic is always non-periodic, i.e., $(x + \sin x)$, $\{x\} + x^2$ are non-periodic functions.

(where $\{.\}$ represents fractional part function.)

ILLUSTRATION 1.136

In each of the following cases find the period of the function if it is periodic.

(i) $f(x) = \sin\frac{\pi x}{\sqrt{2}} + \cos\frac{\pi x}{\sqrt{3}}$ (ii) $f(x) = \sin\frac{\pi x}{\sqrt{3}} + \cos\frac{\pi x}{2\sqrt{3}}$

Sol.

(i) Period of $\sin\frac{\pi x}{\sqrt{2}} = \frac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}$

Period of $\cos\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

Now, L.C.M. of two different kinds of irrational number does not exist.

Therefore, $f(x)$ is not periodic.

(ii) Period of $\sin\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

Period of $\cos\frac{\pi x}{2\sqrt{3}} = \frac{2\pi}{\pi/2\sqrt{3}} = 4\sqrt{3}$

Now, L.C.M. of $(2\sqrt{3}, 4\sqrt{3})$

$= \sqrt{3} \times$ L.C.M. of $(2, 4) = 4\sqrt{3}$

ILLUSTRATION 1.137

Find the period of

(i) $f(x) = \sin \pi x + \{x/3\}$, where $\{.\}$ represents the fractional part.

(ii) $f(x) = |\sin 7x| - \cos^4\frac{3x}{4} + \tan\frac{2x}{3}$

Sol.

(i) $f(x) = \sin \pi x + \{x/3\}$, where $\{.\}$ represents the fractional part

Period of $\sin \pi x$ is $\frac{2\pi}{\pi} = 2$

Period of $\{x/3\}$ is $\frac{1}{1/3} = 3$

Therefore, period of $f(x)$ is L.C.M. of $(2, 3) = 6$

(ii) $f(x) = |\sin 7x| - \cos^4\frac{3x}{4} + \tan\frac{2x}{3}$

Period of $|\sin 7x|$ is $\frac{\pi}{7}$

Period of $\cos^4\frac{3x}{4}$ is $\frac{\pi}{3/4} = \frac{4\pi}{3}$

Period of $\tan\frac{2x}{3}$ is $\frac{\pi}{2/3} = \frac{3\pi}{2}$

Therefore, period of $f(x)$ is L.C.M. of

$$\left(\frac{\pi}{7}, \frac{4\pi}{3}, \frac{3\pi}{2}\right) = \pi \times \frac{L.C.M.\ of\ (1, 4, 3)}{H.C.F.\ of\ (7, 3, 2)}$$

$= 12\pi$

ILLUSTRATION 1.138

Find the period of $f(x) = \sin x + \{x\}$, where $\{x\}$ is the fractional part of x if it is periodic.

Sol. Here, $\sin x$ is periodic with period 2π and $\{x\}$ is periodic with 1. The LCM of 2π (irrational) and 1 (rational) does not exist.

Thus, $f(x)$ is not periodic.

ILLUSTRATION 1.139

If $f(x) = \sin x + \cos ax$ is a periodic function, show that a is a rational number.

Sol. Period of $\sin x = 2\pi = \frac{2\pi}{1}$ and period of $\cos ax = \frac{2\pi}{|a|}$

$\therefore$ Period of $\sin x + \cos ax$ = LCM of $\frac{2\pi}{1}$ and $\frac{2\pi}{|a|}$

$= \frac{\text{LCM of } 2\pi \text{ and } 2\pi}{\text{HCF of 1 and } a}$

$= \frac{2\pi}{\lambda}$

where λ is the HCF of 1 and a.

Since λ is the HCF of 1 and a, $\frac{1}{\lambda}$ and $\frac{|a|}{\lambda}$ should both be integers.

Suppose $\frac{1}{\lambda} = p$ and $\frac{|a|}{\lambda} = q$.

Then, $\frac{\frac{|a|}{\lambda}}{\frac{1}{\lambda}} = \frac{q}{p}$, where $p, q \in Z$

i.e., $|a| = \frac{q}{p}$

Hence, a is the rational number.

ILLUSTRATION 1.140

Find the period of the following functions

(i) $f(x) = |\sin x| + |\cos x|$

(ii) $f(x) = \cos(\cos x) + \cos(\sin x)$

(iii) $f(x) = \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

Sol.

(i) $f(x) = |\sin x| + |\cos x|$

Period of both $|\sin x|$ and $|\cos x|$ is π.

So, we can consider period of $f(x)$ to be π.

But $f\left(\frac{\pi}{2}+x\right) = \left|\sin\left(\frac{\pi}{2}+x\right)\right| + \left|\cos\left(\frac{\pi}{2}+x\right)\right|$

$= |\cos x| + |-\sin x|$

$= |\cos x| + |\sin x|$

$= f(x)$

So, period of $f(x)$ is $\pi/2$.

(ii) $f(x) = \cos(\cos x) + \cos(\sin x)$

Period of $\cos(\cos x)$ is π.

Period of $\cos(\sin x)$ is π.

So, we can consider period of $f(x)$ to be π.

But $f\left(\frac{\pi}{2}+x\right) = \cos\left(\cos\left(\frac{\pi}{2}+x\right)\right) + \cos\left(\sin\left(\frac{\pi}{2}+x\right)\right)$

$= \cos(-\sin x) + \cos(\cos x)$

$= \cos(\sin x) + \cos(\cos x)$

$= f(x)$

So, period of $f(x)$ is $\pi/2$

(iii) $f(x) = \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

$|\sin x + \cos x| = \sqrt{2}\left|\sin\left(x + \frac{\pi}{4}\right)\right|$, which is has period π.

Period of $|\sin x| + |\cos x|$ is $\pi/2$.

Therefore, period of $f(x)$ is L.C.M. of $(\pi, \pi/2) = \pi$

ILLUSTRATION 1.141

For what integral value of n is 3π the period of the function $\cos(nx)\sin\left(\frac{5x}{n}\right)$?

Sol. Let $f(x) = \cos nx \sin\left(\frac{5x}{n}\right)$. $f(x)$ be periodic. Then,

$f(x + \lambda) = f(x)$, where λ is period

or $\cos(nx + n\lambda)\sin\left(\frac{5x + 5\lambda}{n}\right) = \cos nx \sin\left(\frac{5x}{n}\right)$

At $x = 0$,

$\cos n\lambda \sin\left(\frac{5\lambda}{n}\right) = 0$

If $\cos n\lambda = 0$, then

$n\lambda = r\pi + \frac{\pi}{2}, r \in I$

or $n(3\pi) = r\pi + \frac{\pi}{2}$ $(\because \lambda = 3\pi)$

or $3n - r = \frac{1}{2}$ (not possible)

Now, let $\sin\left(\frac{5\lambda}{n}\right) = 0$. Then

$\frac{5\lambda}{n} = p\pi \quad (p \in I)$

or $\frac{5(3\pi)}{n} = p\pi$ $(\because \lambda = 3\pi)$

or $n = \frac{15}{p}$

For $p = \pm1, \pm3, \pm5, \pm15$, we have, respectively,

$n = \pm15, \pm5, \pm3, \pm1$ $(\because n \in I)$

CONCEPT APPLICATION EXERCISE 1.11

1. Which of the following functions is not periodic?
   (a) $|\sin 3x| + \sin^2 x$ **(b)** $\cos\sqrt{x} + \cos^2 x$
   (c) $\cos 4x + \tan^2 x$ **(d)** $\cos 2x + \sin x$
2. Which of the following function/functions is/are periodic?
   (a) $\text{sgn}(e^{-x})$ **(b)** $\sin x + |\sin x|$
   (c) $\min(\sin x, |x|)$ **(d)** $\frac{x}{x}$
3. Find the period of
   (a) $\frac{|\sin 4x| + |\cos 4x|}{|\sin 4x - \cos 4x| + |\sin 4x + \cos 4x|}$
   (b) $f(x) = \sin\frac{\pi x}{n!} - \cos\frac{\pi x}{(n+1)!}$

(c) $f(x) = \sin x + \tan \frac{x}{2} + \sin \frac{x}{2^2} + \tan \frac{x}{2^3} + \cdots$

$+ \sin \frac{x}{2^{n-1}} + \tan \frac{x}{2^n}$

4. Match the column

Column I (Function)	Column II (Period)
p. $f(x) = \sin^3 x + \cos^4 x$	a. $\pi/2$
q. $f(x) = \cos^4 x + \sin^4 x$	b. π
r. $f(x) = \sin^3 x + \cos^3 x$	c. 2π
s. $f(x) = \cos^4 x - \sin^4 x$	

5. Let $[x]$ denotes the greatest integer less than or equal to x. If the function $f(x) = \tan(\sqrt{[n]}\, x)$ has period $\frac{\pi}{3}$, then find the values of n.

6. If $f(x) = \lambda|\sin x| + \lambda^2 |\cos x| + g(\lambda)$ has period equal to $\pi/2$, then find the value of λ.

7. Find the fundamental period of $f(x) = \cos x \cos 2x \cos 3x$.

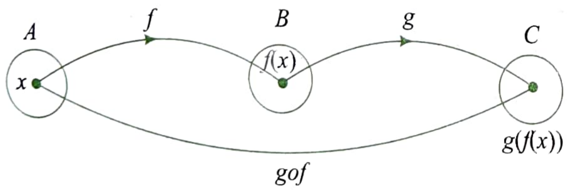
ANSWERS

1. (b) 2. (a),(b),(c)
3. (a) $\pi/8$ (b) $2(n + 1)!$ (c) $2^n\pi$
4. $p \to c$; $q \to a$; $r \to c$; $s \to b$
5. [9, 10) 6. 1 7. π

COMPOSITE FUNCTIONS

Let A, B and C be three non-empty sets.

Let $f: A \to B$ and $g: B \to C$ be two functions then $gof: A \to C$. This function is called composition of f and g, given by $gof(x) = g(f(x))\ \forall x \in A$.



Thus, the image of every $x \in A$ under the function gof is the g-image of the f-image of x.

The gof is defined only if $\forall x \in A$, $f(x)$ is an element of the domain of g so that we can take its g-image.

The range of f must be a subset of or equal to the domain of g in gof.

Note:

- The composition of function is not commutative in general, i.e., $fog \neq gof$.
- The composition of functions is associative. So, if $h: A \to B$, $g: B \to C$ and $f: C \to D$ be three functions, then $(fog)oh = fo(goh)$
- The composition of any function with the identity function is the function itself. So, if $f: A \to B$ then $foI_A = I_B of = f$ where I_A and I_B are the identity functions of domain A and B respectively.

INJECTIVE AND SURJECTIVE NATURE OF COMPOSITE FUNCTION

(i) If $f: A \to B$ and $g: B \to C$ are one-one, then $gof: A \to C$ is also one-one.

Proof:

Suppose $gof(x_1) = gof(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$, as g is one-one

$\Rightarrow x_1 = x_2$, as f is one-one

Hence, gof is one-one.

(ii) If $f: A \to B$ and $g: B \to C$ are onto, then $gof: A \to C$ is also onto.

Proof:

Since g is onto function, given an arbitrary element $z \in C$, there exists a pre-image y of z under g such that $g(y) = z$.

Further, for $y \in B$, there exists an element x in A with $f(x) = y$, since f is onto.

Therefore, $gof(x) = g(f(x)) = g(y) = z$, showing that gof is onto.

(iii) If $f: A \to B$, $g: B \to C$ and $gof(x)$ is one-one then $f(x)$ is necessarily one-one but $g(x)$ may not be one-one.

Proof:

Given that $gof(x)$ is one-one.

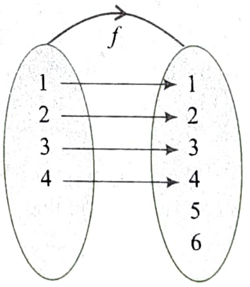
$\therefore$ for $x_1 \neq x_2$; where $x_1, x_2 \in A$, $g(f(x_1)) \neq g(f(x_2))$

$\Rightarrow f(x_1) \neq f(x_2)$ (from the definition of function)

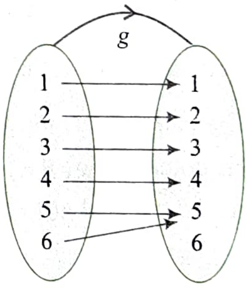
Thus $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Therefore, $f(x)$ is one-one.

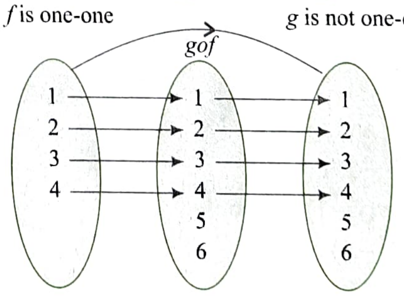
To prove that $g(x)$ may not be one-one, consider following particular example.



f is one-one



g is not one-one



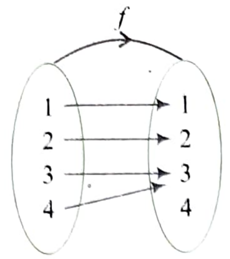
gof is one-one
But g is not one-one

(iv) If $f: A \to B$, $g: B \to C$ and $gof(x)$ is onto then $g(x)$ is necessarily onto but $f(x)$ may not be onto.

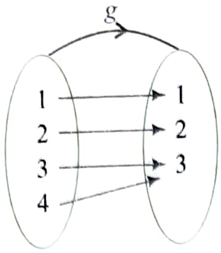
Proof:

Since $gof(x)$ is onto, then for every $c \in C$ there exists $a \in A$ such that $g(f(a)) = c$. Let $b = f(a)$. Then $g(b) = c$ for $b \in B$. Thus, for $c \in C$, there exists $b \in B$. So, $g(x)$ is onto.

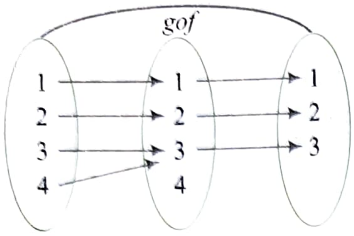
To prove that $f(x)$ may not be onto, consider following particular example.



f is not onto



g is onto



gof is onto but f is not onto.

Note

From the above discussion, we find that if $gof(x)$ is bijective then $f(x)$ is injective (one-one) and $g(x)$ is surjective(onto).

ILLUSTRATION 1.142

Let $f: \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g: \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as $f(2) = 3, f(3) = 4, f(4) = f(5) = 5$, $g(3) = g(4) = 7$, and $g(5) = g(9) = 11$. Find gof.

Sol. We have $gof(2) = g(f(2)) = g(3) = 7$,

$gof(3) = g(f(3)) = g(4) = 7$,

$gof(4) = g(f(4)) = g(5) = 11$

and $gof(5) = g(5) = 11$.

ILLUSTRATION 1.143

Let $f(x)$ and $g(x)$ be bijective functions where $f: \{a, b, c, d\} \to \{1, 2, 3, 4\}$ and $g: \{3, 4, 5, 6\} \to \{w, x, y, z\}$, respectively. Then, find the number of elements in the range set of $g(f(x))$.

Sol. The range of $f(x)$ for which $g(f(x))$ is defined is $\{3, 4\}$. Hence, the domain of $g\{f(x)\}$ has two elements.

Therefore, the range of $g(f(x))$ also has two elements.

ILLUSTRATION 1.144

Suppose that $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$. Then find the function $f(x)$.

Sol. $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$ (1)

$\therefore \quad f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$

Put $1 + \sqrt{x} = y$ or $x = (y - 1)^2$. Then,

$f(y) = 3 + 2(y - 1) + (y - 1)^2 = 2 + y^2$.

$\therefore \quad f(x) = 2 + x^2$

ILLUSTRATION 1.145

The function $f(x)$ is defined in $[0, 1]$. Find the domain of $f(\tan x)$.

Sol. Here, $f(x)$ is defined in $[0, 1]$.

So, $x \in [0, 1]$, i.e., the only values of x that we can substitute lies in $[0, 1]$.

For $f(\tan x)$ to be defined, we must have

$$0 \le \tan x \le 1 \qquad \text{[As } x \text{ is replaced by } \tan x\text{]}$$

i.e., $n\pi \le x \le n\pi + \dfrac{\pi}{4}, n \in Z$ [In general]

Thus, the domain of $f(\tan x)$ is

$$\bigcup_{n \in Z} \left[n\pi, n\pi + \frac{\pi}{4}\right]$$

ILLUSTRATION 1.146

$f(x) = \begin{cases} x+1, & x < 0 \\ x^2, & x \ge 0 \end{cases}$ and $g(x) = \begin{cases} x^3, & x < 1 \\ 2x - 1, & x \ge 1 \end{cases}$. Then find $f(g(x))$ and find its domain and range.

Sol. $f(g(x)) = \begin{cases} g(x)+1, & g(x) < 0 \\ \{g(x)\}^2, & g(x) \ge 0 \end{cases}$

$$= \begin{cases} x^3+1, & x^3 < 0, \ x < 1 \\ 2x-1+1, & 2x-1 < 0, \ x \ge 1 \\ (x^3)^2, & x^3 \ge 0, \ x < 1 \\ (2x-1)^2, & 2x-1 \ge 0, \ x \ge 1 \end{cases}$$

$$= \begin{cases} x^3+1, & x < 0 \\ x^6, & 0 \le x < 1 \\ (2x-1)^2, & x \ge 1 \end{cases}$$

Clearly, domain of function is R.

For $x < 0$, $x^3 + 1 \in (-\infty, 1)$.

For $0 \le x < 1$, $x^6 \in [0, 1)$.

For $x \ge 1$, $(2x - 1)^2 \in [1, \infty)$.

Hence, the range is R.

ILLUSTRATION 1.147

If $f(x) = -1 + |x - 1|, -1 \le x \le 3$ and $g(x) = 2 - |x + 1|$, $-2 \le x \le 2$, then find $fog(x)$ and $gof(x)$.

Sol. $f(g(x)) = -1 + |g(x) - 1|, -1 \le g(x) \le 3, -2 \le x \le 2$

$= -1 + |2 - |x + 1| - 1|, -1 \le 2 - |x + 1| \le 3, -2 \le x \le 2$

$= -1 + |1 - |x + 1||, -1 \le 2 - |x + 1| \le 3, -2 \le x \le 2$

Now $-1 \le 2 - |x + 1| \le 3$

$\Rightarrow -3 \le -|x + 1| \le 1$

$\Rightarrow -1 \le |x + 1| \le 3$

$\Rightarrow 0 \le |x + 1| \le 3$

$\Rightarrow -3 \le x + 1 \le 3$

$\Rightarrow -4 \le x \le 2$

Also $-2 \le x \le 2$

$\therefore \quad fog(x) = -1 + |1 - |x + 1||, -2 \le x \le 2$

$g(f(x)) = 2 - |f(x) + 1|, -2 \le f(x) \le 2, -1 \le x \le 3$

$= 2 - |-1 + |x - 1| + 1|, -2 \le -1 + |x - 1| \le 2, -1 \le x \le 3$

$= 2 - |x - 1|, -2 \le -1 + |x - 1| \le 2, -1 \le x \le 3$

Now $-2 \le -1 + |x - 1| \le 2$

$\Rightarrow -1 \le |x - 1| \le 3$

$\Rightarrow -3 \le x - 1 \le 3$

$\Rightarrow -2 \le x \le 4$

Also $-1 \le x \le 3$

$\therefore \quad g(f(x)) = 2 - |x - 1|, -1 \le x \le 3$

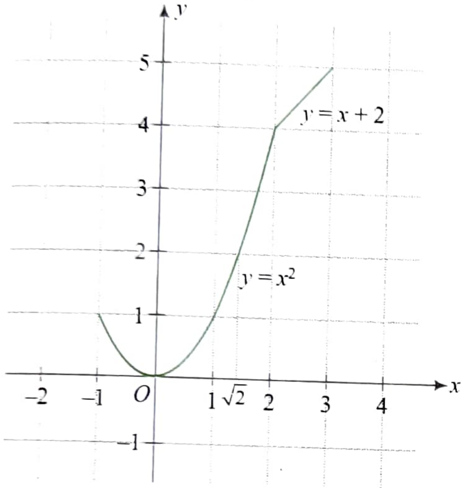
ILLUSTRATION 1.148

Let $f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}$ and $g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}$, find fog($x$).

Sol. We have $f(g(x)) = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$

Here, $g(x)$ is argument of the function $f(g(x))$

So, we draw the graph of $y = g(x)$.



From the graph it is clear that $g(x) \le 1, \forall\, x \in [-1, 1]$

And $\quad 1 < g(x) \le 2, \forall\, x \in (1, \sqrt{2}]$

$$\Rightarrow \quad f(g(x)) = \begin{cases} x^2+1, & -1 \le x \le 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$$

ILLUSTRATION 1.149

Let $f: A \to B$ and $g: C \to D$ be functions for which composite function gof is defined. If each of f and g is onto, then is gof is necessarily onto?

Sol. Let $z \in D$, then there is some $y \in C$ such that $g(y) = z$

$\therefore \quad$ Now if $y \notin B$, then there is no x for which $f(x) = y$

$\therefore \quad$ gof is not onto.

ILLUSTRATION 1.150

Suppose $f: A \to B$ and $g: B \to C$.

(i) Prove that if f is onto and g is not one-one, then gof is not one-to-one

(ii) Prove that if f is not onto and g is one-one, then gof is not onto.

Sol.

(i) Suppose that f is onto and g is not one-one. We can find elements in B to witness the fact that g is not one-to-one, namely we can find $x, y \in B$ with $x \ne y$ and $g(x) = g(y)$. Since f is onto, we can then find $a, b \in A$ with $f(a) = x$ and $f(b) = y$. Since $x \ne y$, the definition of a function tells us that $a \ne b$. Therefore, we have that $(gof)\,(a) = g(f(a)) = g(x) = g(y) = g(f(b)) = (gof)\,(b)$, but $a \ne b$ and so gof is not one-one.

(ii) Suppose that f is not onto and g is one-one. We can find an $x \in B$ which witnesses the fact that f is not onto, i.e., $x \notin$ Range(f). In order to prove that gof is not onto, we now want to show that for $a \in A$, $(gof)\,(a) \ne g(x)$, demonstrating that Range $(gof) \subset C$. Given any $a \in A$, we know that $f(a) \ne x$ since x is not in the range of f. Since g is one-one, $f(a) \ne x$ tells us that $g(f(a)) \ne g(x)$ as well, which is what we were trying to prove.

CONCEPT APPLICATION EXERCISE 1.12

1. Let $f(x) = \begin{cases} 1+|x|, & x < -1 \\ [x], & x \ge -1 \end{cases}$, where [.] denotes the greatest integer function. Then find the value of $f(f(-2.3))$.

2. If $f(x) = \log\left[\dfrac{1+x}{1-x}\right]$, then prove that $f\left[\dfrac{2x}{1+x^2}\right] = 2f(x)$.

3. Let $f(x) = \dfrac{\alpha x}{x+1}, x \ne -1$. Then for what value of α is $f(f(x)) = x$?

4. If the domain of $y = f(x)$ is $[-3, 2]$, then find the domain of $g(x) = f(|[x]|)$, where [ ] denotes the greatest integer function.

5. Let f be a function defined on $[0, 2]$. Then prove that the domain of function $g(x)$ is $f(9x^2 - 1)$.

6. A function f has domain $[-1, 2]$ and range $[0, 1]$. Find the domain and range of the function g defined by $g(x) = 1 - f(x + 1)$.

7. Let $f(x) = \tan x$ and $g(f(x)) = f\left(x - \dfrac{\pi}{4}\right)$, where $f(x)$ and $g(x)$ are real-valued functions. Prove that $f(g(x)) = \tan\left(\dfrac{x-1}{x+1}\right)$.

8. Let $g(x) = 1 + x - [x]$, where [.] denotes the greatest integer function and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$. Then for all x, find $f(g(x))$.

9. $f(x) = \begin{cases} \log_e x, & 0 < x < 1 \\ x^2 - 1, & x \ge 1 \end{cases}$ and $g(x) = \begin{cases} x+1, & x < 2 \\ x^2 - 1, & x \ge 2 \end{cases}$.

Then find $g(f(x))$.

ANSWERS

1. 3 **3.** -1 **4.** $[-2, 3)$

5. $\left[-\dfrac{1}{\sqrt{3}}, -\dfrac{1}{3}\right] \cup \left[\dfrac{1}{3}, \dfrac{1}{\sqrt{3}}\right]$

6. Domain: $[-2, 1]$; Range: $[0, 1]$

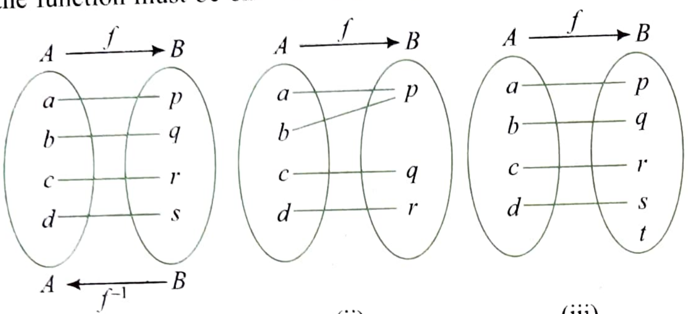
8. $f(g(x)) = 1$ for all x

9. $g(f(x)) = \begin{cases} 1+\ln x, & 0 < x < 1 \\ x^2, & 1 \le x < \sqrt{3} \\ (x^2-1)^2 - 1, & x \ge \sqrt{3} \end{cases}$

INVERSE FUNCTIONS

If $f: A \to B$ be a function defined by $y = f(x)$ such that f is both one-one(injective) and onto(surjective), i.e., bijective, then there exists a unique function $g: B \to A$ such that for each $y \in B$, $g(y) = x$ if and only if $y = f(x)$. The function g so defined is called the inverse of f and denoted by f^{-1}. Also, if g is the inverse of f, then f is the inverse of g and the two functions f and g are said to be inverses of each other.

The condition for existence of inverse of a function is that the function must be one-one and onto.



(i) (ii) (iii)

Mapping in figure (i) is one-one (injective) and surjective (onto), i.e., bijective. Clearly function f^{-1} is defined from set B to A.

Here $f(a) = p$ then $f^{-1}(p) = a$, $f(b) = q$ then $f^{-1}(q) = b$ and so on.

Mappings in figure (ii) and (iii) are not suitable for f^{-1} as relation from set B to A in each case does not follow the rules of function.

Thus, if function f is many-one or onto, f^{-1} does not exist.

Whenever an inverse function is defined, the range of the original function becomes the domain of the inverse function and domain of the original function becomes the range of the inverse function.

To, find the inverse of given function $y = f(x)$, we get 'x' as expression in y.

ILLUSTRATION 1.151

Which of the following functions have inverse?

(a) $f: Z \to Z$ defined by $f(x) = x + 2$

(b) $f: Z \to Z$ defined by $f(x) = 2x$

(c) $f: Z \to Z$ defined by $f(x) = x$

(d) $f: Z \to Z$ defined by $f(x) = |x|$

Sol. $f: Z \to Z$, $f(x) = x + 2$ is both one-one and onto. Hence invertible.

$f: Z \to Z$, $f(x) = 2x$ is one-one but not onto as range is set

$\{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$. Hence not invertible.

$f: Z \to Z$, $f(x) = x$ is one-one and onto. Hence invertible.

$f: Z \to Z$, $f(x) = |x|$ is many-one. Also, range is $\{0, 1, 2, 3, \ldots\}$. So, into. Hence not invertible.

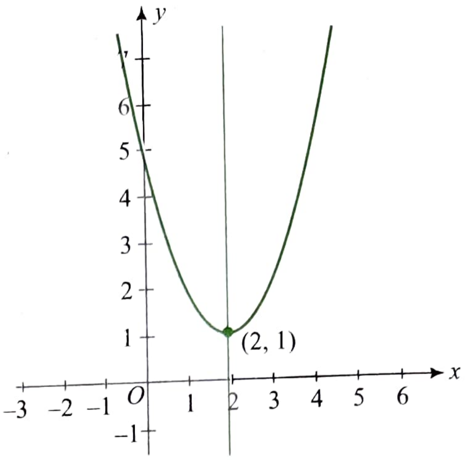
ILLUSTRATION 1.152

Let $f: R \to [1, \infty)$, $f(x) = x^2 - 4x + 5$. Then find the largest possible intervals for which $f^{-1}(x)$ is defined and find corresponding $f^{-1}(x)$.

Sol. We have $f: R \to [1, \infty)$, $f(x) = x^2 - 4x + 5$

or $\quad f(x) = (x - 2)^2 + 1$.

Graph of the function is as shown in the following figure.



Clearly $f(x)$ is many-one function.

However, in the interval $(-\infty, 2]$ or $[2, \infty)$, $f(x)$ is one-one.

For $\quad f: (-\infty, 2] \to [1, \infty)$, $y = (x - 2)^2 + 1$

$\therefore \quad (x - 2)^2 = y - 1$

$\Rightarrow \quad x - 2 = -\sqrt{y - 1}$

$\Rightarrow \quad x = 2 - \sqrt{y - 1}$

Thus $\quad f^{-1}: [1, \infty) \to (-\infty, 2]$, $f^{-1}(x) = 2 - \sqrt{x - 1}$

For $\quad f: [2, \infty) \to [1, \infty)$, $y = (x - 2)^2 + 1$

$\Rightarrow \quad x = 2 + \sqrt{y - 1}$

Thus $\quad f^{-1}: [1, \infty) \to [2, \infty)$, $f^{-1}(x) = \sqrt{x - 1} + 2$

ILLUSTRATION 1.153

Let $A = R - \{3\}$, $B = R - \{1\}$ and let $f: A \to B$ be defined by $f(x) = \dfrac{x - 2}{x - 3}$. Prove that '$f$' is invertible. Also find the inverse function.

Sol. Let $x_1, x_2 \in A$

and let $f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow \quad x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$$

$$\Rightarrow \quad x_1 = x_2$$

So, f is one-one.

Now $y = \dfrac{x - 2}{x - 3}$.

Clearly for any value of y other than '1' there exists unique x.

So, range of function is $R - \{1\}$.

Thus, $f(x)$ is onto.

Therefore, $f(x)$ is invertible.

Now $\quad y = \dfrac{x - 2}{x - 3}$

$\Rightarrow \quad xy - 3y = x - 2$

$\Rightarrow \quad x(y - 1) = 3y - 2$

$\Rightarrow \quad x = \dfrac{3y-2}{y-1}$

Therefore, $f^{-1}: A \to B$, $f^{-1}(x) = \dfrac{3x-2}{x-1}$

ILLUSTRATION 1.154

Let $f: R \to R$ be defined by $f(x) = e^x - e^{-x}$. Prove that $f(x)$ is invertible. Also find the inverse function.

Sol. Let $p, q \in R$ and let $f(p) = f(q)$

$\Rightarrow \quad e^p - e^{-p} = e^q - e^{-q}$

$\Rightarrow \quad e^p - e^q = e^{-p} - e^{-q}$

$\Rightarrow \quad e^p - e^q = \dfrac{1}{e^p} - \dfrac{1}{e^q}$

$\Rightarrow \quad e^p - e^q = \dfrac{e^q - e^p}{e^p e^q}$

$\Rightarrow \quad e^p - e^q = 0$ or $e^p e^q = -1$ (not possible)

$\Rightarrow \quad e^p = e^q$

$\Rightarrow \quad p = q$

Thus, $f(x)$ is one-one.

Lets us find the range of the function.

Clearly when x approaches to infinity $(e^x - e^{-x})$ approaches to infinity and when x approaches to negative infinity, $(e^x - e^{-x})$ approaches to negative infinity.

Also, $(e^x - e^{-x})$ continuously exists.

So, range of function is R.

Thus $f(x)$ bijective and so invertible.

Let $\quad e^x - e^{-x} = y$

$\Rightarrow \quad e^{2x} - ye^x - 1 = 0$

$\Rightarrow \quad e^x = \dfrac{y \pm \sqrt{y^2+4}}{2}$

$\Rightarrow \quad e^x = \dfrac{y + \sqrt{y^2+4}}{2}$ (as $y - \sqrt{y^2+4} < 0$ for all y and e^x is always positive)

$\Rightarrow \quad x = \log_e\left(\dfrac{y+\sqrt{y^2+4}}{2}\right)$

$\Rightarrow \quad f^{-1}(x) = \log_e\left(\dfrac{x+\sqrt{x^2+4}}{2}\right)$

ILLUSTRATION 1.155

Find the inverse of $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$

Sol. Given $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$

Let $\quad f(x) = y$

or $\quad x = f^{-1}(y) \quad (1)$

$\therefore \quad x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le \sqrt{y} \le 4 \\ y^2/64, & y^2/64 > 4 \end{cases} = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y > 16 \end{cases}$

$f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y > 16 \end{cases}$ [From (1)]

Hence, $f^{-1}(x) = \begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ x^2/64, & x > 16 \end{cases}$

ILLUSTRATION 1.156

Find the inverse of the function $f: [-1, 1] \to [-1, 1]$, $f(x) = x^2 \times sgn(x)$.

Sol. $f(x) = x^2 \times sgn(x)$

$= \begin{cases} x^2(1), & x > 0 \\ 0, & x = 0 \\ x^2(-1), & x < 0 \end{cases}$

$\therefore \quad f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ -x^2, & -1 \le x < 0 \end{cases}$

Now $\quad y = x^2, 0 \le x \le 1$

Since $\quad y \in [0, 1]$, we have $x = \sqrt{y}$

For $\quad y = -x^2, -1 \le x \le 0$.

Since $\quad y \in [-1, 0]$, we have $x = -\sqrt{-y}$

Thus, $\quad f^{-1}(x) = \begin{cases} \sqrt{x}, & 0 \le x \le 1 \\ -\sqrt{-x}, & -1 \le x < 0 \end{cases}$

ILLUSTRATION 1.157

Find the inverse of the function

$f: \left[-\dfrac{\pi}{2} - \tan^{-1}\dfrac{3}{4}, \dfrac{\pi}{2} - \tan^{-1}\dfrac{3}{4}\right] \to [-1, 1]$,

$f(x) = 3\cos x + 4\sin x + 7$.

Sol. We have,

$f(x) = y = 3\cos x + 4\sin x + 7$

$\Rightarrow \quad y = \sqrt{3^2+4^2}\left[\dfrac{3}{\sqrt{3^2+4^2}}\cos x + \dfrac{4}{\sqrt{3^2+4^2}}\sin x\right] + 7$

$= 5\left[\dfrac{3}{5}\cos x + \dfrac{4}{5}\sin x\right] + 7$

$= 5[\sin\theta\cos x + \cos\theta\sin x] + 7$, where $\tan\theta = \dfrac{3}{4}$

$= 5\sin(x+\theta) + 7$

$\Rightarrow \quad y - 7 = 5\sin(x+\theta)$

$\Rightarrow \quad \dfrac{y-7}{5} = \sin(x+\theta)$

$\Rightarrow \quad \sin^{-1}\dfrac{y-7}{5} = x+\theta$

$\Rightarrow \quad x = \sin^{-1}\dfrac{y-7}{5} - \theta$

$\Rightarrow \quad f^{-1}(x) = \sin^{-1}\dfrac{x-7}{5} - \tan^{-1}\dfrac{3}{4}$

Properties of Inverse Function

We have following properties of inverse function.

- The inverse of bijective function is unique and bijective.
- Let $f: A \to B$ be a function such that f is bijective and $g: B \to A$ is inverse of f, then $fog = I_B$ = Identity function of set B and $gof = I_A$ = Identity function of set A.
- Let the functions $f: A \to B$ and $g: B \to A$ satisfy both $gof = I_A$ and $fog = I_B$.

  Then function g is unique and $g = f^{-1}$.
- If f and g are two bijective functions such that $f: A \to B$ and $g: B \to C$ then $gof: A \to C$ is bijective and $(gof)^{-1} = f^{-1} o\, g^{-1}$

  Proof:

  To show that $(gof)^{-1} = f^{-1}og^{-1}$, it is enough to show that $(f^{-1}og^{-1})\, o\, (gof) = I_A$ and $(gof)\, o\, (f^{-1}og^{-1}) = I_C$.

  Now, $(f^{-1}og^{-1})o(gof) = ((f^{-1}og^{-1})\, og)\, of$
  $= (f^{-1}o(g^{-1}og))\, of$
  $= (f^{-1}\, oI_B)\, of$
  $= (f^{-1}\, o f)$
  $= I_A$.

  Similarly, it can be shown that $(gof)o(f^{-1}og^{-1}) = I_C$.
- Graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetrical about the line $y = x$.

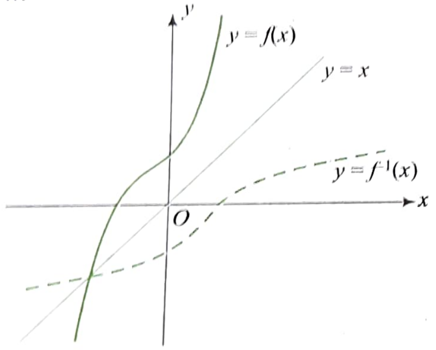


ILLUSTRATION 1.158

If $f(x) = 3x - 2$ and $(gof)^{-1}(x) = x - 2$ then find function $g(x)$ without finding $gof(x)$.

Sol. We have $f(x) = 3x - 2$

$\therefore \quad f^{-1}(x) = \dfrac{x+2}{3}$

Now $\quad (gof)^{-1}(x) = x - 2$

$\Rightarrow \quad f^{-1}og^{-1}(x) = x - 2$

$\Rightarrow \quad f^{-1}(g^{-1}(x)) = x - 2$

$\Rightarrow \quad \dfrac{g^{-1}(x)+2}{3} = x - 2$

$\Rightarrow \quad g^{-1}(x) = 3x - 8$

$\Rightarrow \quad g(x) = \dfrac{x+8}{3}$

Concept Application Exercise 1.13

Find the inverse of the following functions:

1. $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} + 2$
2. $f: R \to (-\infty, 1)$ given by $f(x) = 1 - 2^{-x}$
3. $f: (2, 3) \to (0, 1)$ defined by $f(x) = x - [x]$, where [.] represents the greatest integer function
4. $f: Z \to Z$ defined by $f(x) = [x + 1]$, where [.] denotes the greatest integer function.
5. $f(x) = \begin{cases} x^3 - 1, & x < 2 \\ x^2 + 3, & x \ge 2 \end{cases}$
6. $f: [-1, 1] \to [-1, 1]$ defined by $f(x) = x|x|$
7. $f: (-\infty, 1] \to \left[\dfrac{1}{2}, \infty\right)$, where $f(x) = 2^{x(x-2)}$

ANSWERS

1. $f^{-1}(x) = \log_e\left(\dfrac{x-1}{3-x}\right)^{1/2}$
2. $f^{-1}(x) = -\log_2(1-x)$
3. $f^{-1}(x) = x + 2$
4. $f^{-1}(x) = x - 1$
5. $f^{-1}(x) = \begin{cases} (x+1)^{1/3}, & x < 7 \\ (x-3)^{1/2}, & x \ge 7 \end{cases}$
6. $f^{-1}(x) = \begin{cases} -\sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$
7. $f^{-1}(x) = 1 - \sqrt{1 + \log_2 x}$

FUNCTIONAL EQUATIONS

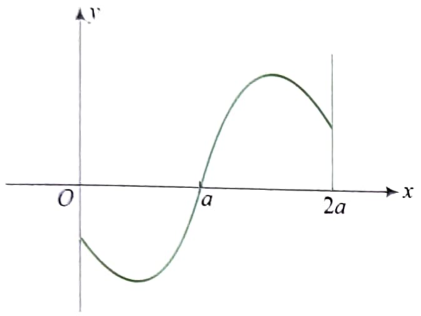
Functional equation (relation) is any equation that specifies a function in implicit form. Often, the equation relates the value of a function (or functions) at some point with its values at other points. Some functional equations satisfied by commonly used functions are as follows.

Function	Functional relation
$f(x) = a^x$, $a > 0$, $a \ne 1$	$f(x) \times f(y) = f(x+y)$ as $a^x a^y = a^{x+y}$ $\dfrac{f(x)}{f(y)} = f(x-y)$ as $\dfrac{a^x}{a^y} = a^{x-y}$
$f(x) = \log_a x$, $a > 0$, $a \ne 1$ and $x > 0$.	$f(x) + f(y) = f(xy)$ as $\log_a x + \log_a y = \log_a(xy)$ $f(x) - f(y) = f\left(\dfrac{x}{y}\right)$ as $\log_a x - \log_a y = \log_a \dfrac{x}{y}$
$f(x) = \sin^{-1}x$	$f(x) \pm f(y) = f(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$
$f(x) = \cos^{-1}x$	$f(x) \pm f(y) = f(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$

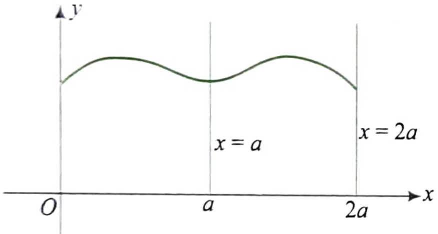
$f(x) = \tan^{-1}x$	$f(x) \pm f(y) = f\left(\dfrac{x \pm y}{1 \mp xy}\right)$
$f(x) \cdot f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$	$f(x) = \pm x^n + 1$

Functional equations resulting from properties of functions

- If function $f(x)$ is odd or graph of $f(x)$ is symmetrical about origin, then $f(x) + f(-x) = 0$, $\forall\, x \in D_f$, where D_f is domain of function.
- If function $f(x)$ is even or graph of $f(x)$ is symmetrical about y-axis then $f(x) = f(-x)$, $\forall\, x \in D_f$
- If graph of function $f(x)$ is symmetrical about point $(a, 0)$ then $f(a - x) = -f(a + x)$, $\forall\, x \in D_f$.



- If graph of function $f(x)$ is symmetric about line $x = a$, then $f(a - x) = f(a + x)$, $\forall\, x \in D_f$



- If function $f(x)$ is periodic having period $T\,(>0)$ then $f(x) = f(x + T)$, $\forall\, x \in D_f$ for least positive value T.

Problems Based on Finding the Value of Function at Some Point

Illustration 1.159

Let $f(x) = x + f(x - 1)$ for $\forall\, x \in R$. If $f(0) = 1$, find $f(100)$.

Sol. Given $f(x) = x + f(x - 1)$ and $f(0) = 1$

Put $x = 1$. Then,

$f(1) = 1 + f(0) = 2$

Put $x = 2$. Then,

$f(2) = 2 + f(1) = 4$

Put $x = 3$. Then,

$f(3) = 3 + f(2) = 7$

Thus, $f(0), f(1), f(2), \ldots$ form a series $1, 2, 4, 7, \ldots$.

Let $S = 1 + 2 + 4 + 7 + \cdots + f(n - 1)$

$S = \quad 1 + 2 + 4 + \cdots + f(n - 2) + f(n - 1)$

Subtracting, we get

$0 = (1 + 1 + 2 + 3 + \cdots + n \text{ terms}) - f(n - 1)$

$\therefore \quad f(n - 1) = \dfrac{n(n + 1)}{2}$

$\therefore \quad f(100) = 5051$

Illustration 1.160

The function $f(x)$ is defined for all real x. If $f(a + b) = f(ab)\ \forall\, a$ and b and $f\left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}$ then find the value of $f(1005)$.

Sol. We have $f(a + b) = f(ab)\ \forall\, a$ and b

Put $a = 0$

$\therefore \quad f(b) = f(0) = k$ (say)

Putting $b = 0$ in given relation,

$f(a) = f(0) = k$

Thus $f(a) = f(b) = k\ \forall\, a, b$

Hence, $f(x)$ is a constant function.

$\therefore \quad f(1005) = f\left(-\dfrac{1}{2}\right)$

$= -\dfrac{1}{2}$

Illustration 1.161

Let a function $f(x)$ satisfies $f(x) + f(2x) + f(2 - x) + f(1 + x) = x\ \forall\, x \in R$. Then find the value of $f(0)$.

Sol. $f(x) + f(2x) + f(2 - x) + f(1 + x) = x$

Put $x = 0$. Then,

$f(0) + f(0) + f(2) + f(1) = 0$

or $\quad 2f(0) + f(1) + f(2) = 0 \qquad (1)$

Put $x = 1$. Then,

$f(1) + f(2) + f(1) + f(2) = 1$

$\therefore \quad f(1) + f(2) = 1/2$

So, from (1)

$2f(0) + 1/2 = 0$

$\therefore \quad f(0) = -1/4$

Illustration 1.162

Let f be a function satisfying $f(xy) = \dfrac{f(x)}{y}$ for all positive real numbers x and y. If $f(30) = 20$, then find the value of $f(40)$.

Sol. Given $f(xy) = \dfrac{f(x)}{y}$

or $\quad f(y) = \dfrac{f(1)}{y} \qquad$ (Putting $x = 1$)

or $\quad f(30) = \dfrac{f(1)}{30}$

or $\quad f(1) = 30 \times f(30) = 30 \times 20 = 600$

$\therefore \quad f(40) = \dfrac{f(1)}{40}$

$= \dfrac{600}{40} = 15$

Illustration 1.163

If $f(x)$ is a polynomial function satisfying

$f(x)\cdot f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$ and $f(4) = 65$, then find $f(6)$.

Sol. Polynomial function satisfying

$$f(x)\cdot f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$$

is $f(x) = \pm x^n + 1$

$\therefore\quad f(4) = \pm 4^n + 1 = 65$

or $\quad 4^n + 1 = 65$

or $\quad 4^n = 64$

or $\quad n = 3$

So, $f(x) = x^3 + 1$.

Hence, $f(6) = 6^3 + 1 = 217$

Illustration 1.164

Let $f(x)=\frac{9^x}{9^x+3}$. Show that $f(x) + f(1 - x) = 1$ and, hence, evaluate

$$f\left(\frac{1}{1996}\right)+f\left(\frac{2}{1996}\right)+f\left(\frac{3}{1996}\right)+\cdots+f\left(\frac{1995}{1996}\right)$$

Sol. $f(x)=\frac{9^x}{9^x+3}$ (1)

and $\quad f(1-x)=\frac{9^{1-x}}{9^{1-x}+3}$

or $\quad f(1-x)=\frac{\frac{9}{9^x}}{\frac{9}{9^x}+3}=\frac{9}{9+3\cdot 9^x}$

or $\quad f(1-x)=\frac{3}{(3+9^x)}$ (2)

Adding (1) and (2), we get

$$f(x)+f(1-x)=\frac{9^x}{9^x+3}+\frac{3}{(3+9^x)}=1$$

or $\quad f(x)+f(1-x)=1$ (3)

Now, putting $x=\frac{1}{1996},\frac{2}{1996},\frac{3}{1996},\ldots,\frac{998}{1996}$ in (3), we get

$$f\left(\frac{1}{1996}\right)+f\left(\frac{1995}{1996}\right)=1,\ f\left(\frac{2}{1996}\right)+f\left(\frac{1994}{1996}\right)=1,$$

$$f\left(\frac{3}{1996}\right)+f\left(\frac{1993}{1996}\right)=1$$

$$\vdots$$

$$f\left(\frac{997}{1996}\right)+f\left(\frac{999}{1996}\right)=1,\ f\left(\frac{998}{1996}\right)+f\left(\frac{998}{1996}\right)=1$$

$$\therefore\quad f\left(\frac{998}{1996}\right)=\frac{1}{2}$$

Adding all the above expressions, we get

$$f\left(\frac{1}{1996}\right)+f\left(\frac{2}{1996}\right)+\cdots+f\left(\frac{1995}{1996}\right)$$

$$=(1+1+1+\cdots+997\text{ times})+\frac{1}{2}$$

$$=997+\frac{1}{2}=997.5$$

Problems Based on Finding Function

Illustration 1.165

Consider a real-valued function $f(x)$ satisfying $2f(xy) = (f(x))^y + (f(y))^x\ \forall\ x, y \in R$ and $f(1) = a$, where $a \neq 1$. Prove that $(a-1)\sum_{i=1}^{n} f(i)=a^{n+1}-a$.

Sol. We have $2f(xy) = (f(x))^y + (f(y))^x$.

Replacing y by 1, we get

$$2f(x) = f(x) + (f(1))^x \text{ or } f(x) = a^x$$

or $\quad \sum_{i=1}^{n} f(i)=a+a^2+\cdots+a^n=\frac{a^{n+1}-a}{a-1}$

or $\quad (a-1)\sum_{i=1}^{n} f(i)=a^{n+1}-a$

Illustration 1.166

Let f be a real-valued function such that $f(x)+2f\left(\frac{2002}{x}\right)=3x$. Then find $f(x)$.

Sol. $f(x)+2f\left(\frac{2002}{x}\right)=3x$ (1)

Replacing x by $\frac{2002}{x}$, we get

$$f\left(\frac{2002}{x}\right)+2f(x)=\frac{6006}{x} \quad (2)$$

Solving (1) and (2), for $f(x)$, we get

$$f(x)=\frac{4004}{x}-x$$

Illustration 1.167

If $f: R \to R$ is an odd function such that

(i) $f(1 + x) = 1 + f(x)$

(ii) $x^2 f\left(\frac{1}{x}\right)=f(x),\quad x \neq 0$ then find $f(x)$.

Sol. $x^2 f\left(\frac{1}{x}\right)=f(x),\quad x\neq 0$

Replace x by $x + 1$. Then,

$$(x+1)^2 f\left(\frac{1}{1+x}\right)=f(x+1)$$

$$f\left(\frac{1}{1+x}\right)=\frac{1+f(x)}{(1+x)^2} \quad (1)$$

$$\therefore \quad f\left(\frac{1}{1+x}\right)=f\left(1-\frac{x}{1+x}\right)$$

$$=1+f\left(-\frac{x}{1+x}\right)$$

$$=1-f\left(\frac{x}{1+x}\right) \quad (\because f(x) \text{ is odd})$$

$$=1-\left(\frac{x}{1+x}\right)^2 f\left(\frac{1+x}{x}\right)$$

$$=1-\left(\frac{x}{1+x}\right)^2 f\left(1+\frac{1}{x}\right)$$

$$=1-\left(\frac{x}{1+x}\right)^2\left(1+\frac{f(x)}{x^2}\right) \quad (2)$$

From (1) and (2),

$$\frac{1+f(x)}{(1+x)^2}=1-\left(\frac{x}{1+x}\right)^2\left(1+\frac{f(x)}{x^2}\right)$$

or $\quad 1+f(x)=(1+x)^2-x^2-f(x)$

or $\quad f(x)=x$

ILLUSTRATION 1.168

Let $f: R^+ \to R$ be a function which satisfies $f(x)\times f(y)=f(xy)+2\left(\frac{1}{x}+\frac{1}{y}+1\right)$ for x, $y>0$, then find $f(x)$.

Sol. We have $f(x)\times f(y)=f(xy)+2\left(\frac{1}{x}+\frac{1}{y}+1\right)$ ...(1)

To get $f(x)$ we put $y=1$,

$$\therefore \quad f(x)\times f(1)=f(x)+2\left(\frac{1}{x}+2\right)$$

$$=f(x)+2\left(\frac{2x+1}{x}\right)$$

$$\Rightarrow \quad f(x)\,(f(1)-1)=\frac{2(2x+1)}{x}$$

$$\Rightarrow \quad f(x)=\frac{2(2x+1)}{x(f(1)-1)} \quad ...(2)$$

Now, we need the value of f(1).

Put $x=1$ and $y=1$ in (1), we get

$$(f(1))^2-f(1)-6=0$$

$$\Rightarrow \quad f(1)=3 \text{ or } f(1)=-2$$

For $\quad f(1)=3, f(x)=\frac{2x+1}{x}$

and for $f(1)=-2, f(x)=\frac{2(2x+1)}{-3x}$

ILLUSTRATION 1.169

A continuous function $f(x)$ on $R\to R$ satisfies the relation $f(x)+f(2x+y)+5xy=f(3x-y)+2x^2+1\ \forall\ x, y\in R$ then find $f(x)$.

Sol. We have,

$$f(x)+f(2x+y)+5xy=f(3x-y)+2x^2+1\ \forall\ x,y\in R \quad ...(1)$$

To get function we should eliminate $f(2x+y)$ and $f(3x-y)$.

For that let $2x+y=3x-y$

$$\Rightarrow \quad y=\frac{x}{2}$$

So, putting $y=\frac{x}{2}$, we get

$$f(x)+f\left(\frac{5x}{2}\right)+\frac{5x^2}{2}=f\left(\frac{5x}{2}\right)+2x^2+1$$

$$\Rightarrow \quad f(x)=1-\frac{x^2}{2}$$

ILLUSTRATION 1.170

If for all real values of u and υ, $2f(u)\cos\upsilon=f(u+\upsilon)+f(u-\upsilon)$, prove that for all real values of x,

(i) $f(x)+f(-x)=2a\cos x$.

(ii) $f(\pi-x)+f(-x)=0$.

(iii) $f(\pi-x)+f(x)=2b\sin x$.

Deduce that $f(x)=a\cos x+b\sin x$, where a, b are arbitrary constants.

Sol. Given $2f(u)\cos\upsilon=f(u+\upsilon)+f(u-\upsilon)$ (1)

Putting $u=0$ and $\upsilon=x$ in (1), we get

$$f(x)+f(-x)=2f(0)\cos x=2a\cos x \quad (2)$$

a is an arbitrary constant.

Now, putting $u=\frac{\pi}{2}-x$ and $\upsilon=\frac{\pi}{2}$ in (1), we get

$$f(\pi-x)+f(-x)=0 \quad (3)$$

Again, putting $u=\pi/2$ and $\upsilon=\pi/2-x$ in (1), we get

$$f(\pi-x)+f(x)=2f(\pi/2)\sin x=2b\sin x \quad (4)$$

b is an arbitrary constant.

Adding (2) and (4), we get

$$2f(x)+f(\pi-x)+f(-x)=2a\cos x+2b\sin x$$

or $\quad 2f(x)+0=2a\cos x+2b\sin x$ [From (3)]

$\therefore \quad f(x)=a\cos x+b\sin x$

PROBLEMS BASED ON PROPERTIES OF FUNCTION

ILLUSTRATION 1.171

Prove that $f(x)$ given by $f(x+y)=f(x)+f(y)\ \forall\ x\in R$ is an odd function.

Sol. Given $f(x+y)=f(x)+f(y)\ \forall x\in R$ (1)

Replacing y by $-x$, we have

$$f(x-x)=f(x)+f(-x)$$

or $\quad f(x)+f(-x)=f(0)$ (2

Now, putting $x = y = 0$ in (1), we have

$$f(0+0) = f(0) + f(0)$$

or $\quad f(0) = 0$

Then from (2), $f(x) + f(-x) = 0$. Hence, $f(x)$ is an odd function.

ILLUSTRATION 1.172

If $f(x+y) = f(x).f(y)$ for all real x, y and $f(0) \neq 0$, then prove that the function $g(x) = \dfrac{f(x)}{1+\{f(x)\}^2}$ is an even function.

Sol. Given $f(x+y) = f(x)\cdot f(y)$. (1)

Put $\quad x = y = 0$. Then $f(0) = 1$.

Putting $y = -x$ in (1), we get

$$f(0) = f(x)\,f(-x)$$

or $\quad f(-x) =$

Now, $\quad g(x) =$

$$\therefore \quad g(-x) = \frac{f(-x)}{1+\{f(-x)\}^2} = \frac{\dfrac{1}{f(x)}}{1+\dfrac{1}{\{f(x)\}^2}} = \frac{f(x)}{1+\{f(x)\}^2} = g(x)$$

ILLUSTRATION 1.173

Let $f(x)$ be be a function and k be a positive real number such that $f(x+k) + f(x) = 0$ for all $x \in R$. Prove that $f(x)$ is periodic with period $2k$.

We have $f(x+k) + f(x) = 0 \forall x \in R$

or $\quad f(x+k) = -f(x) \forall x \in R$

Put $x = x + k$. Then,

or $\quad f(x+2k) = -f(x+k)\ \forall\ x \in R$

or $\quad f(x+2k) = f(x),\ \forall\ x \in R \qquad$ [As $f(x+k) = -f(x)$]

which clearly shows that $f(x)$ is periodic with period $2k$.

ILLUSTRATION 1.174

If $f(x)$ satisfies the relation $f(x) + f(x+4) = f(x+2) + f(x+6)$ for all x, then prove that $f(x)$ is periodic and find its period.

Given $f(x) + f(x+4) = f(x+2) + f(x+6)$ (1)

Replace x by $x + 2$. Then

$$f(x+2) + f(x+6) = f(x+4) + f(x+8) \qquad (2)$$

From (1) and (2), we have $f(x) = f(x+8)$.

Hence, $f(x)$ is periodic with period 8.

ILLUSTRATION 1.175

An odd function is symmetrical about the vertical line $x = a$, $(a > 0)$, and if $= 8$, then find the value of $f(1)$.

$f(x)$ is an odd function. Therefore,

$$f(x) = -f(-x) \qquad (1)$$

$f(x)$ is symmetrical about the line $x = a$. Therefore,

$$f(a-x) = f(a+x) \qquad (2)$$

$\therefore \quad f(2a-x) = f(x) \qquad$ (Replacing x by $a-x$)

or $\quad f(2a+x) = f(-x) \qquad$ (Replacing x by $-x$)

or $\quad f(2a+x) = -f(x) \qquad$ ($\because f$ is odd)

or $\quad f(x+4a) = -f(x+2a) \qquad$ (Replacing x by $x+2a$)

or $\quad f(x+4a) = f(x)$

i.e., $\quad f$ is periodic with period $4a$. So,

or $\quad f(1+4r) = f(1)$

Now $\quad \displaystyle\sum_{r=0}^{\infty} [f(1)]^r = 8$

or $\quad \dfrac{1}{1-f(1)} = 8$

or $\quad f(1) = 7/8$

ILLUSTRATION 1.176

Check whether the function defined by $f(x+\lambda) = 1 + \sqrt{2f(x) - f^2(x)}\ \forall\ x \in R$ is periodic or not. If yes, then find its period $(\lambda > 0)$.

Sol. For the function to be true,

$$2f(x) - f^2(x) \geq 0$$

or $\quad f(x)\,[f(x) - 2] \leq 0$ or $0 \leq f(x) \leq 2 \qquad$ (1)

and from the given function,

$$f(x+\lambda) \geq 1 \text{ or } f(x) \geq 1 \qquad (2)$$

From (1) and (2), we have $1 \leq f(x) \leq 2$.

Again, we have

$$\{f(x+\lambda) - 1\}^2 = 2f(x) - f^2(x)$$

or $\quad \{f(x+\lambda) - 1\}^2 = 1 + \{2f(x) - f^2(x) - 1\}$

or $\quad \{f(x+\lambda) - 1\}^2 = 1 - \{f(x) - 1\}^2 \qquad$ (3)

Replacing x by $x + \lambda$, we get

$$\{f(x+2\lambda) - 1\}^2 = 1 - \{f(x+\lambda) - 1\}^2 \qquad (4)$$

Subtracting (3) from (4), we get

$$\{f(x+2\lambda) - 1\}^2 = \{f(x) - 1\}^2$$

or $\quad f(x+2\lambda) - 1 = f(x) - 1 \qquad$ ($\because 1 \leq f(x) \leq 2$)

$\therefore \quad f(x+2\lambda) = f(x)$

So, $\quad f$ is periodic with period 2λ.

CONCEPT APPLICATION EXERCISE 1.14

1. If $f(x+y+1) = \{\sqrt{f(x)} + \sqrt{f(y)}\}^2$ and $f(0) = 1\ \forall x, y \in R$, determine $f(n)$, $n \in N$.
2. Let $g(x)$ be a function such that $g(a+b) = g(a)\cdot g(b)\ \forall\ a, b \in R$. If zero is not an element in the range of g, then find the value of $g(x)\cdot g(-x)$.
3. If $f(x+2a) = f(x-2a)$, where $a > 0$, then prove that $f(x)$ is periodic.
4. If $f(x + f(y)) = f(x) + y\ \forall\ x, y \in R$ and $f(0) = 1$, then find the value of $f(7)$.
5. If $f: R^+ \to R$, $f(x) + 3xf\left(\dfrac{1}{x}\right) = 2(x+1)$, then find $f(x)$.

6. $f: R \to R, f(x^2+x+3)+2f(x^2-3x+5)=6x^2-10x+17$ $\forall x \in R$, then find the function $f(x)$.
7. Consider $f: R^+ \to R$ such that $f(3)=1$ for $x \in R^+$ and $f(x)$.
$f(y)+f\left(\frac{3}{x}\right)f\left(\frac{3}{y}\right)=2f(xy)\ \forall\ x, y \in R^+$. Then find $f(x)$.
8. Determine all functions $f: R \to R$ such that $f(x-f(y))$ $=f(f(y))+xf(y)+f(x)-1\ \forall\ x, y \geq \in R$.
9. Determine the function satisfying $f^2(x+y)=f^2(x)+f^2(y)\ \forall\ x, y \in R$.
10. If $f: R \to R$ is a function satisfying the equation $f(2x+3)+f(2x+7)=2\ \forall\ x \in R$, then find the fundamental period of $f(x)$.
11. If $f(x)$ is an even function and satisfies the relation $x^2 . f(x)-2f\left(\frac{1}{x}\right)=g(x)$, where $g(x)$ is an odd function, then find the value of $f(5)$.
12. If $f(a-x)=f(a+x)$ and $f(b-x)=f(b+x)$ for all real x, where a, b $(a>b>0)$ are constants, then prove that $f(x)$ is a periodic function.
13. A real-valued function $f(x)$ satisfies the functional equation $f(x-y)=f(x)f(y)-f(a-x)f(a+y)$, where a is given constant and $f(0)=1$. Then prove that $f(x)$ is symmetrical about point $(a, 0)$.

ANSWERS

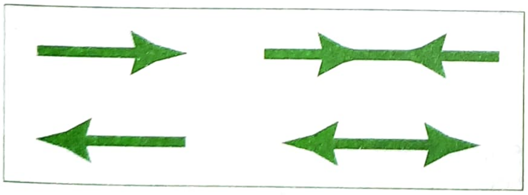
1. $f(n)=(n+1)^2$	2. 1	3. $4a$	4. 1
5. $f(x)=(x+1)/2$	6. $f(x)=2x-3$		7. $f(x)=1$
8. $f(x)=1-\frac{x^2}{2}$	9. $f(x)=0$		10. 8
11. 0			

TRANSFORMATION OF GRAPHS

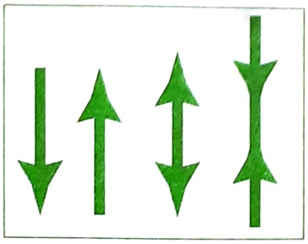
Transformations cause functions to change in some way. constants are used to either translate or stretch a function's graph.

Direction of transformation:

If the constant is inside the function, with the x, then it causes transformation in the direction of x-axis.



If the constant is outside the function, operating on $f(x)$, then it causes transformation in the direction of y-axis.



If the graph of $y=f(x)$ is given then we have following transformations

Translate (a shift):

For $y=f(x)+a$, shift the graph of $y=f(x)$ up and down

For $y=f(x+a)$, shift the graph of $y=f(x)$ left or right

Stretch:

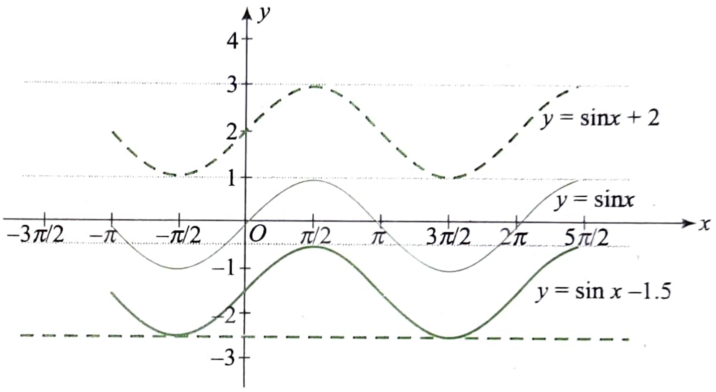
For $y=f(ax)$, where $a>0$ stretch or squeeze the graph of $y=f(x)$ parallel to x-axis

For $y=af(x)$, where $a>0$ stretch or squeeze the graph of $y=f(x)$ parallel to y-axis.

If $a<0$ then we first flip the graph of $y=f(x)$ horizontally for $y=f(ax)$ and vertically for $y=af(x)$ and then we do the above transformations.

VERTICAL SHIFT

Consider the graph of $y=\sin x$, $y=\sin x+2$ and $y=\sin x-1.5$



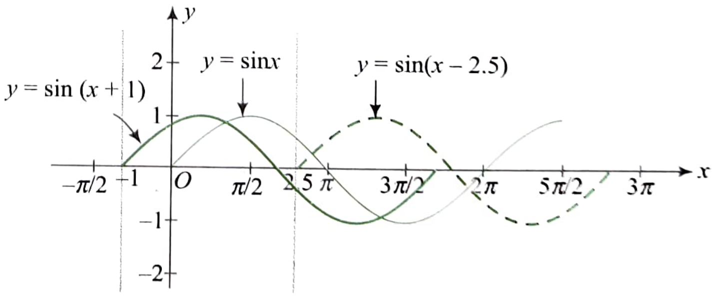
From the above graphs we can say that

- to get the graph of $y=f(x)+a$ $(a>0)$ shift the graph of $y=f(x)$ 'a' unit upward.
- to get the graph of $y=f(x)-a$ $(a>0)$ shift the graph of $y=f(x)$, 'a' unit downward.

Here domain of function remains the same, but range changes. Also, for periodic function period remains the same.

HORIZONTAL SHIFT

Consider the graph of $y=\sin x$, $y=\sin(x-2.5)$ and $y=\sin(x+1)$



Here we observe that the starting point of graph of $y=\sin x$ is $(0, 0)$. Starting point of graph of $y=\sin(x+1)$ is $(-1, 0)$ and that of $y=\sin(x-2.5)$ is $(2.5, 0)$.

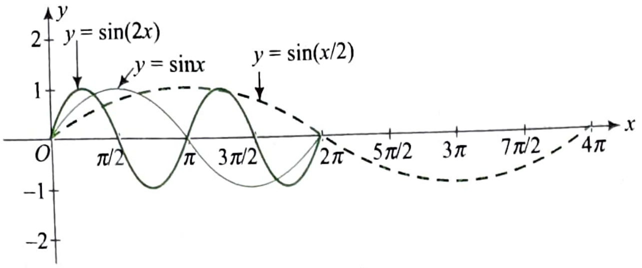
From the above graphs we can say that

- to get the graph of $y=f(x+a)$ $(a>0)$ shift the graph of $y=f(x)$ 'a' unit to the left.
- to get the graph of $y=f(x-a)$ $(a>0)$ shift the graph of $y=f(x)$, 'a' unit to the right.

Here domain the function changes but range remains the same. Also, for periodic function period remains the same.

Horizontal Stretch or Squeeze

Consider the graph of $y = \sin x$, $y = \sin(2x)$ and $y = \sin(x/2)$



From the above graphs we can say that

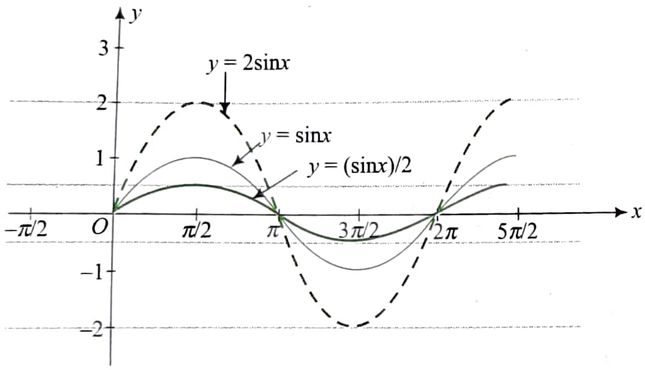
- to get the graph of $y = f(ax)$ $(a > 1)$ squeeze the graph of $y = f(x)$ horizontally by dividing each of its x-coordinates by k.
- to get the graph of $y = f(ax)$ $(0 < a < 1)$, stretch the graph of $y = f(x)$ horizontally by dividing each of its x-coordinates by k.

Here domain of function changes, but range remains the same.

For periodic function, period changes. So, for $y = f(ax)$, period is $T/|a|$, where T is period of $y = f(x)$.

Vertical Stretch or Squeeze

Consider the graph of $y = \sin x$, $y = 2\sin x$ and $y = (\sin x)/2$



From the above graphs we can say that

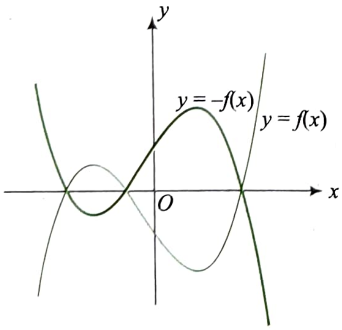
- to get the graph of $y = af(x)$ $(a > 1)$ stretch the graph of $y=f(x)$ vertically by by multiplying each of its y-coordinates by k.
- to get the graph of $y = af(x)$ $(0 < a < 1)$, squeeze the graph of $y = f(x)$ vertically by multiplying each of its y-coordinates by k.

Here domain the function remains the same but range changes.

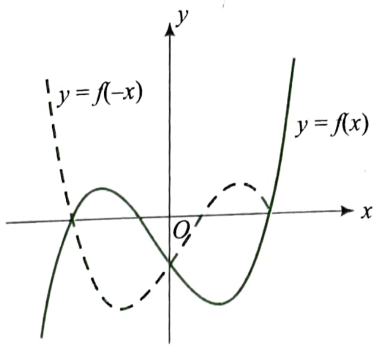
For periodic function, period remains the same.

Vertical and Horizontal Flip

To get the graph of $y = -f(x)$, flip vertically the graph of $y = f(x)$, i.e., reflect the graph of $y = f(x)$ over x-axis.



To get the graph of $y = f(-x)$, flip horizontally the graph of $y = f(x)$, i.e., reflect the graph of $y = f(x)$ over y-axis.



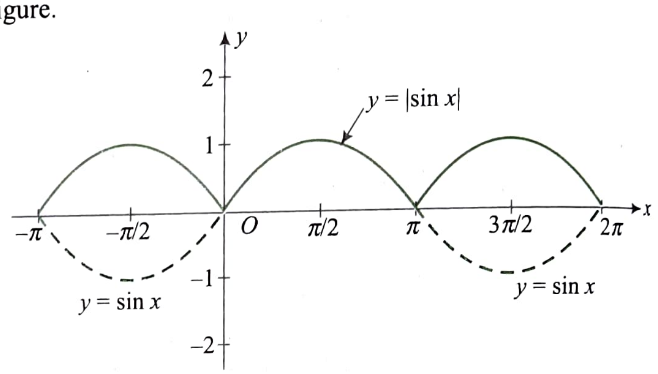
Transformation Related to Modulus

Graph of $y = |f(x)|$ from the graph of $y = f(x)$

We know that $|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$

For $f(x) > 0$, the graph of $y = f(x)$ and that of $y = |f(x)|$ coincide. For $f(x) < 0$, the graph of $y = f(x)$ is reflected over x-axis to the get the graph of $y = |f(x)|$.

The graph of $y = \sin x$ and $y = |\sin x|$ are shown in the following figure.



We observe that in the intervals $(0, \pi)$, where $\sin x > 0$, graphs of $y = \sin x$ and that of $y = |\sin x|$ coincide.

In the intervals $(-\pi, 0)$ and $(\pi, 2\pi)$, where $\sin x < 0$, graph of $y = \sin x$ is reflected over x-axis to get the graph of $y = |\sin x|$.

Graph of $y = f(|x|)$ from the graph of $y = f(x)$

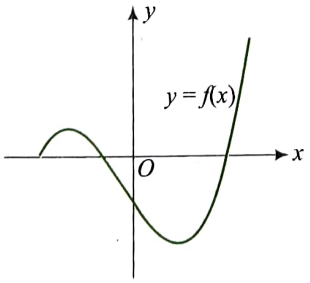
We know that $f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$

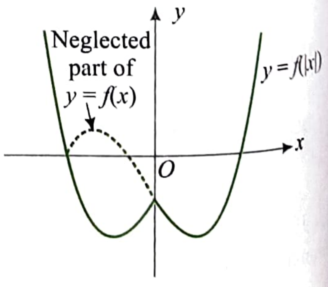
Thus, for $x > 0$, the graph of $y = f(x)$ and that of $y = f(|x|)$ coincide.

For $x < 0$, the graph of $y = f(x)$ for $x > 0$ is reflected over y-axis to the get the graph of $y = f(|x|)$.

In this process we neglect the initial graph of $y = f(x)$ for $x < 0$.

Also, $y = f(|x|)$ is an even function. So, graph is always symmetrical about y-axis.





Thus, to draw the graph of $y = f(|x|)$ from the given graph of $y = f(x)$, neglect the graph of $y = f(x)$ for $x < 0$ and reflect the graph of $y = f(x)$ for $x \geq 0$ over y-axis while retailing this branch of the graph.

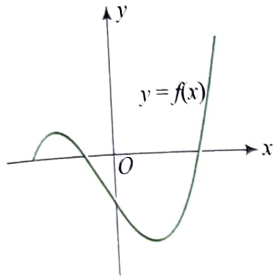
Graph of $|y| = f(x)$ from the graph of $y = f(x)$

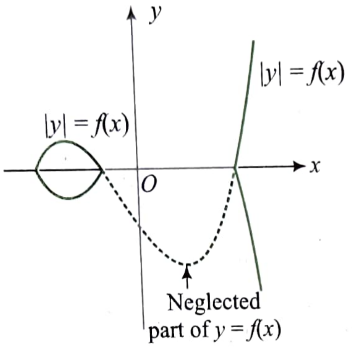
Equation $|y| = f(x)$ represent relation.

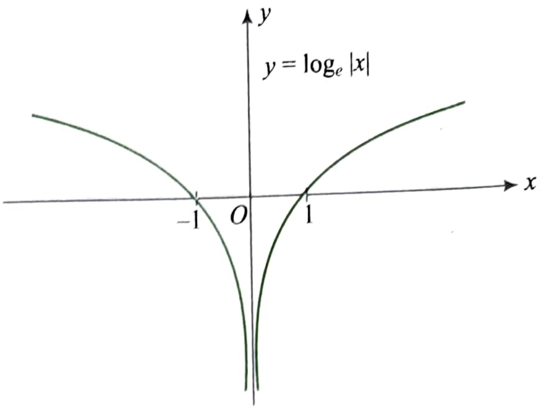
Clearly, equation is meaningful if $f(x) \geq 0$.

Then it represents two functions $y = \pm f(x)$ for $f(x) \geq 0$

So, to draw the graph of $|y| = f(x)$, we neglect the graph of $y = f(x)$, for $f(x) < 0$ and reflect the graph of $y = f(x)$ for $x > 0$ over x-axis, while retaining this branch of the graph.







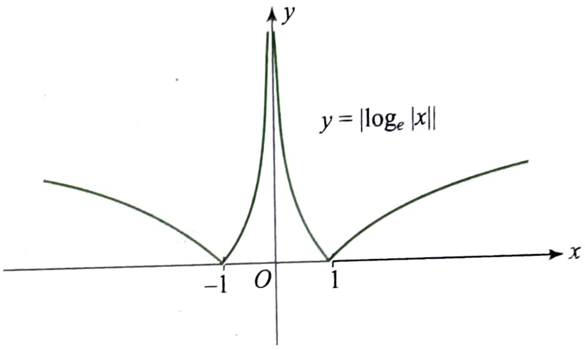
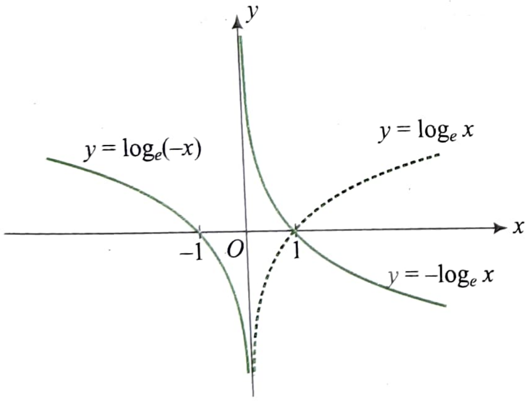


ILLUSTRATION 1.177

Draw the graphs of $y = \log_e(-x)$, $-\log_e x$, $y = |\log_e x|$, $y = \log_e|x|$ and $y = |\log_e|x||$ transforming the graph of $y = \log_e x$.

Sol.



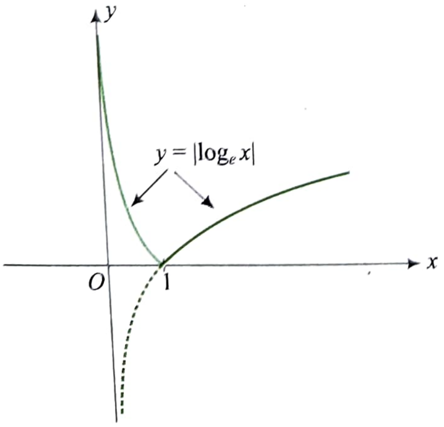
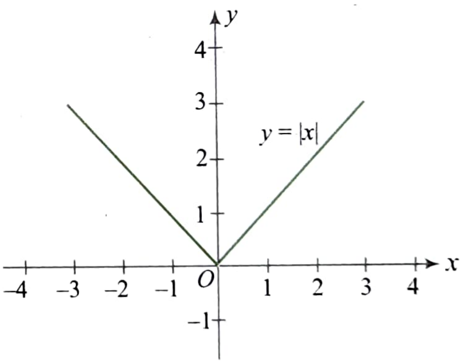
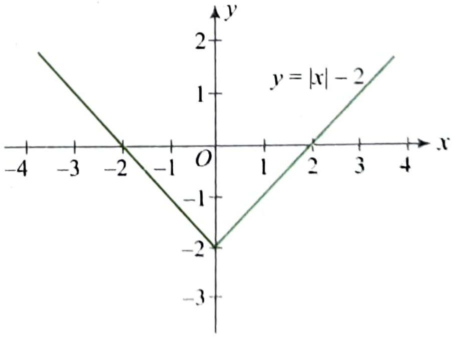


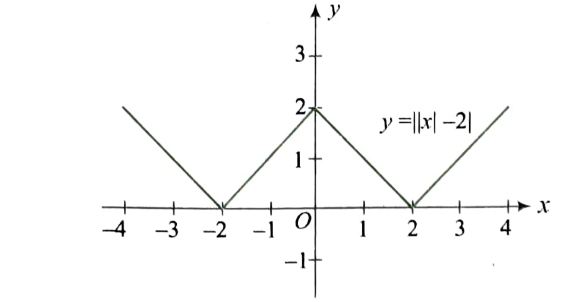
ILLUSTRATION 1.178

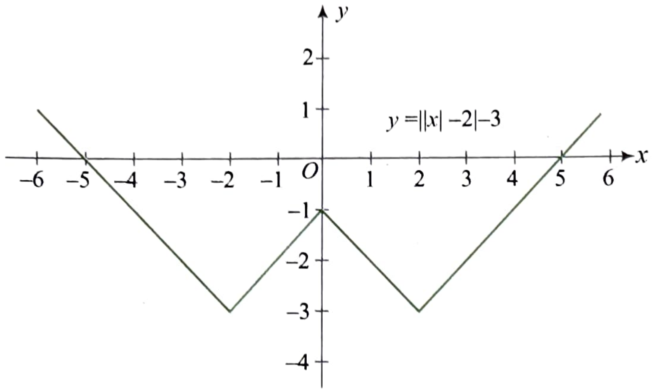
Draw the graph of $y = |||x| - 2| - 3|$ by transforming the graph of $y = |x|$

Sol.









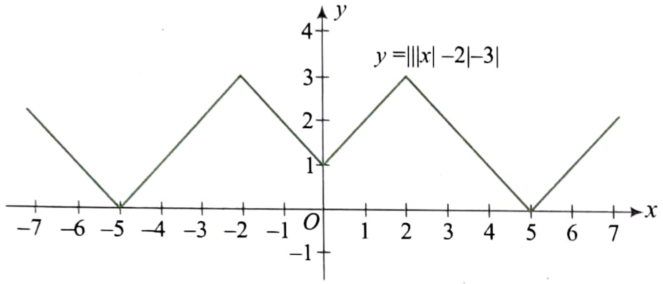


ILLUSTRATION 1.179

Consider the function $f(x)=\begin{cases}2x+3, & x\le 1\\ -x^2+6, & x>1\end{cases}$.

Then draw the graph of the function $y = f(x)$, $y = f(|x|)$, $y = |f(x)|$, and $y = |f(|x|)|$.

Sol.

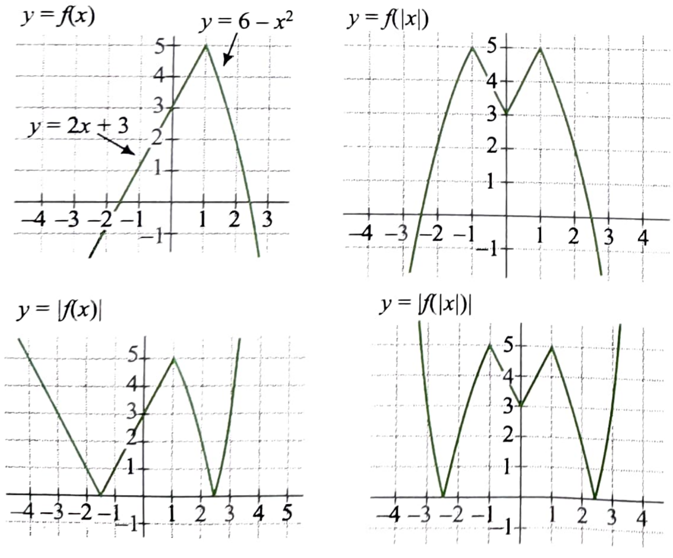
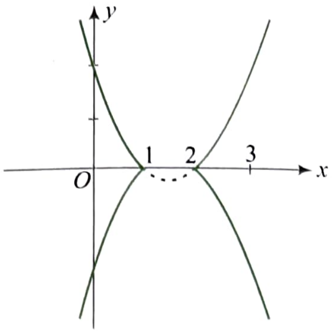


ILLUSTRATION 1.180

Sketch the curve $|y| = (x-1)(x-2)$.

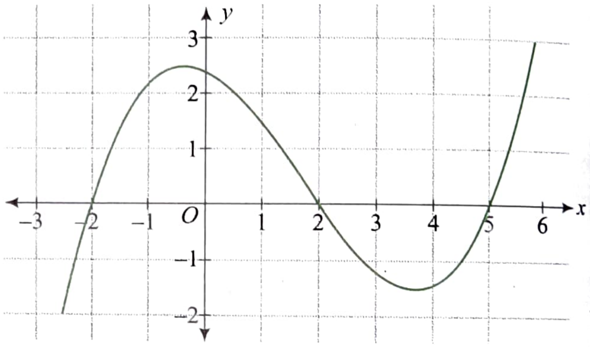
Sol.



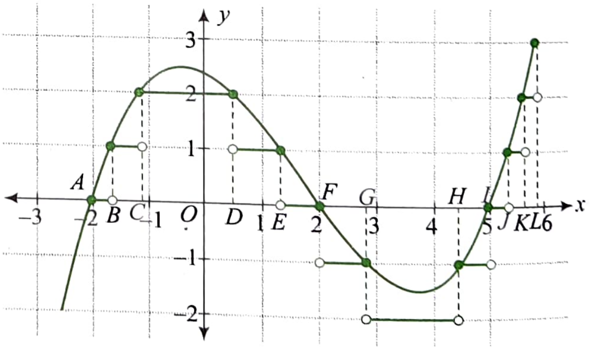
GRAPH OF $y = [f(x)]$ FROM THE KNOWN GRAPH OF $y = f(x)$, WHERE [.] REPRESENTS THE GREATEST INTEGER FUNCTION

We know that if $n \le f(x) < n + 1$, $n \in I$ then $[f(x)] = n$. Thus, we would draw lines parallel to the x-axis passing through different integral points. Hence values of x can be obtained so that $f(x)$ lies between two successive integers.

Consider the following graph of $y = f(x)$.



Draw perpendicular from each of the points where lines $y = n$, $(n \in Z)$ intersect the curve on the x-axis as shown in the following figure.



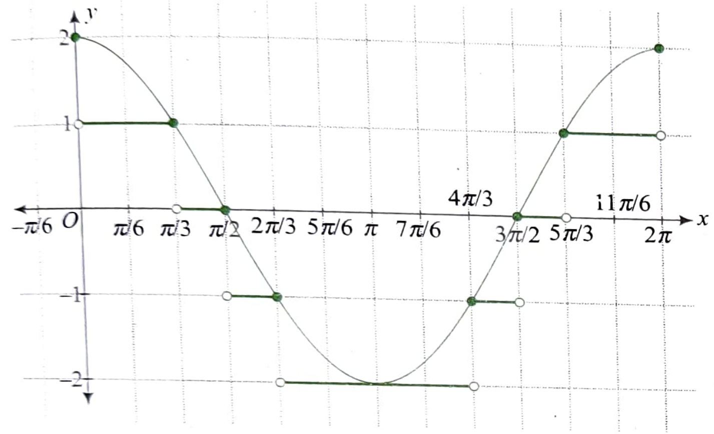
Values of x lying between points	Values of y lying between consecutive integers	Value of $y = [f(x)]$
A and B	0 and 1	0
B and C	1 and 2	1
C and D	2 and 3	2

D and E	1 and 2	1
E and F	0 and 1	0
F and G	–1 and 0	–1
G and H	–2 and –1	–2
H and I	–1 and 0	–1
I and J	0 and 1	0
J and K	1 and 2	1
K and L	2 and 3	2

ILLUSTRATION 1.181

Draw the graph of $f(x) = [2\cos x]$, $x \in [0, 2\pi]$, ([·] represents greatest integer function).

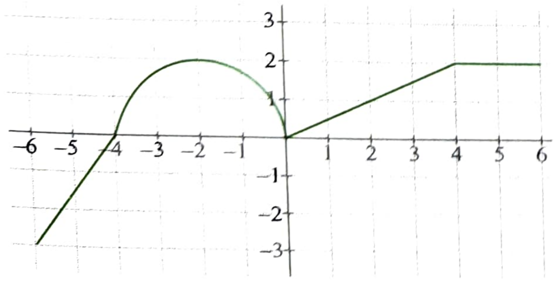
Sol. $f(x) = [2 \cos x]$



CONCEPT APPLICATION EXERCISE 1.15

Draw the graph of the following (Q. 1 to 4):

1. $f(x) = \sin |x|$
2. $|f(x)| = \tan x$
3. $f(x) = |x^2 - 3|x| + 2|$
4. $f(x) = -|x - 1|^{1/2}$
5. Find the total number of solutions of $\sin \pi x = |\ln |x||$.
6. Solve $\left|\dfrac{x^2}{x-1}\right| \le 1$ using the graphical method.
7. Given the graph of $f(x)$, draw the graph each one of the following functions:



(a) $y = f(x) + 3$ **(b)** $y = -f(x) + 2$
(c) $y = f(x + 1) - 2$ **(d)** $y = -f(x - 1)$
(e) $y = f(-x)$ **(f)** $y = f(|x|)$
(g) $y = f(1 - x)$

8. Draw the graph of function $f(x) = [x^2 - x - 1]$, $x \in [-1, 2]$, ([·] represents greatest integer function)

ANSWERS

5. 6 **6.** $\left[\dfrac{-1-\sqrt{5}}{2}, \dfrac{-1+\sqrt{5}}{2}\right]$

Solved Examples

EXAMPLE 1.1

Find the range of $f(x) = \sqrt{\cos(\sin x)} + \sqrt{\sin(\cos x)}$.

Sol. $f(x) = \sqrt{\cos(\sin x)} + \sqrt{\sin(\cos x)}$

Period of $f(x)$ is 2π.

Also, $\sin(\cos x) \ge 0 \Rightarrow \cos x \in [0, 1]$

$$\Rightarrow \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

∴ x lies in 1st and 4th quadrants.

Also $f(-x) = f(x)$

∴ $f(x)$ is even.

∴ we need to find the range in $\left[0, \dfrac{\pi}{2}\right]$ only

In $\left[0, \dfrac{\pi}{2}\right]$, $\sin x$ increases, but $\cos x$ decreases

∴ both $\cos(\sin x)$ and $\sin(\cos x)$ decreases.

Hence $f(x)$ decreases.

∴ Range is $[f(\pi/2), f(0)] \equiv [\sqrt{\cos 1}, 1 + \sqrt{\sin 1}]$

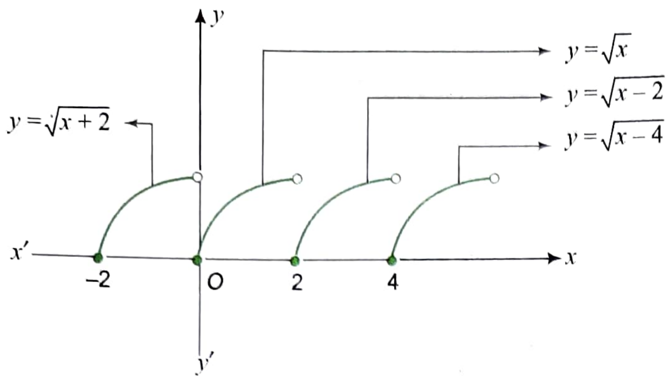
EXAMPLE 1.2

Let $g(x) = \sqrt{x - 2k}$, $\forall\ 2k \le x < 2(k + 1)$, where $k \in Z$. Check whether $g(x)$ is periodic or not.

Sol. $g(x) = \sqrt{x - 2k}$, $\forall\ 2k \le x < 2(k + 1)$, where $k \in Z$.

$$= \begin{cases} \ldots \\ \ldots \\ \sqrt{x+2}, & -2 \le x < 0 \\ \sqrt{x}, & 0 \le x < 2 \\ \sqrt{x-2}, & 2 \le x < 4 \\ \sqrt{x-4}, & 4 \le x < 6 \\ \ldots \\ \ldots \end{cases}$$

Graph of the function is as shown in the figure.



Example 1.3

Let $f(x) = x^2 - 2x, x \in R$, and $g(x) = f(f(x) - 1) + f(5 - f(x))$. Show that $g(x) \geq 0 \,\forall\, x \in R$.

Sol. Given $f(x) = x^2 - 2x = (x-1)^2 - 1$

or $g(x) = f(f(x) - 1) + f(5 - f(x))$

$= f((x-1)^2 - 2) + f(6 - (x-1)^2)$

$= [(x-1)^2 - 2 - 1]^2 - 1 + [6 - (x-1)^2 - 1]^2 - 1$

$= (x-1)^4 - 6(x-1)^2 + 9 - 1 + (x-1)^4$

$- 10(x-1)^2 + 25 - 1$

$= 2(x-1)^4 - 16(x-1)^2 + 32$

$= 2[(x-1)^4 - 8(x-1)^2 + 16]$

$= 2[(x-1)^2 - 4]^2 \geq 0 \,\forall\, x \in R$

Example 1.4

If f and g are two distinct linear functions defined on R such that they map $[-1, 1]$ onto $[0, 2]$ and $h: R - \{-1, 0, 1\} \to R$ defined by $h(x) = \dfrac{f(x)}{g(x)}$, then show that $|h(h(x)) + h(h(1/x))| > 2$.

Sol. Let two linear functions be

$f(x) = ax + b$ and $g(x) = cx + d$.

They map $[-1, 1] \to [0, 2]$ and mapping is onto.

Therefore, $f(-1) = 0$ and $f(1) = 2$

and $g(-1) = 2$ and $g(1) = 0$, i.e.,

$-a + b = 0$ and $a + b = 2$ (1)

and $-c + d = 2$ and $c + d = 0$ (2)

or $a = b = 1$ and $c = -1, d = 1$

or $f(x) = x + 1$ and $g(x) = -x + 1$

or $h(x) = \dfrac{x+1}{1-x}$ or $h(h(x)) = \dfrac{\frac{x+1}{1-x}+1}{\frac{x+1}{1-x}-1} = \dfrac{1}{x}$

or $h(h(1/x)) = x$

or $|h(h(x)) + h(h(1/x))| = |x + 1/x| > 2$

Example 1.5

Let $f(x) = \log_2\log_3\log_4\log_5(\sin x + a^2)$. Find the set of values of a for which the domain of $f(x)$ is R.

Sol. Given $f(x) = \log_2\log_3\log_4\log_5(\sin x + a^2)$.

$f(x)$ is defined only if

$\log_3\log_4\log_5(\sin x + a^2) > 0 \,\forall\, x \in R$

or $\log_4\log_5(\sin x + a^2) > 1 \,\forall\, x \in R$

or $\log_5(\sin x + a^2) > 4 \,\forall\, x \in R$

or $(\sin x + a^2) > 5^4 \,\forall\, x \in R$

or $a^2 > 625 - \sin x \,\forall\, x \in R$

Therefore, a^2 must be greater than maximum value of $625 - \sin x$ which is 626 (when $\sin x = -1$). Therefore,

$a^2 > 626$

or $a \in (-\infty, -\sqrt{626}) \cup (\sqrt{626}, \infty)$

Example 1.6

If f is a polynomial function satisfying $2 + f(x)f(y) = f(x) + f(y) + f(xy) \,\forall\, x, y \in R$ and if $f(2) = 5$, then find the value of $f(f(2))$.

Sol. Put $y = \dfrac{1}{x}$. Then

$$2 + f(x)\, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) + f(1) \quad (1)$$

Now, put $x = 1$. Then

$2 + (f(1))^2 = 3f(1)$

$\therefore$ $f(1) = 1$ or 2

But $f(1) \neq 1$, otherwise from the given relation, $2 + f(x) f(1) = f(x) + f(1) + f(x)$ or $f(x) = 1$, which is not possible as given that $f(2) = 5$.

Hence, $f(1) = 2$.

Therefore, from (1), we have

$$f(x)\, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

or $f(x) = \pm x^n + 1$

$\therefore$ $f(2) = \pm 2^n + 1 = 5$

or $2^n = 4$ or $n = 2$

or $f(x) = x^2 + 1$

or $f(f(2)) = f(5) = 26$

Example 1.7

Let $f: X \to Y$ be a function defined by

$f(x) = a \sin\left(x + \dfrac{\pi}{4}\right) + b\cos x + c$. If f is both one-one and onto, find set Y and one of the possible sets X.

Sol. $f(x) = a\sin\left(x + \dfrac{\pi}{4}\right) + b\cos x + c$

$$= a\left\{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right\} + b\cos x + c$$

$$= \frac{a}{\sqrt{2}}\sin x + \left(\frac{a}{\sqrt{2}} + b\right)\cos x + c \quad (1)$$

Let $\left(\dfrac{a}{\sqrt{2}}\right) = r\cos\alpha,\ \left(\dfrac{a}{\sqrt{2}} + b\right) = r\sin\alpha$

$\therefore$ $f(x) = r[\cos\alpha \sin x + \sin\alpha\cos x] + c$

$= r[\sin(x + \alpha)] + c$

where $r = \sqrt{a^2 + \sqrt{2}ab + b^2}$

and $\alpha = \tan^{-1}\left(\frac{a + b\sqrt{2}}{a}\right)$ (2)

For f to be one-one, we must have $-\pi/2 \le x + \alpha \le \pi/2$. Thus, domain is $\left[\frac{-\pi}{2} - \alpha, \frac{\pi}{2} - \alpha\right]$ and range is $[c - r, c + r]$

or $X \equiv \left[\frac{-\pi}{2} - \alpha, \frac{\pi}{2} - \alpha\right]$ and $Y \equiv [c - r, c + r]$

EXAMPLE 1.8

Let $f: R \to R$, $f(x) = \frac{x-a}{(x-b)(x-c)}$, $b > c$. If f is onto, then prove that $a \in (b, c)$.

Sol. $f(x) = y = \frac{x-a}{(x-b)(x-c)}$

Since $f(x)$ is onto, range must be R.

Now, $yx^2 - [(b + c)y + 1]x + bcy + a = 0$

Now, x is real. Therefore,

$D \ge 0$

or $[(b + c)y + 1]^2 - 4y(bcy + a) \ge 0 \ \forall \ y \in R$

[as given that $f(x)$ is an onto function]

or $(b - c)^2 y^2 + 2(b + c - 2a)y + 1 \ge 0 \ \forall \ y \in R$

$\therefore$ $D \le 0$

or $4(b + c - 2a)^2 - 4(b - c)^2 \le 0$

or $(b + c - 2a - b + c)(b + c - 2a + b - c) \le 0$

or $(c - a)(b - a) \le 0$

i.e., $c \le a$ and $b \ge a$ or $c \ge a$ and $b \le a$

or $c \le a \le b$ (as $b > c$)

or $a \in (c, b)$

EXAMPLE 1.9

If p, q are positive integers, f is a function defined for positive numbers and attains only positive values such that $f(xf(y)) = x^p y^q$, then prove that $p^2 = q$.

Sol. Given $f(xf(y)) = x^p y^q$

or $x = \frac{\{f(xf(y))\}^{1/p}}{y^{q/p}}$ (1)

Let $xf(y) = 1$ or $x = \frac{1}{f(y)}$. Then from (1),

$f(y) = \frac{y^{q/p}}{\{f(1)\}^{1/p}}$

or $f(1) = \frac{1}{\{f(1)\}^{1/p}}$

$\therefore$ $f(1) = 1$

$\therefore$ $f(y) = y^{q/p}$ (2)

Now, $f(x\, y^{q/p}) = x^p y^q$. Put $y^{q/p} = z$. Then

$f(xz) = (xz)^p$

or $f(x) = x^p$ (3)

From (2) and (3), $x^p = x^{q/p}$ or $p^2 = q$.

EXAMPLE 1.10

If $f: R \to [0, \infty)$ is a function such that $f(x - 1) + f(x + 1) = \sqrt{3} f(x)$, then prove that $f(x)$ is periodic and find its period.

Sol. $f(x - 1) + f(x + 1) = \sqrt{3} f(x)$ (1)

Putting $x + 2$ for x in relation (1), we get

$f(x + 1) + f(x + 3) = \sqrt{3} f(x + 2)$ (2)

From (1) and (2), we get

$f(x - 1) + 2f(x + 1) + f(x + 3) = \sqrt{3}(f(x) + f(x + 2))$

$= \sqrt{3}(\sqrt{3} f(x + 1))$

$= 3f(x + 1)$

or $f(x - 1) + f(x + 3) = f(x + 1)$ (3)

Putting $x + 2$ for x in (3), we get

$f(x + 1) + f(x + 5) = f(x + 3)$ (4)

Adding (3) and (4), we get $f(x - 1) = -f(x + 5)$.

Now, put $x + 1$ for x. Then $f(x) = -f(x + 6)$ (5)

Put $x + 6$ in place of x in (5). Then $f(x + 6) = -f(x + 12)$.

Therefore, from (5) again, $f(x) = -[-f(x + 12)] = f(x + 12)$.

Hence, the period of $f(x)$ is 12.

EXAMPLE 1.11

If a, b are two fixed positive integers such that $f(a + x) = b + [b^3 + 1 - 3b^2 f(x) + 3b\{f(x)\}^2 - \{f(x)\}^3]^{1/3}$ for all real x, then prove that $f(x)$ is periodic and find its period.

Sol. $f(a + x) = b + [1 + b^3 - 3b^2 f(x) + 3b\{f(x)\}^2 - \{f(x)\}^3]^{1/3}$

$= b + [1 + \{b - f(x)\}^3]^{1/3}$

or $f(a + x) - b = [1 - \{f(x) - b\}^3]^{1/3}$

or $\phi(a + x) = [1 - \{\phi(x)\}^3]^{1/3}$ (1)

where $\phi(x) = f(x) - b$

or $\phi(2a + x) = [1 - \{\phi(x + a)\}^3]^{1/3} = \phi(x)$ [From (1)]

or $f(x + 2a) - b = f(x) - b$

or $f(x + 2a) = f(x)$

Therefore, $f(x)$ is periodic with period $2a$.

EXAMPLE 1.12

Let $f(x, y)$ be a periodic function, satisfying the condition $f(x, y) = f(2x + 2y, 2y - 2x)$ $\forall$ $x, y \in R$ and let $g(x)$ be a function defined as $g(x) = f(2^x, 0)$. Prove that $g(x)$ is a periodic function and find its period.

Sol. $f(x, y) = f(2x + 2y, 2y - 2x)$

(Replacing x by $2x + 2y$ and y by $2y - 2x$)

$= f(2(2x + 2y) + 2(2y - 2x), 2(2y - 2x) - 2(2x + 2y))$

$\therefore$ $f(x, y) = f(8y, -8x) = f(8(-8x), -8(8y))$

$$= f(-64x, -64y)$$
$$= f(-64(-64x), -64y(-64y))$$
$$= f(2^{12}x, 2^{12}y)$$

$\therefore \quad f(x, 0) = f(2^{12}x, 0)$

$\therefore \quad f(2^y, 0) = f(2^{12} \cdot 2^y, 0) = f(2^{12+y}, 0)$

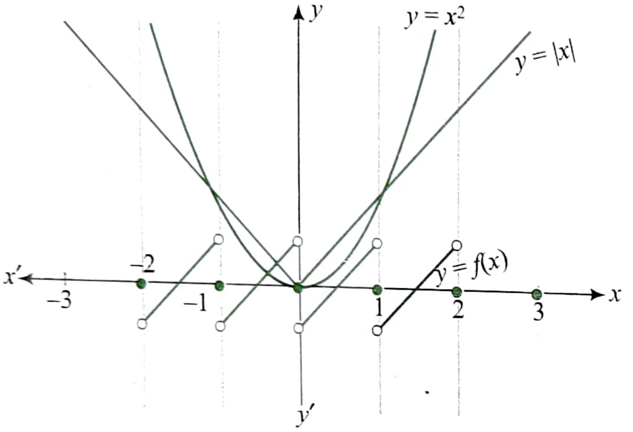
or $\quad g(y) = g(y + 12)$

Hence, $g(x)$ is periodic and its period is 12.

Example 1.13

Consider the function $f(x) = \begin{cases} x - [x] - \dfrac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$, where [.] denotes the fractional integral function and I is the set of integers. Then find $g(x) = \max.\{x^2, f(x), |x|\}$; $-2 \le x \le 2$.

Sol. Clearly, from graph,



$$g(x) = \begin{cases} x^2, & -2 \le x \le -1 \\ -x, & -1 < x \le -1/4 \\ \frac{1}{2} + x, & -\frac{1}{4} < x < 0 \\ x, & 0 \le x < 1 \\ x^2, & 1 \le x \le 2 \end{cases}$$

Example 1.14

Let $f(x)$ be defined on $[-2, 2]$ and be given by

$f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases}$ and $g(x) = f(|x|) + |f(x)|$.

Then find $g(x)$.

Sol. We have

$$f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases}$$

or $$f(|x|) = \begin{cases} -1, & -2 \le |x| \le 0 \\ |x| - 1, & 0 \le |x| \le 2 \end{cases}$$

$$= |x| - 1, \ 0 \le |x| \le 2$$

(As $-2 \le |x| < 0$ is not possible)

$$= \begin{cases} -x - 1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases} \qquad (1)$$

Again, $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases}$

or $$|f(x)| = \begin{cases} |-1|, & -2 \le x \le 0 \\ |x - 1|, & 0 < x \le 2 \end{cases}$$

or $$|f(x)| = \begin{cases} 1, & -2 \le x \le 0 \\ -(x-1), & 0 < x \le 1 \\ +(x-1), & 1 < x \le 2 \end{cases} \qquad (2$$

Therefore, $g(x) = f(|x|) + |f(x)|$ can be expressed as

$$g(x) = \begin{cases} (-x-1) + 1, & -2 \le x \le 0 \\ (x-1) + (1-x), & 0 < x \le 1 \\ (x-1) + (x-1), & 1 < x \le 2 \end{cases} \quad \text{[Using (1) and (2)]}$$

$$= \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 < x \le 1 \\ 2(x-1), & 1 < x \le 2 \end{cases}$$

Example 1.15

If $f(x) = \dfrac{a^x}{a^x + \sqrt{a}}$, $(a > 0)$, then find the value of $\displaystyle\sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right)$.

Sol. $$f(x) = \frac{a^x}{a^x + \sqrt{a}}$$

or $$f(1-x) = \frac{a^{1-x}}{a^{1-x} + \sqrt{a}}$$

$$= \frac{a^1}{a^1 + \sqrt{a}a^x}$$

$$= \frac{\sqrt{a}}{\sqrt{a} + a^x}$$

or $$f(x) + f(1-x) = 1.$$

$\therefore$ $$f\left(\frac{1}{2}\right) = \frac{1}{2}$$

so, $$\sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right) = 2\left[\begin{array}{l} f\left(\frac{1}{2n}\right) + f\left(\frac{2}{2n}\right) + \cdots + f\left(\frac{n-1}{2n}\right) \\ + f\left(\frac{n}{2n}\right) + f\left(\frac{n+1}{2n}\right) + \cdots + f\left(\frac{2n-1}{2n}\right) \end{array}\right]$$

$$= 2\left\{\begin{array}{l} \left[f\left(\frac{1}{2n}\right) + f\left(\frac{2n-1}{2n}\right)\right] + \left[f\left(\frac{2}{2n}\right) + f\left(\frac{2n-2}{2n}\right)\right] \\ + \cdots + \left[f\left(\frac{n-1}{2n}\right) + f\left(\frac{n+1}{2n}\right)\right] + f\left(\frac{1}{2}\right) \end{array}\right\}$$

$$= 2\left\{\begin{array}{l} \left[f\left(\frac{1}{2n}\right) + f\left(1 - \frac{1}{2n}\right)\right] + \left[f\left(\frac{2}{2n}\right) + f\left(1 - \frac{2}{2n}\right)\right] \\ + \cdots + \left[f\left(\frac{n-1}{2n}\right) + f\left(1 - \frac{n-1}{2n}\right)\right] + \frac{1}{2} \end{array}\right\}$$

$$= 2[1 + 1 + 1 + \cdots + (n-1) \text{ times}] + 1$$

$$= 2n - 1$$

Exercises

Single Correct Answer Type

1. The relation $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on set $A = \{1, 2, 3\}$ is
   (1) Reflexive but not symmetric
   (2) Reflexive but not transitive
   (3) Symmetric and transitive
   (4) Neither symmetric nor transitive

2. Let $P = \{(x, y) \mid x^2 + y^2 = 1, x, y \in R\}$. Then, P is
   (1) Reflexive (2) Symmetric
   (3) Transitive (4) Anti-symmetric

3. Let R be an equivalence relation on a finite set A having n elements. Then, the number of ordered pairs in R is
   (1) Less than n (2) Greater than or equal to n
   (3) Less than or equal to n (4) None of these

4. A relation R on the set of complex numbers defined by $Z_1 R Z_2 \Leftrightarrow \dfrac{Z_1 - Z_2}{Z_1 + Z_2}$ is real then which of the following is not true?
   (1) R is reflexive (2) R is symmetric
   (3) R is transitive (4) R is not equivalence

5. Which one of the following relations on R is an equivalence relation?
   (1) $a R_1 b \Leftrightarrow |a| = |b|$ (2) $a R_2 b \Leftrightarrow a \ge b$
   (3) $a R_3 b \Leftrightarrow a$ divides b (4) $a R_4 b \Leftrightarrow a < b$

6. Let R be the relation on the set of all real numbers defined by aRb iff $|a - b| \le 1$. Then, R is
   (1) Reflexive and symmetric
   (2) Symmetric only
   (3) Transitive only
   (4) None of these

7. The function $f: N \to N$ (N is the set of natural numbers) defined by $f(n) = 2n + 3$ is
   (1) surjective only (2) injective only
   (3) bijective (4) none of these

8. $f: N \to N$, where $f(x) = x - (-1)^x$. Then f is
   (1) one-one and into
   (2) many-one and into
   (3) one-one and onto
   (4) many-one and onto

9. Let S be the set of all triangles and R^+ be the set of positive real numbers. Then the function $f: S \to R^+$, $f(\Delta)$ = area of Δ, where $\Delta \in S$, is
   (1) injective but not surjective
   (2) surjective but not injective
   (3) injective as well as surjective
   (4) neither injective nor surjective

10. The function $f: (-\infty, -1] \to (0, e^5]$ defined by $f(x) = e^{x^3 - 3x + 2}$ is
    (1) many-one and onto (2) many-one and into
    (3) one-one and onto (4) one-one and into

11. Let $f: N \to N$ be defined by $f(x) = x^2 + x + 1, x \in N$. Then f is
    (1) one-one onto (2) many-one onto
    (3) one-one but not onto (4) none of these

12. Let $X = \{a_1, a_2, \ldots, a_6\}$ and $Y = \{b_1, b_2, b_3\}$. The number of functions f from X to Y such that it is onto and there are exactly three elements x in X such that $f(x) = b_1$ is
    (1) 75 (2) 90 (3) 100 (4) 120

13. Which of the following functions is an injective (one-one) function in its respective domain?
    (1) $f(x) = 2x + \sin 3x$
    (2) $f(x) = x \cdot [x]$, (where [.] denotes the G.I.F)
    (3) $f(x) = \dfrac{2^x - 1}{4^x + 1}$
    (4) $f(x) = \dfrac{2^x + 1}{4^x - 1}$

14. Given the function $f(x) = \dfrac{a^x + a^{-x}}{2}$ (where $a > 2$). Then $f(x + y) + f(x - y) =$
    (1) $2f(x) \cdot f(y)$ (2) $f(x) \cdot f(y)$
    (3) $\dfrac{f(x)}{f(y)}$ (4) none of these

15. If $f(x) = \cos(\log_e x)$, then $f(x)f(y) - \dfrac{1}{2}\left[f\left(\dfrac{x}{y}\right) + f(xy)\right]$ has value
    (1) -1 (2) 1/2
    (3) -2 (4) 0

16. The domain of the function $f(x) = \dfrac{1}{\sqrt{{}^{10}C_{x-1} - 3 \times {}^{10}C_x}}$ is
    (1) $\{9, 10, 11\}$ (2) $\{9, 10, 12\}$
    (3) all natural numbers (4) $\{9, 10\}$

17. The domain of the function $f(x) = \dfrac{\sin^{-1}(3 - x)}{\ln(|x| - 2)}$ is
    (1) $[2, 4]$ (2) $(2, 3) \cup (3, 4]$
    (3) $[2, \infty)$ (4) $(-\infty, -3) \cup [2, \infty)$

18. The domain of $f(x) = \dfrac{\log_2(x + 3)}{x^2 + 3x + 2}$ is
    (1) $R - \{-1, -2\}$ (2) $(-2, \infty)$
    (3) $R - \{-1, -2, -3\}$ (4) $(-3, \infty) - \{-1, -2\}$

19. The domain of the function $f(x) = \sqrt{x^2 - [x]^2}$, where $[x]$ is the greatest integer less than or equal to x, is
    (1) R (2) $[0, +\infty)$
    (3) $(-\infty, 0]$ (4) none of these

20. The domain of the function $f(x) = \log_{3+x}(x^2 - 1)$ is
(1) $(-3, -1) \cup (1, \infty)$
(2) $[-3, -1) \cup [1, \infty)$
(3) $(-3, -2) \cup (-2, -1) \cup (1, \infty)$
(4) $[-3, -2) \cup (-2, -1) \cup [1, \infty)$

21. The domain of the function $f(x) = \left[\log_{10}\left(\dfrac{5x - x^2}{4}\right)\right]^{1/2}$ is
(1) $-\infty < x < \infty$ (2) $1 \le x \le 4$
(3) $4 \le x \le 16$ (4) $-1 \le x \le 1$

22. The domain of $f(x) = \log|\log x|$ is
(1) $(0, \infty)$ (2) $(1, \infty)$
(3) $(0, 1) \cup (1, \infty)$ (4) $(-\infty, 1)$

23. If $x^3 f(x) = \sqrt{1 + \cos 2x} + |f(x)|, \dfrac{-3\pi}{4} < x < \dfrac{-\pi}{2}$ and $f(x) = \dfrac{\alpha \cos x}{1 + x^3}$, then the value of α is
(1) 2 (2) $-\sqrt{2}$
(3) $\sqrt{2}$ (4) 1

24. The function $f(x) = \dfrac{\sec^{-1} x}{\sqrt{x - [x]}}$, where $[x]$ denotes the greatest integer less than or equal to x, is defined for all $x \in$
(1) R (2) $R - \{(-1, 1) \cup \{n \mid n \in Z\}\}$
(3) $R^+ - (0, 1)$ (4) $R^+ - \{n \mid n \in N\}$

25. The domain of definition of the function $f(x)$ given by the equation $2^x + 2^y = 2$ is
(1) $0 < x \le 1$ (2) $0 \le x \le 1$
(3) $-\infty < x \le 0$ (4) $-\infty < x < 1$

26. The domain of $f(x) = \cos^{-1}\left(\dfrac{2 - |x|}{4}\right) + [\log(3 - x)]^{-1}$ is
(1) $[-2, 6]$ (2) $[-6, 2) \cup (2, 3)$
(3) $[-6, 2]$ (4) $[-2, 2] \cup (2, 3)$

27. The domain of the function $f(x) = \sqrt{\log\left(\dfrac{1}{|\sin x|}\right)}$
(1) $R - \{-\pi, \pi\}$ (2) $R - \{n\pi \mid n \in Z\}$
(3) $R - \{2n\pi \mid n \in z\}$ (4) $(-\infty, \infty)$

28. The domain of the following function is
$f(x) = \log_2\left(-\log_{1/2}\left(1 + \dfrac{1}{x^{1/4}}\right) - 1\right)$
(1) $(0, 1)$ (2) $(0, 1]$
(3) $[1, \infty)$ (4) $(1, \infty)$

29. The number of real solutions of the $\log_{0.5}|x| = 2|x|$ is
(1) 1 (2) 2
(3) 0 (4) none of these

30. Let $f: R \to \left[0, \dfrac{\pi}{2}\right)$ be defined by $f(x) = \tan^{-1}(x^2 + x + a)$.
Then the set of values of a for which f is onto is
(1) $[0, \infty)$ (2) $[2, 1]$
(3) $\left[\dfrac{1}{4}, \infty\right)$ (4) none of these

31. The domain of the function $f(x) = \sqrt{\ln_{(|x|-1)}(x^2 + 4x + 4)}$ is
(1) $[-3, -1] \cup [1, 2]$
(2) $(-2, -1) \cup [2, \infty)$
(3) $(-\infty, -3] \cup (-2, -1) \cup (2, \infty)$
(4) none of these

32. The domain of $f(x) = \ln(ax^3 + (a + b)x^2 + (b + c)x + c)$, where $a > 0$, $b^2 - 4ac = 0$, is (where $[\cdot]$ represents greatest integer function)
(1) $(-1, \infty) \sim \left\{-\dfrac{b}{2a}\right\}$ (2) $(1, \infty) \sim \left\{-\dfrac{b}{2a}\right\}$
(3) $(-1, 1) \sim \left\{-\dfrac{b}{2a}\right\}$ (4) none of these

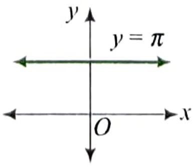
33. The domain of the function $f(x) = \dfrac{1}{\sqrt{4x - |x^2 - 10x + 9|}}$ is
(1) $(7 - \sqrt{40}, 7 + \sqrt{40})$ (2) $(0, 7 + \sqrt{40})$
(3) $(7 - \sqrt{40}, \infty)$ (4) none of these

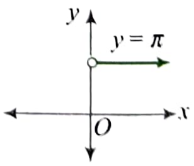
34. The domain of $f(x) = \dfrac{1}{\sqrt{|\cos x| + \cos x}}$ is
(1) $[-2n\pi, 2n\pi], n \in Z$
(2) $(2n\pi, \overline{2n + 1}\,\pi), n \in Z$
(3) $\left(\dfrac{(4n+1)\pi}{2}, \dfrac{(4n+3)\pi}{2}\right), n \in Z$
(4) $\left(\dfrac{(4n-1)\pi}{2}, \dfrac{(4n+1)\pi}{2}\right), n \in Z$

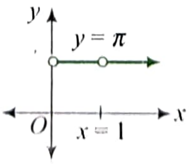
35. The exhaustive domain of the following function is
$f(x) = \sqrt{x^{12} - x^9 + x^4 - x + 1}$
(1) $[0, 1]$ (2) $[1, \infty)$ (3) $(-\infty, 1]$ (4) R

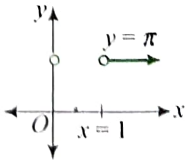
36. The domain of the function
$f(x) = \sqrt{\sin x + \cos x} + \sqrt{7x - x^2 - 6}$ is
(1) $[1, 6]$ (2) $\left[1, \dfrac{3\pi}{4}\right] \cup \left[\dfrac{7\pi}{4}, 6\right]$
(3) $[1, \pi] \cup \left[\dfrac{7\pi}{4}, 6\right]$ (4) none of these

37. Which one of following best represent the graph of $y = x^{\log_x \pi}$?

(1)


(2)


(3)


(4)


38. If x is real, then the values of the expression $\frac{x^2+14x+9}{x^2+2x+3}$ are
(1) [4, 5] (2) [–4, 5]
(3) [–5, 4] (4) none of these

39. The range of the function $f(x)=|x-1|+|x-2|, -1 \le x \le 3$, is
(1) [1, 3] (2) [1, 5]
(3) [3, 5] (4) none of these

40. The function $f: R \to R$ is defined by $f(x)=\cos^2 x+\sin^4 x$ for $x \in R$. Then the range of $f(x)$ is
(1) $\left(\frac{3}{4}, 1\right]$ (2) $\left[\frac{3}{4}, 1\right)$
(3) $\left[\frac{3}{4}, 1\right]$ (4) $\left(\frac{3}{4}, 1\right)$

41. The range of $f(x)=[\,|\sin x|+|\cos x|\,]$, where [.] denotes the greatest integer function, is
(1) {0} (2) {0, 1}
(3) {1} (4) none of these

42. The range of the function $f(x)={}^{7-x}P_{x-3}$ is
(1) {1, 2, 3} (2) {1, 2, 3, 4, 5, 6}
(3) {1, 2, 3, 4} (4) {1, 2, 3, 4, 5}

43. The range of $f(x)=\sin^{-1}\left(\frac{x^2+1}{x^2+2}\right)$ is
(1) $[0, \pi/2]$ (2) $(0, \pi/6)$
(3) $[\pi/6, \pi/2)$ (4) none of these

44. The range of the function $f(x)=\frac{e^x-e^{|x|}}{e^x+e^{|x|}}$ is
(1) $(-\infty, \infty)$ (2) [0, 1)
(3) (–1, 0] (4) (–1, 1)

45. Domain (D) and range (R) of $f(x)=\sin^{-1}(\cos^{-1}[x])$, where [.] denotes the greatest integer function, is
(1) $D \equiv x \in [1, 2), R \equiv \{0\}$
(2) $D \equiv x \in [0, 1], R \equiv \{-1, 0, 1\}$
(3) $D \equiv x \in [-1, 1], R \equiv \left\{0, \sin^{-1}\left(\frac{\pi}{2}\right), \sin^{-1}(\pi)\right\}$
(4) $D \equiv x \in [-1, 1], R \equiv \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$

46. The range of the function f defined by $f(x)=\left[\frac{1}{\sin\{x\}}\right]$ (where [.] and {.}, respectively, denote the greatest integer and the fractional part functions) is
(1) I, the set of integers
(2) N, the set of natural numbers
(3) W, the set of whole numbers
(4) {1, 2, 3, 4, ...}

47. Range of the function $f(x)=\cos(K\sin x)$ is [–1, 1], then the least positive integral value of K will be
(1) 1 (2) 2 (3) 3 (4) 4

48. Let $f(x)=\sqrt{|x|-\{x\}}$ (where {.} denotes the fractional part of x) and X, Y are its domain and range, respectively). Then
(1) $x \in \left(-\infty, \frac{1}{2}\right]$ and $Y \in \left[\frac{1}{2}, \infty\right)$
(2) $x \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $Y \in \left[\frac{1}{2}, \infty\right)$
(3) $X \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $Y \in [0, \infty)$
(4) none of these

49. The range of $f(x)=\cos^{-1}\left(\frac{1+x^2}{2x}\right)+\sqrt{2-x^2}$ is
(1) $\left\{0, 1+\frac{\pi}{2}\right\}$ (2) $\{0, 1+\pi\}$
(3) $\left\{1, 1+\frac{\pi}{2}\right\}$ (4) $\{1, 1+\pi\}$

50. The range of the following function is
$$f(x)=\sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{\cdots\infty}}}}$$
(1) [0, 1] (2) [0, 1/2]
(3) [0, 2) (4) none of these

51. The range of $f(x)=(x+1)(x+2)(x+3)(x+4)+5$ for $x \in [-6, 6]$ is
(1) [4, 5045] (2) [0, 5045]
(3) [–20, 5045] (4) none of these

52. The range of $f(x)=\sec^{-1}(\log_3 \tan x+\log_{\tan x} 3)$ is
(1) $\left[\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$
(2) $\left[0, \frac{\pi}{2}\right)$
(3) $\left(\frac{2\pi}{3}, \pi\right]$
(4) none of these

53. The domain of definition of the function $f(x)=\{x\}^{\{x\}}+[x]^{[x]}$ is (where {·} represents fractional part and [·] represents greatest integral function)
(1) $R-I$ (2) $R-[0, 1)$
(3) $R-\{I \cup (0, 1)\}$ (4) $I \cup (0, 1)$

54. If $[x^2-2x+a]=0$ has no solution then, where [·] represents greatest integer function,
(1) $-\infty < a < 1$ (2) $2 \le a < \infty$
(3) $1 < a < 2$ (4) $a \in R$

55. If $[x]$ and $\{x\}$ represent the integral and fractional parts of x, respectively, then the value of $\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}$ is
(1) x (2) $[x]$
(3) $\{x\}$ (4) $x+2001$

56. If $f(x)=\dfrac{\sin([x]\pi)}{x^2+x+1}$, where [.] denotes the greatest integer function, then

(1) f is one-one
(2) f is not one-one and non-constant
(3) f is a constant function
(4) none of these

57. Let $f(x)=([a]^2-5[a]+4)x^3-(6\{a\}^2-5\{a\}+1)x-(\tan x)\times \operatorname{sgn} x$ be an even function for all $x\in R$. Then the sum of all possible values of a is (where $[\cdot]$ and $\{\cdot\}$ denote greatest integer function and fractional part function, respectively)

(1) $\dfrac{17}{6}$ (2) $\dfrac{53}{6}$
(3) $\dfrac{31}{3}$ (4) $\dfrac{35}{3}$

58. The solution set for $[x]\ \{x\} = 1$ (where $\{x\}$ and $[x]$ are, respectively, fractional part function and greatest integer function) is

(1) $R^+-(0,1)$
(2) $R^+-\{1\}$
(3) $\left\{m+\dfrac{1}{m}\ /\ m\in I-\{0\}\right\}$
(4) $\left\{m+\dfrac{1}{m}\ /\ m\in N-\{1\}\right\}$

59. Let $[x]$ represent the greatest integer less than or equal to x. If $[\sqrt{n^2+\lambda}]=[\sqrt{n^2+1}]+2$, where $\lambda, n\in N$, then λ can assume

(1) $(2n+4)$ different values
(2) $(2n+5)$ different values
(3) $(2n+3)$ different values
(4) $(2n+6)$ different values

60. The number of roots of $x^2-2=[\sin x]$, where $[\cdot]$ stands for the greatest integer function, is

(1) 0 (2) 1 (3) 2 (4) 3

61. The domain of $f(x)=\sin^{-1}[2x^2-3]$, where [.] denotes the greatest integer function, is

(1) $\left(-\sqrt{\dfrac{3}{2}},\sqrt{\dfrac{3}{2}}\right)$
(2) $\left(-\sqrt{\dfrac{3}{2}},-1\right]\cup\left(-\sqrt{\dfrac{5}{2}},\sqrt{\dfrac{5}{2}}\right)$
(3) $\left(-\sqrt{\dfrac{5}{2}},\sqrt{\dfrac{5}{2}}\right)$
(4) $\left(-\sqrt{\dfrac{5}{2}},-1\right]\cup\left[1,\sqrt{\dfrac{5}{2}}\right)$

62. The domain of $f(x)=\sqrt{2\{x\}^2-3\{x\}+1}$, where {.} denotes the fractional part in [– 1, 1], is

(1) $[-1,1]\sim\left(\dfrac{1}{2},1\right)$
(2) $\left[-1,-\dfrac{1}{2}\right]\cup\left[0,\dfrac{1}{2}\right]\cup\{1\}$
(3) $\left[-1,\dfrac{1}{2}\right]$
(4) $\left[-\dfrac{1}{2},1\right]$

63. The range of $\sin^{-1}\left[x^2+\dfrac{1}{2}\right]+\cos^{-1}\left[x^2-\dfrac{1}{2}\right]$, where [.] denotes the greatest integer function, is

(1) $\left\{\dfrac{\pi}{2},\pi\right\}$ (2) $\{\pi\}$
(3) $\left\{\dfrac{\pi}{2}\right\}$ (4) none of these

64. Let $f(x)=e^{\{e^{|x|}\operatorname{sgn} x\}}$ and $g(x)=e^{[e^{|x|}\operatorname{sgn} x]}$, $x\in R$, where { } and [ ] denote the fractional and integral part functions, respectively. Also, $h(x)=\log(f(x))+\log(g(x))$. Then for real x, $h(x)$ is

(1) an odd function
(2) an even function
(3) neither an odd nor an even function
(4) both odd and even function

65. The number of solutions of the equation $[y+[y]]=2\cos x$, where $y=\dfrac{1}{3}[\sin x+[\sin x+[\sin x]]]$ (where [.] denotes the greatest integer function) is

(1) 4 (2) 2 (3) 3 (4) 0

66. The function $f(x)=\sin\left(\log(x+\sqrt{1+x^2})\right)$ is

(1) even function (2) odd function
(3) neither even nor odd (4) periodic function

67. If $f(x)=\sqrt[n]{x^m}$, $m\in N$, is an even function, then m is

(1) even integer (2) odd integer
(3) any integer (4) $f(x)$-even is not possible

68. If $f(x)=\begin{cases}x^2\sin\dfrac{\pi x}{2}, & |x|<1\\ x|x|, & |x|\ge 1\end{cases}$, then $f(x)$ is

(1) an even function (2) an odd function
(3) a periodic function (4) none of these

69. If the graph of the function $f(x)=\dfrac{a^x-1}{x^n(a^x+1)}$ is symmetrical about the y-axis, then n can be

(1) 2 (2) $\dfrac{2}{3}$ (3) $\dfrac{4}{3}$ (4) $-\dfrac{1}{3}$

0. If $f: R \to R$ is an invertible function such that $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = -x$, then
   (1) $f(x)$ is odd
   (2) $f(x)$ and $f^{-1}(x)$ may not be symmetric about the line $y = x$
   (3) $f(x)$ may not be odd
   (4) none of these

71. If $f(x) = ax^7 + bx^3 + cx - 5$, a, b, c are real constants, and $f(-7) = 7$, then the range of $f(7) + 17 \cos x$ is
   (1) $[-34, 0]$ (2) $[0, 34]$
   (3) $[-34, 34]$ (4) none of these

72. If $g: [-2, 2] \to R$, where $f(x) = x^3 + \tan x + \left[\dfrac{x^2+1}{P}\right]$ is an odd function, then the value of parametric P, where [.] denotes the greatest integer function, is
   (1) $-5 < P < 5$ (2) $P < 5$
   (3) $P > 5$ (4) none of these

73. Let $f: [-10, 10] \to R$, where $f(x) = \sin x + [x^2/a]$, be an odd function. Then the set of values of parameter a is/are
   (1) $(-10, 10) \sim \{0\}$ (2) $(0, 10)$
   (3) $[100, \infty)$ (4) $(100, \infty)$

74. $f(x) = \dfrac{\cos x}{\left[\dfrac{2x}{\pi}\right] + \dfrac{1}{2}}$, where x is not an integral multiple of π and [.] denotes the greatest integer function, is
   (1) an odd function (2) an even function
   (3) neither odd nor even (4) none of these

75. Let $f(x) = \begin{cases} \sin x + \cos x, & 0 < x < \dfrac{\pi}{2} \\ a, & x = \pi/2 \\ \tan^2 x + \operatorname{cosec} x, & \pi/2 < x < \pi \end{cases}$.

Then its odd extension is

(1) $\begin{cases} -\tan^2 x - \operatorname{cosec} x, & -\pi < x < -\dfrac{\pi}{2} \\ -a, & x = -\dfrac{\pi}{2} \\ -\sin x + \cos x, & -\dfrac{\pi}{2} < x < 0 \end{cases}$

(2) $\begin{cases} -\tan^2 x + \operatorname{cosec} x, & -\pi < x < -\dfrac{\pi}{2} \\ -a, & x = -\dfrac{\pi}{2} \\ \sin x - \cos x, & -\dfrac{\pi}{2} < x < 0 \end{cases}$

(3) $\begin{cases} -\tan^2 x + \operatorname{cosec} x, & -\pi < x < -\dfrac{\pi}{2} \\ a, & x = -\dfrac{\pi}{2} \\ \sin x - \cos x, & -\dfrac{\pi}{2} < x < 0 \end{cases}$

(4) $\begin{cases} \tan^2 x + \cos x, & -\pi < x < -\dfrac{\pi}{2} \\ -a, & x = -\dfrac{\pi}{2} \\ \sin x + \cos x, & -\dfrac{\pi}{2} < x < 0 \end{cases}$

76. The period of the function $\left|\sin^3 \dfrac{x}{2}\right| + \left|\cos^5 \dfrac{x}{5}\right|$ is
   (1) 2π (2) 10π (3) 8π (4) 5π

77. If f is periodic, g is polynomial function, $f(g(x))$ is periodic, $g(2) = 3$, and $g(4) = 7$, then $g(6)$ is
   (1) 13 (2) 15
   (3) 11 (4) none of these

78. The period of function $2^{\{x\}} + \sin \pi x + 3^{\{x/2\}} + \cos 2\pi x$ (where $\{x\}$ denotes the fractional part of x) is
   (1) 2 (2) 1
   (3) 3 (4) none of these

79. The period of the function $f(x) = [6x + 7] + \cos \pi x - 6x$, where [.] denotes the greatest integer function, is
   (1) 3 (2) 2π
   (3) 2 (4) none of these

80. If $f(x)$ and $g(x)$ are periodic functions with periods 7 and 11, respectively, then the period of $F(x) = f(x)\, g\left(\dfrac{x}{5}\right) - g(x)\, f\left(\dfrac{x}{3}\right)$ is
   (1) 177 (2) 222 (3) 433 (4) 1155

81. The fundamental period of the function
$f(x) = c^{\sin^2 x + \sin^2\left(x+\frac{\pi}{3}\right) + \cos x \cos\left(x+\frac{\pi}{3}\right)}$ is (where c is constant)
   (1) 1 (2) $\dfrac{\pi}{2}$
   (3) π (4) None of these

82. Let $f(x) = \{(0.1)3^{[x]}\}$. (where [.] denotes greatest integer function and {.} denotes fractional part) If $f(x + T) = f(x)\ \forall\ x \geq 0$, where T is a fixed positive number then the least value of T is
   (1) 2 (2) 4
   (3) 6 (4) none of these

83. If the period of $\dfrac{\cos(\sin(nx))}{\tan\left(\dfrac{x}{n}\right)}$, $n \in N$, is 6π, then $n =$
   (1) 3 (2) 2 (3) 6 (4) 1

84. The period of $f(x) = [x] + [2x] + [3x] + [4x] + \cdots [nx] - \dfrac{n(n+1)}{2}x$, where $n \in N$, is (where [·] represents greatest integer function)
   (1) n (2) 1
   (3) $\dfrac{1}{n}$ (4) none of these

85. If $f(x) = (-1)^{\left[\frac{2x}{\pi}\right]}$, $g(x) = |\sin x| - |\cos x|$, and $\phi(x) = f(x)\, g(x)$ (where [.] denotes the greatest integer function), then the respective fundamental periods of $f(x)$, $g(x)$, and $\phi(x)$ are

(1) π, π, π (2) $\pi, 2\pi, \pi$

(3) $\pi, \pi, \frac{\pi}{2}$ (4) $\pi, \frac{\pi}{2}, \pi$

86. If $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x^2}$, and $h(x) = x^2$, then

(1) $fog(x) = x^2, x \neq 0, h(g(x)) = \frac{1}{x^2}$

(2) $h(g(x)) = \frac{1}{x^2}, x \neq 0, fog(x) = x^2$

(3) $fog(x) = x^2, x \neq 0, h(g(x)) = (g(x))^2, x \neq 0$

(4) none of these

87. If $f(x) = \begin{cases} x^2, & \text{for } x \geq 0 \\ x, & \text{for } x < 0 \end{cases}$, then $fof(x)$ is given by

(1) x^2 for $x \geq 0$, x for $x < 0$

(2) x^4 for $x \geq 0$, x^2 for $x < 0$

(3) x^4 for $x \geq 0$, $-x^2$ for $x < 0$

(4) x^4 for $x \geq 0$, x for $x < 0$

88. Let $f(x) = \sin x$ and $g(x) = \log_e|x|$. If the ranges of the composition functions fog and gof are R_1 and R_2, respectively, then

(1) $R_1 = \{u: -1 \leq u < 1\}, R_2 = \{v: -\infty < v < 0\}$

(2) $R_1 = \{u: -\infty < u < 0\}, R_2 = \{v: -\infty < v < 0\}$

(3) $R_1 = \{u: -1 < u < 1\}, R_2 = \{v: -\infty < v < 0\}$

(4) $R_1 = \{u: -1 \leq u \leq 1\}, R_2 = \{v: -\infty < v \leq 0\}$

89. If $f(x) = \begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases}$, then $f(f(x))$ is

(1) $x \,\forall\, x \in R$

(2) $\begin{cases} x, & x \text{ is irrational} \\ 1-x, & x \text{ is rational} \end{cases}$

(3) $\begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases}$

(4) none of these

90. If f and g are one-one functions, then

(1) $f + g$ is one-one (2) fg is one-one

(3) fog is one-one (4) none of these

91. The domain of $f(x)$ is $(0, 1)$. Then the domain of $f(e^x) + f(\ln |x|)$ is

(1) $(-1, e)$ (2) $(1, e)$

(3) $(-e, -1)$ (4) $(-e, 1)$

92. Let $h(x) = |kx + 5|$, the domain of $f(x)$ be $[-5, 7]$, the domain of $f(h(x))$ be $[-6, 1]$, and the range of $h(x)$ be the same as the domain of $f(x)$. Then the value of k is

(1) 1 (2) 2 (3) 3 (4) 4

93. If $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$, then $g(f(x))$ is invertible in the domain

(1) $\left[0, \frac{\pi}{2}\right]$ (2) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(3) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (4) $[0, \pi]$

94. If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is

(1) $\left(\frac{1}{2}\right)^{x(x-1)}$ (2) $\frac{1}{2}\left(1 + \sqrt{1+4 \log_2 x}\right)$

(3) $\frac{1}{2}\left(1 - \sqrt{1 + 4\log_2 x}\right)$ (4) not defined

95. Let $f(x) = (x + 1)^2 - 1, x \geq -1$. Then the set $\{x: f(x) = f^{-1}(x)\}$ is

(1) $\left\{0, -1, \frac{-3 + i\sqrt{3}}{2}, \frac{-3 - i\sqrt{3}}{2}\right\}$

(2) $\{0, 1, -1\}$

(3) $\{0, -1\}$

(4) empty

96. If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals

(1) $\frac{\left(x + \sqrt{x^2 - 4}\right)}{2}$ (2) $\frac{x}{1+x^2}$

(3) $\frac{\left(x - \sqrt{x^2 - 4}\right)}{2}$ (4) $1 + \sqrt{x^2 - 4}$

97. Suppose $f(x) = (x + 1)^2$ for $x \geq -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals

(1) $1 - \sqrt{x} - 1, x \geq 0$ (2) $\frac{1}{(x+1)^2}, x > -1$

(3) $\sqrt{x+1}, x \geq -1$ (4) $\sqrt{x} - 1, x \geq 0$

98. Let $f: \left[-\frac{\pi}{3}, \frac{2\pi}{3}\right] \to [0, 4]$ be a function defined as $f(x) = \sqrt{3} \sin x - \cos x + 2$. Then $f^{-1}(x)$ is given by

(1) $\sin^{-1}\left(\frac{x-2}{2}\right) - \frac{\pi}{6}$

(2) $\sin^{-1}\left(\frac{x-2}{2}\right) + \frac{\pi}{6}$

(3) $\frac{2\pi}{3} + \cos^{-1}\left(\frac{x-2}{2}\right)$

(4) none of these

99. Which of the following functions is the inverse of itself?

(1) $f(x) = \frac{1-x}{1+x}$ (2) $f(x) = 5^{\log x}$

(3) $f(x) = 2^{x(x-1)}$ (4) None of these

). If $g(x) = x^2 + x - 2$ and $\frac{1}{2} gof(x) = 2x^2 - 5x + 2$, then which is not a possible $f(x)$?

(1) $2x - 3$ (2) $-2x + 2$

(3) $x - 3$ (4) None of these

1. Let $f: X \to Y, f(x) = \sin x + \cos x + 2\sqrt{2}$ be invertible. Then which $X \to Y$ is not possible?

(1) $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \to [\sqrt{2}, 3\sqrt{2}]$

(2) $\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right] \to [\sqrt{2}, 3\sqrt{2}]$

(3) $\left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right] \to [\sqrt{2}, 3\sqrt{2}]$

(4) none of these

2. If $f(x)$ is an invertible function and $g(x) = 2f(x) + 5$, then the value of $g^{-1}(x)$ is

(1) $2f^{-1}(x) - 5$ (2) $\frac{1}{2f^{-1}(x) + 5}$

(3) $\frac{1}{2} f^{-1}(x) + 5$ (4) $f^{-1}\left(\frac{x-5}{2}\right)$

3. Let $f(x) = [x] + \sqrt{\{x\}}$; where [.] denotes the integral part of x and $\{x\}$ denotes the fractional part of x. Then $f^{-1}(x)$ is

(1) $[x] + \sqrt{\{x\}}$ (2) $[x] + \{x\}^2$

(3) $[x]^2 + \{x\}$ (4) $\{x\} + \sqrt{\{x\}}$

4. If f is a function such that $f(0) = 2$, $f(1) = 3$, and $f(x + 2) = 2f(x) - f(x + 1)$ for every real x, then $f(5)$ is

(1) 7 (2) 13 (3) 1 (4) 5

5. A function $F(x)$ satisfies the functional equation $x^2F(x) + F(1 - x) = 2x - x^4$ for all real x. $F(x)$ must be

(1) x^2 (2) $1 - x^2$

(3) $1 + x^2$ (4) $x^2 + x + 1$

6. If $f(x)$ is a polynomial satisfying $f(x)f(1/x) = f(x) + f(1/x)$ and $f(3) = 28$, then $f(4)$ is equal to

(1) 63 (2) 65

(3) 17 (4) none of these

7. If $f\left(2x + \frac{y}{8}, 2x - \frac{y}{8}\right) = xy$, then $f(m, n) + f(n, m) = 0$

(1) only when $m = n$ (2) only when $m \neq n$

(3) only when $m = -n$ (4) for all m and n

8. Let $f: R \to R$ be a function such that $f(0) = 1$ and for any $x, y \in R$, $f(xy + 1) = f(x)f(y) - f(y) - x + 2$. Then f is

(1) one-one and onto (2) one-one but not onto

(3) many one but onto (4) many one and into

9. If $f(x + y) = f(x) + f(y) - xy - 1 \ \forall\, x, y \in R$ and $f(1) = 1$, then the number of solutions of $f(n) = n, n \in N$, is

(1) 0 (2) 1

(3) 2 (4) more than 2

110. The function f satisfies the functional equation $3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30$ for all real $x \neq 1$. The value of $f(7)$ is

(1) 8 (2) 4 (3) -8 (4) 11

111. Let $f: R \to R$ be a continuous and differentiable function such that $\left(f(x^2+1)\right)^{\sqrt{x}} = 5$ for $\forall\, x \in (0, \infty)$. Then the value of $\left(f\left(\frac{16+y^2}{y^2}\right)\right)^{\frac{4}{\sqrt{y}}}$ for $y \in (0, \infty)$ is equal to

(1) 5 (2) 25 (3) 125 (4) 625

112. Let $g(x) = f(x) - 1$. If $f(x) + f(1 - x) = 2 \ \forall\, x \in R$, then $g(x)$ is symmetrical about

(1) the origin (2) the line $x = \frac{1}{2}$

(3) the point (1, 0) (4) the point $\left(\frac{1}{2}, 0\right)$

113. If $f(x + 1) + f(x - 1) = 2f(x)$ and $f(0) = 0$, then $f(n)$, $n \in N$, is

(1) $nf(1)$ (2) $\{f(1)\}^n$

(3) 0 (4) none of these

114. If $f\left(x + \frac{1}{2}\right) + f\left(x - \frac{1}{2}\right) = f(x)$ for all $x \in R$, then the period of $f(x)$ is

(1) 1 (2) 2 (3) 3 (4) 4

115. If $af(x + 1) + bf\left(\frac{1}{x+1}\right) = x$, $x \neq -1$, $a \neq b$, then $f(2)$ is equal to

(1) $\frac{2a+b}{2(a^2-b^2)}$ (2) $\frac{a}{a^2-b^2}$

(3) $\frac{a+2b}{a^2-b^2}$ (4) none of these

116. If $f(3x + 2) + f(3x + 29) = 0 \ \forall\, x \in R$, then the period of $f(x)$ is

(1) 7 (2) 8

(3) 10 (4) none of these

117. If the graph of $y = f(x)$ is symmetrical about the lines $x = 1$ and $x = 2$, then which of the following is true?

(1) $f(x + 1) = f(x)$ (2) $f(x + 3) = f(x)$

(3) $f(x + 2) = f(x)$ (4) None of these

118. If $f(x) = \text{maximum}\left\{x^3, x^2, \frac{1}{64}\right\} \ \forall\, x \in [0, \infty)$, then

(1) $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ x^3, & x > 1 \end{cases}$ (2) $f(x) = \begin{cases} \frac{1}{64}, & 0 \le x \le \frac{1}{4} \\ x^2, & \frac{1}{4} < x \le 1 \\ x^3, & x > 1 \end{cases}$

(3) $f(x)=\begin{cases}\frac{1}{64}, 0\le x\le \frac{1}{8}\\ x^2, \frac{1}{8}<x\le 1\\ x^3, x>1\end{cases}$ (4) $f(x)=\begin{cases}\frac{1}{64}, 0\le x\le \frac{1}{8}\\ x^3, x>1/8\end{cases}$

119. The equation $||x-2|+a|=4$ can have four distinct real solutions for x if a belongs to the interval

(1) $(-\infty,-4)$ (2) $(-\infty,0]$
(3) $[4,\infty)$ (4) none of these

120. Number of integral values of k for which the equation $4\cos^{-1}(-|x|)=k$ has exactly two solutions is

(1) 4 (2) 5 (3) 6 (4) 7

121. If $f(x)$ is a real-valued function defined as $f(x)=\ln(1-\sin x)$, then the graph of $f(x)$ is

(1) symmetric about the line $x=\pi$
(2) symmetric about the y-axis
(3) symmetric about the line $x=\frac{\pi}{2}$
(4) symmetric about the origin

122. Let $f(x)=x+2|x+1|+2|x-1|$. If $f(x)=k$ has exactly one real solution, then the value of k is

(1) 3 (2) 0 (3) 1 (4) 2

123. The number of solutions of $2\cos x=|\sin x|$, $0\le x\le 4\pi$, is

(1) 0 (2) 2 (3) 4 (4) infinite

124. Let $f_1(x)=\begin{cases}x, 0\le x\le 1\\ 1, \quad x>1\\ 0, \text{ otherwise}\end{cases}$

$f_2(x)=f_1(-x)$ for all x
$f_3(x)=-f_2(x)$ for all x
$f_4(x)=f_3(-x)$ for all x

Which of the following is necessarily true?

(1) $f_4(x)=f_1(x)$ for all x
(2) $f_1(x)=-f_3(-x)$ for all x
(3) $f_2(-x)=f_4(x)$ for all x
(4) $f_1(x)+f_3(x)=0$ for all x

125. If $\log_4\left(\frac{2f(x)}{1-f(x)}\right)=x$, then $(f(2010)+f(-2009))$ is equal to

(1) 0 (2) -1 (3) 1 (4) 2

Multiple Correct Answers Type

1. Let $f(x)=\sec^{-1}[1+\cos^2 x]$, where [.] denotes the greatest integer function. Then the

(1) domain of f is R
(2) domain of f is $[1, 2]$
(3) domain of f is $[1, 2]$
(4) range of f is $\{\sec^{-1}1, \sec^{-1}2\}$

2. $f: R\to[-1,\infty)$ and $f(x)=\ln([|\sin 2x|+|\cos 2x|])$ (where [.] is the greatest integer function.) Then,

(1) $f(x)$ has range Z
(2) Range of $f(x)$ is singleton set
(3) $f(x)$ is invertible in $\left[0,\frac{\pi}{4}\right]$
(4) $f(x)$ is into function

3. If $f: R\to N\cup\{0\}$, where f(area of triangle joining points $P(5,0)$, $Q(8,4)$ and $R(x,y)$ such that angle PRQ is a right angle) = number of triangles, then which of the following is true?

(1) $f(5)=4$ (2) $f(7)=0$
(3) $f(6.25)=2$ (4) $f(4.5)=4$

4. The domain of the function

$$f(x)=\log_e\left\{\log_{|\sin x|}(x^2-8x+23)-\frac{3}{\log_2|\sin x|}\right\}$$

contains which of the following interval(s)?

(1) $(3,\pi)$ (2) $\left(\pi,\frac{3\pi}{2}\right)$
(3) $\left(\frac{3\pi}{2},5\right)$ (4) None of these

5. Let $f(x)=\text{sgn}(\cot^{-1}x)+\tan\left(\frac{\pi}{2}[x]\right)$, where $[x]$ is the greatest integer function less than or equal to x. Then which of the following alternatives is/are true?

(1) $f(x)$ is many-one but not an even function.
(2) $f(x)$ is a periodic function.
(3) $f(x)$ is a bounded function.
(4) The graph of $f(x)$ remains above the x-axis.

6. $f(x)=\sqrt{1-\sin^2 x}+\sqrt{1+\tan^2 x}$ then

(1) fundamental period of $f(x)$ is π
(2) range of $f(x)$ is $[2,\infty)$
(3) domain of $f(x)$ is R
(4) $f(x)=2$ has 3 solution in $[0, 2\pi]$

7. If the following functions are defined from $[-1, 1]$ to $[-1, 1]$, select those which are not bijective.

(1) $\sin(\sin^{-1}x)$ (2) $\frac{2}{\pi}\sin^{-1}(\sin x)$
(3) $(\text{sgn}(x))\ln(e^x)$ (4) $x^3(\text{sgn}(x))$

8. Let $f(x)=\begin{cases}x^2-4x+3, & x<3\\ x-4, & x\ge 3\end{cases}$

and $g(x)=\begin{cases}x-3, & x<4\\ x^2+2x+2, & x\ge 4\end{cases}$

Then which of the following is/are true?

(1) $(f+g)(3.5)=0$ (2) $f(g(3))=3$
(3) $(fg)(2)=1$ (4) $(f-g)(4)=0$

9. Let $f(x) = \max\{1 + \sin x, 1, 1 - \cos x\}, x \in [0, 2\pi]$, and $g(x) = \max\{1, |x - 1|\}, x \in R$. Then

(1) $g(f(0)) = 1$ (2) $g(f(1)) = 1$

(3) $f(f(1)) = 1$ (4) $f(g(0)) = 1 + \sin 1$

10. Consider the function $y = f(x)$ satisfying the condition $f\left(x+\frac{1}{x}\right) = x^2 + \frac{1}{x^2} (x \neq 0)$. Then the

(1) domain of $f(x)$ is R

(2) domain of $f(x)$ is $R - (-2, 2)$

(3) range of $f(x)$ is $[-2, \infty)$

(4) range of $f(x)$ is $[2, \infty)$

11. Consider the real-valued function satisfying $2f(\sin x) + f(\cos x) = x$. Then the

(1) domain of $f(x)$ is R

(2) domain of $f(x)$ is $[-1, 1]$

(3) range of $f(x)$ is $\left[-\frac{2\pi}{3}, \frac{\pi}{3}\right]$

(4) range of $f(x)$ is R

12. If $f: R^+ \to R^+$ is a polynomial function satisfying the functional equation $f(f(x)) = 6x - f(x)$, then $f(17)$ is equal to

(1) 17 (2) −51 (3) 34 (4) −34

13. $f(x) = x^2 - 2ax + a(a + 1), f: [a, \infty) \to [a, \infty)$. If one of the solutions of the equation $f(x) = f^{-1}(x)$ is 5049, then the other may be

(1) 5051 (2) 5048 (3) 5052 (4) 5050

14. Which of the following function is/are periodic?

(1) $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

(2) $f(x) = \begin{cases} x-[x]; & 2n \le x < 2n+1 \\ \frac{1}{2}; & 2n+1 \le x < 2n+2 \end{cases}$,

where [.] denotes the greatest integer function, $n \in Z$

(3) $f(x) = (-1)^{\left[\frac{2x}{\pi}\right]}$, where [.] denotes the greatest integer function

(4) $f(x) = x - [x + 3] + \tan\left(\frac{\pi x}{2}\right)$, where [.] denotes the greatest integer function, and a is a rational number

15. Let $f(x) = \frac{3}{4}x + 1$, $f^n(x)$ be defined as $f^2(x) = f(f(x))$, and for $n \ge 2, f^{n+1}(x) = f(f^n(x))$. If $\lambda = \lim\limits_{n\to\infty} f^n(x)$, then

(1) λ is independent of x

(2) λ is a linear polynomial in x

(3) the line $y = \lambda$ has slope 0

(4) the line $4y = \lambda$ touches the unit circle with center at the origin.

16. If the fundamental period of function $f(x) = \sin x + \cos(\sqrt{4-a^2})\, x$ is 4π, then the value of a is / are

(1) $\frac{\sqrt{15}}{2}$ (2) $-\frac{\sqrt{15}}{2}$

(3) $\frac{\sqrt{7}}{2}$ (4) $-\frac{\sqrt{7}}{2}$

17. $f(x) = \sin^{-1}[e^x] + \sin^{-1}[e^{-x}]$, where [.] greatest integer function then

(1) domain of $f(x)$ is $(-\log_e 2, \log_e 2)$

(2) range of $f(x) = \{\pi\}$

(3) Range of $f(x)$ is $\left\{\frac{\pi}{2}, \pi\right\}$

(4) $f(x) = \cos^{-1} x$ has only one solution

18. $[2x] - 2[x] = \lambda$ where [·] represents greatest integer function and {·} represents fractional part of a real number then

(1) $\lambda = 1 \ \forall\, x \in R$ (2) $\lambda = 0 \ \forall\, x \in R$

(3) $\lambda = 1 \ \forall\, \{x\} \ge \frac{1}{2}$ (4) $\lambda = 0 \ \forall\, \{x\} < \frac{1}{2}$

19. The set of all values of x satisfying $\{x\} = x\,[x]$ where $[x]$ represents greatest integer function $\{x\}$ represents fractional part of x

(1) 0 (2) $-\frac{1}{2}$

(3) $-1 < x < 1$ (4) R

20. The function 'g' defined by $g(x) = \sin(\sin^{-1}\{\sqrt{x}\}) + \cos(\sin^{-1}\{\sqrt{x}\}) - 1$ where $\{x\}$ denotes the functional part function is

(1) an even function (2) periodic function

(3) odd function (4) Neither even nor odd

21. If the function f satisfies the relation $f(x + y) + f(x - y) = 2f(x) f(y) \ \forall\, x, y \in R$ and $f(0) \neq 0$, then

(1) $f(x)$ is an even function

(2) $f(x)$ is an odd function

(3) If $f(2) = a$, then $f(-2) = a$

(4) If $f(4) = b$, then $f(-4) = -b$

22. Let $f(x) + f(y) = f(x\sqrt{1-y^2} + y\sqrt{1-x^2})$ [$f(x)$ is not identically zero]. Then

(1) $f(4x^3 - 3x) + 3f(x) = 0$

(2) $f(4x^3 - 3x) = 3f(x)$

(3) $f(2x\sqrt{1-x^2}) + 2f(x) = 0$

(4) $f(2x\sqrt{1-x^2}) = 2f(x)$

23. Let $f: R \to R$ be a function defined by $f(x + 1) = \frac{f(x)-5}{f(x)-3}$ $\forall\, x \in R$. Then which of the following statement(s) is/are true?

(1) $f(2008) = f(2004)$ (2) $f(2006) = f(2010)$

(3) $f(2006) = f(2002)$ (4) $f(2006) = f(2018)$

24. Let a function $f(x)$, $x \neq 0$ be such that

$f(x)+f\left(\frac{1}{x}\right)=f(x)\cdot f\left(\frac{1}{x}\right)$ then $f(x)$ can be

(1) $1-x^{2013}$ (2) $\sqrt{|x|}+1$

(3) $\frac{\pi}{2\tan^{-1}|x|}$ (4) $\frac{2}{1+k\ln|x|}$

25. Let f be a differential function such that $f(x)=f(2-x)$ and $g(x)=f(1+x)$ then

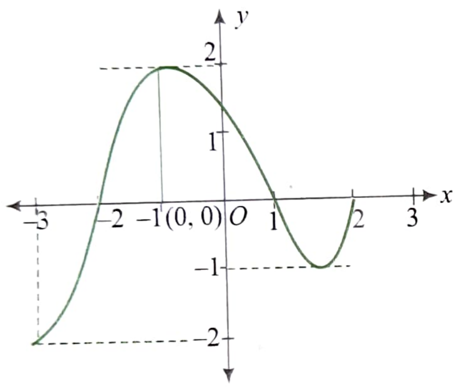
(1) $g(x)$ is an odd function

(2) $g(x)$ is an even function

(3) Graph of $f(x)$ is symmetrical about the line $x=1$

(4) $f'(1)=0$

26. The figure illustrates the graph of the function $y=f(x)$ defined in $[-3, 2]$.



Identify the correct statement(s) ?

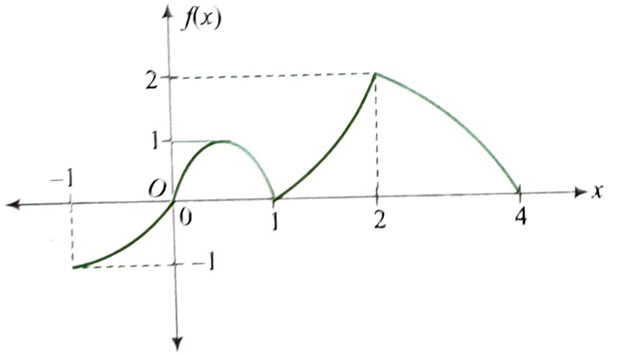
(1) Range of $y=f(-|x|)$ is $[-2, 2]$

(2) Domain of $y=f(|x|)$ is $[-2, 2]$

(3) Domain of $y=f(|x|+1)$ is $[-1, 1]$

(4) Range of $y=f(|x|+1)$ is $[-1, 0]$

27. If graph of a function $f(x)$ which is defined in $[-1, 4]$ is shown in the following figure then identify the correct statement(s).



(1) domain of $f(|x|-1)$ is $[-5, 5]$

(2) range of $f(|x|+1)$ is $[0, 2]$

(3) range of $f(-|x|)$ is $[-1, 0]$

(4) domain of $f(|x|)$ is $[-3, 3]$

Linked Comprehension Type

For Problems 1–3

Consider the functions

$$f(x)=\begin{cases} x+1, & x\le 1 \\ 2x+1, & 1<x\le 2\end{cases} \text{ and } g(x)=\begin{cases} x^2, & -1\le x<2 \\ x+2, & 2\le x\le 3\end{cases}$$

1. The domain of the function $f(g(x))$ is

(1) $[0, \sqrt{2}]$ (2) $[-1, 2]$

(3) $[-1, \sqrt{2}]$ (4) none of these

2. The range of the function $f(g(x))$ is

(1) $[1, 5]$ (2) $[2, 3]$

(3) $[1, 2]\cup(3, 5]$ (4) none of these

3. The number of roots of the equation $f(g(x))=2$ is

(1) 1 (2) 2

(3) 4 (4) none of these

For Problems 4–6

Consider the function $f(x)$ satisfying the identity $f(x)+f\left(\frac{x-1}{x}\right)=1+x\ \forall\ x\in R-\{0, 1\}$, and $g(x)=2f(x)-x+1$.

4. The domain of $y=\sqrt{g(x)}$ is

(1) $\left(-\infty, \frac{1-\sqrt{5}}{2}\right]\cup\left[1, \frac{1+\sqrt{5}}{2}\right]$

(2) $\left(-\infty, \frac{1-\sqrt{5}}{2}\right]\cup(0, 1)\cup\left[\frac{1+\sqrt{5}}{2}, \infty\right)$

(3) $\left[\frac{-1-\sqrt{5}}{2}, 0\right]\cup\left[\frac{-1+\sqrt{5}}{2}, 1\right]$

(4) none of these

5. The range of $y=g(x)$ is

(1) $(-\infty, 5]$ (2) $[1, \infty)$

(3) $(-\infty, 1)\cup[5, \infty)$ (4) none of these

6. The number of roots of the equation $g(x)=1$ is

(1) 2 (2) 1 (3) 3 (4) 0

For Problems 7–9

If $(f(x))^2\times f\left(\frac{1-x}{1+x}\right)=64x\ \forall\ x\in D_f$, then

7. $f(x)$ is equal to

(1) $4x^{2/3}\left(\frac{1+x}{1-x}\right)^{1/3}$ (2) $x^{1/3}\left(\frac{1-x}{1+x}\right)^{1/3}$

(3) $x^{1/3}\left(\frac{1-x}{1+x}\right)^{1/3}$ (4) $x\left(\frac{1+x}{1-x}\right)^{1/3}$

8. The domain of $f(x)$ is

(1) $[0, \infty)$ (2) $R-\{1\}$

(3) $(-\infty, \infty)$ (4) none of these

9. The value of $f(9/7)$ is

(1) $8(7/9)^{2/3}$ (2) $4(9/7)^{1/3}$

(3) $-8(9/7)^{2/3}$ (4) none of these

For Problems 10–12

$f(x)=\begin{cases}x-1; -1\le x\le 0\\ x^2, 0\le x\le 1\end{cases}$ and $g(x)=\sin x$

Consider the functions $h_1(x)=f(|g(x)|)$ and $h_2(x)=|f(g(x))|$.

10. Which of the following is not true about $h_1(x)$?
(1) It is a periodic function with period π.
(2) The range is [0, 1].
(3) The domain is R.
(4) None of these.

11. Which of the following is not true about $h_2(x)$?
(1) The domain is R
(2) It is periodic with period 2π.
(3) The range is [0, 1].
(4) None of these.

12. If for $h_1(x)$ and $h_2(x)$ are identical functions, then which of the following is not true?
(1) Domain of $h_1(x)$ and $h_2(x)$ is $x\in[2n\pi,(2n+1)\pi], n\in Z$.
(2) Range of $h_1(x)$ and $h_2(x)$ is [0, 1].
(3) Period of $h_1(x)$ and $h_2(x)$ is π.
(4) None of these.

For Problems 13–15

If $a_0=x$, $a_{n+1}=f(a_n)$, where $n=0, 1, 2, \ldots$, then answer the following questions.

13. If $f(x)=\sqrt[m]{(a-x^m)}$, $x>0$, $m\ge 2$, $m\in N$, then
(1) $a_n=x$, $n=2k+1$, where k is an integer
(2) $a_n=f(x)$ if $n=2k$, where k is an integer
(3) The inverse of a_n exists for any value of n and m
(4) none of these

14. If $f(x)=\dfrac{1}{1-x}$, then which of the following is not true?
(1) $a_n=\dfrac{1}{1-x}$ if $n=3k+1$
(2) $a_n=\dfrac{x-1}{x}$ if $n=3k+2$
(3) $a_n=x$ if $n=3k$
(4) None of these

15. If $f: R\to R$ is given by $f(x)=3+4x$ and $a_n=A+Bx$, then which of the following is not true?
(1) $A+B+1=2^{2n+1}$
(2) $|A-B|=1$
(3) $\lim_{n\to\infty}\dfrac{A}{B}=-1$
(4) None of these

For Problems 16–18

Let $f(x)=f_1(x)-2f_2(x)$, where

where $f_1(x)=\begin{cases}\min\{x^2,|x|\}, & |x|\le 1\\ \max\{x^2,|x|\}, & |x|>1\end{cases}$

and $f_2(x)=\begin{cases}\min\{x^2,|x|\}, & |x|>1\\ \max\{x^2,|x|\}, & |x|\le 1\end{cases}$

and let $g(x)=\begin{cases}\min\{f(t): -3\le t\le x, & -3\le x<0\}\\ \max\{f(t): 0\le t\le x, & 0\le x\le 3\}\end{cases}$.

16. For $-3\le x\le -1$, the range of $g(x)$ is
(1) $[-1, 3]$
(2) $[-1, -15]$
(3) $[-1, 9]$
(4) none of these

17. For $x\in(-1, 0)$, $f(x)+g(x)$ is
(1) x^2-2x+1
(2) x^2+2x-1
(3) x^2+2x+1
(4) x^2-2x-1

18. The graph of $y=g(x)$ in its domain is broken at
(1) 1 point
(2) 2 points
(3) 3 points
(4) none of these

For Problems 19–21

Let $f(x)=\begin{cases}2x+a, & x\ge -1\\ bx^2+3, & x<-1\end{cases}$

and $g(x)=\begin{cases}x+4, & 0\le x\le 4\\ -3x-2, & -2<x<0\end{cases}$

19. $g(f(x))$ is not defined if
(1) $a\in(10,\infty)$, $b\in(5,\infty)$
(2) $a\in(4, 10)$, $b\in(5,\infty)$
(3) $a\in(10,\infty)$, $b\in(0, 1)$
(4) $a\in(4, 10)$, $b\in(1, 5)$

20. If the domain of $g(f(x))$ is $[-1, 4]$, then
(1) $a=1, b>5$
(2) $a=2, b>7$
(3) $a=2, b>10$
(4) $a=0, b\in R$

21. If $a=2$ and $b=3$, then the range of $g(f(x))$ is
(1) $(-2, 8]$
(2) $(0, 8]$
(3) $[4, 8]$
(4) $[-1, 8]$

For Problems 22–24

Let $f: R\to R$ be a function satisfying $f(2-x)=f(2+x)$ and $f(20-x)=f(x)\forall x\in R$. For this function f, answer the following.

22. If $f(0)=5$, then the minimum possible number of values of x satisfying $f(x)=5$, for $x\in[0, 170]$, is
(1) 21
(2) 12
(3) 11
(4) 22

23. The graph of $y=f(x)$ is not symmetrial about
(1) symmetrical about $x=2$
(2) symmetrical about $x=10$
(3) symmetrical about $x=8$
(4) none of these

24. If $f(2)\ne f(6)$, then the
(1) fundamental period of $f(x)$ is 1
(2) fundamental period of $f(x)$ may be 1
(3) period of $f(x)$ cannot be 1
(4) fundamental period of $f(x)$ is 8

For Problems 25–27

Consider two functions

$f(x)=\begin{cases}[x], & -2\le x\le -1\\ |x|+1, & -1<x\le 2\end{cases}$ and $g(x)=\begin{cases}[x], & -\pi\le x<0\\ \sin x, & 0\le x\le \pi\end{cases}$

where [.] denotes the greatest integer function.

25. The exhaustive domain of $g(f(x))$ is

(1) [0, 2] (2) [–2, 0]

(3) [–2, 2] (4) [–1, 2]

26. The range of $g(f(x))$ is

(1) [sin 3, sin 1]

(2) $[\sin 3, 1]\cup\{-2,-1,0\}$

(3) $[\sin 3, 1]\cup\{-2,-1\}$

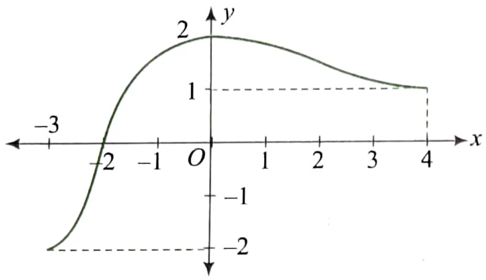
(4) [sin 1, 1]

27. The number of integral points in the range of $g(f(x))$ is

(1) 2 (2) 4 (3) 3 (4) 5

For Problems 28 and 29

Consider a function f whose domain is [–3, 4] and range is [– 2, 2] with following graph.



28. Domain and range of $g(x) = f(|x|)$ is $[a, b]$ and $[c, d]$ respectively, then $(b - a + c + d)$ is

(1) 11 (2) 10 (3) 8 (4) 7

29. If $h(x)=\left|f(x)-\dfrac{3}{2}\right|$ has range $[e, f]$ and n be number of real solutions of $h(x)=\dfrac{1}{4}$, then $(n + e + 2f)$ is

(1) 8 (2) 9 (3) 10 (4) 11

For Problems 30–32

Consider a differentiable $f : R \to R$ for which $f(1) = 2$ and $f(x + y) = 2^x f(y) + 4^y f(x)\ \forall\, x, y \in R$.

30. The value of f(4) is

(1) 160 (2) 240

(3) 200 (4) none of these

31. The minimum value of $f(x)$ is

(1) 1 (2) $-\dfrac{1}{2}$

(3) $-\dfrac{1}{4}$ (4) none of these

32. The number of solutions of $f(x) = 2$ is

(1) 0 (2) 1

(3) 2 (4) infinite

Matrix Match Type

1. The function $f(x)$ is defined on the interval [0, 1]. Now, match the following lists:

Lisr I: Function	List II: Domain
a. $f(\tan x)$	**p.** $\left[2n\pi-\dfrac{\pi}{2}, 2n\pi+\dfrac{\pi}{2}\right], n\in Z$
b. $f(\sin x)$	**q.** $\left[2n\pi, 2n\pi+\dfrac{\pi}{6}\right]\cup\left[2n\pi+\dfrac{5\pi}{6}, (2n+1)\pi\right]$, $n\in Z$
c. $f(\cos x)$	**r.** $[2n\pi, (2n+1)\pi], n\in Z$
d. $f(2\sin x)$	**s.** $\left[n\pi, n\pi+\dfrac{\pi}{4}\right], n\in Z$

2. Match the following lists:

List I: Function	List II: Type of function
a. $f(x)=\{(\operatorname{sgn} x)^{\operatorname{sgn} x}\}^n$; $x\neq 0$, n is an odd integer	**p.** odd function
b. $f(x)=\dfrac{x}{e^x-1}+\dfrac{x}{2}+1$	**q.** even function
c. $f(x)=\begin{cases}0, & \text{if } x \text{ is rational}\\ 1, & \text{if } x \text{ is irrational}\end{cases}$	**r.** neither odd nor even function
d. $f(x)=\max\{\tan x, \cot x\}$	**s.** periodic

3. Let $f(x)=\dfrac{x^2-6x+5}{x^2-5x+6}$. Then match the expressions/statements in List I with expressions/statements in List II.

List I	List II
a. If $-1<x<1$, then $f(x)$ satisfies	**p.** $0<f(x)<1$
b. If $1<x<2$, then $f(x)$ satisfies	**q.** $f(x)<0$
c. If $3<x<5$, then $f(x)$ satisfies	**r.** $f(x)>0$
d. If $x>5$, then $f(x)$ satisfies	**s.** $f(x)<1$

4. Match the following lists:

List I: Function	List II: Values of x for which both the functions in any option of List I are identical
a. $f(x)=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, $g(x)=2\tan^{-1}x$	**p.** $x\in\{-1, 1\}$
b. $f(x)=\sin^{-1}(\sin x)$ and $g(x)=\sin(\sin^{-1}x)$	**q.** $x\in[-1, 1]$
c. $f(x)=\log_{x^2}25$ and $g(x)=\log_x 5$	**r.** $x\in(-1, 1)$
d. $f(x)=\sec^{-1}x+\operatorname{cosec}^{-1}x$, $g(x)=\sin^{-1}x+\cos^{-1}x$	**s.** $x\in(0, 1)$

5. Match the following lists:

List I	List II
a. $f: R \to \left[\frac{3\pi}{4}, \pi\right)$ and $f(x) = \cot^{-1}(2x - x^2 - 2)$. Then $f(x)$ is	p. one-one
b. $f: R \to R$ and $f(x) = e^x \sin x$. Then $f(x)$ is	q. into
c. $f: R^+ \to [4, \infty]$ and $f(x) = 4 + 3x^2$. Then $f(x)$ is	r. many-one
d. $f: X \to X$ and $f(f(x)) = x \ \forall x \in X$. Then $f(x)$ is	s. onto

6. Match the following lists:

List I: Function	List II: Fundamental Period
a. $f(x) = \cos(\lvert\sin x\rvert - \lvert\cos x\rvert)$	p. π
b. $f(x) = \cos(\tan x + \cot x) \times \cos(\tan x - \cot x)$	q. $\pi/2$
c. $f(x) = \sin^{-1}(\sin x) + e^{\tan x}$	r. 4π
d. $f(x) = \sin^3 x \sin 3x$	s. 2π

7. {.} denotes the fractional part function and [.] denotes the greatest integer function. Now, match the following lists:

List I: Function	List II: Period
a. $f(x) = e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$	p. 1/3
b. $f(x) = \cos 2\pi\{2x\} + \sin 2\pi\{2x\}$	q. 1/4
c. $f(x) = \sin 3\pi\{x\} + \tan \pi[x]$	r. 1/2
d. $f(x) = 3x - [3x + a] - b$, where $a, b \in R^+$	s. 1

8. Match the following lists and then choose the correct code.

List I: Function	List II: Range
a. $f(x) = \log_3(5 + 4x - x^2)$	p. Function not defined
b. $f(x) = \log_3(x^2 - 4x - 5)$	q. $[0, \infty)$
c. $f(x) = \log_3(x^2 - 4x + 5)$	r. $(-\infty, 2]$
d. $f(x) = \log_3(4x - 5 - x^2)$	s. R

Codes:

	a	b	c	d
(1)	p	r	s	q
(2)	r	s	q	p
(3)	r	q	s	p
(4)	p	q	s	r

9. Match the following lists and then choose the correct code.

List I: Equation	List II: Number of roots
a. $x^2 \tan x = 1, x \in [0, 2\pi]$	p. 5
b. $2^{\cos x} = \lvert\sin x\rvert, x \in [0, 2\pi]$	q. 2
c. If $f(x)$ is a polynomial of degree 5 with real coefficients such that $f(\lvert x\rvert) = 0$ has 8 real roots, then the number of roots of $f(x) = 0$	r. 3
d. $7^{\lvert x\rvert}(\lvert 5 - \lvert x\rvert\rvert) = 1$	s. 4

Codes:

	a	b	c	d
(1)	q	s	r	p
(2)	p	s	r	s
(3)	q	s	p	s
(4)	s	p	q	r

Numerical Value Type

1. Let f be a real-valued invertible function such that $f\left(\frac{2x-3}{x-2}\right) = 5x - 2, x \neq 2$. Then the value of $f^{-1}(13)$ is ______.

2. Let $f(x) = 3x^2 - 7x + c$, where c is a variable coefficient and $x > \frac{7}{6}$. Then the value of $[c]$ such that $f(x)$ touches $f^{-1}(x)$ is (where [·] represents greatest integer function)______.

3. The number of points on the real line where the function $f(x) = \log_{\lvert x^2-1\rvert} \lvert x-3\rvert$ is not defined is______.

4. The number of integral values of x for which $\frac{\left(2^{\frac{\pi}{\tan^{-1}x}} - 4\right)(x-4)(x-10)}{x! - (x-1)!} < 0$ is ______.

5. If $f(x) = \begin{cases} x\cos x + \log_e\left(\frac{1-x}{1+x}\right); & x \neq 0 \\ a & ; x = 0 \end{cases}$ is odd, then $a =$ ______.

6. The number of integers in the range of the function $f(x) = \left|4\frac{(\sqrt{\cos x} - \sqrt{\sin x})(\sqrt{\cos x} + \sqrt{\sin x})}{(\cos x + \sin x)}\right|$ is ______.

7. The number of integers in the domain of function, satisfying $f(x) + f(x^{-1}) = \frac{x^3+1}{x}$, is ______

8. If a polynomial function $f(x)$ satisfies $f(f(f(x))) = 8x + 21$, where p and q are real numbers, then $p + q$ is equal to______.

9. If $f(x)$ is an odd function, $f(1) = 3$, and $f(x + 2) = f(x) + f(2)$, then the value of $f(3)$ is ______.

10. Let $f: R \to R$ be a continuous onto function satisfying $f(x) + f(-x) = 0 \ \forall \in R$. If $f(-3) = 2$ and $f(5) = 4$ in $[-5, 5]$, then the minimum number of roots of the equation $f(x) = 0$ is ______.

11. The number of integral values of x for which the function $\sqrt{\sin x + \cos x} + \sqrt{7x - x^2 - 6}$ is defined is ______.

12. Suppose that f is an even, periodic function with period 2, and that $f(x) = x$ for all x in the interval [0, 1]. Then the value of $f(3.14)$ is ______.

13. If $f(x) = \sqrt{4 - x^2} + \sqrt{x^2 - 1}$, then the maximum value of $(f(x))^2$ is ______.

14. The function $f(x) = \frac{x+1}{x^3+1}$ can be written as the sum of an even function $g(x)$ and an odd function $h(x)$. Then the value of $\lvert g(0)\rvert$ is ______

15. If T is the period of the function $f(x) = [8x + 7] + \lvert\tan 2\pi x + \cot 2\pi x\rvert - 8x$ (where [·] denotes the greatest integer function), then the value of T is______.

16. An even polynomial function $f(x)$ satisfies a relation $f(2x)\left(1-f\left(\frac{1}{2x}\right)\right) + f(16x^2y) = f(-2) - f(4xy)\ \forall\ x, y \in R-\{0\}$ and $f(4) = -255, f(0) = 1$. Then the value of $f(2)$ is ________.

17. If $f(x) = \sin^2 x + \sin^2\left(x+\frac{\pi}{3}\right) + \cos x \cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$, then $(gof)(x)$ is ________

18. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. Then the number of onto functions from E to F, is ________.

19. The function f is continuous and has the property $f(f(x)) = 1 - x$. Then the value of $f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)$ is ________.

20. A function f from integers to integers is defined as $f(n) = \begin{cases} n+3, & n \in \text{odd} \\ n/2, & n \in \text{even} \end{cases}$.

Suppose $k \in$ odd and $f(f(f(k))) = 27$. Then the value of k is ________

21. If θ is the fundamental period of the function $f(x) = \sin^{99} x + \sin^{99}\left(x+\frac{2\pi}{3}\right) + \sin^{99}\left(x+\frac{4\pi}{3}\right)$, then the complex number $z = |z|(\cos\theta + i\sin\theta)$ lies in the quadrant number ________.

22. If $x = \frac{4}{9}$ satisfies the equation $\log_a(x^2 - x + 2) > \log_a(-x^2 + 2x + 3)$, then the number of integral values of x is ________.

23. If $4^x - 2^{x+2} + 5 + ||b-1|-3| = |\sin y|$, $x, y, b \in R$, then the positive value of b is ________.

24. If $f: N \to N$, and $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)\ \forall\ x_1, x_2 \in N$ and $f(f(n)) = 3n\ \forall\ n \in N$, then $f(2) =$ ________.

25. The number of integral values of a for which $f(x) = \log(\log_{1/3}(\log_7(\sin x + a)))$ is defined for every real value of x is ________.

26. Let $f(x) = \sin^{23} x - \cos^{22} x$ and $g(x) = 1 + \frac{1}{2}\tan^{-1}|x|$. Then the number of values of x in the interval $[-10\pi, 8\pi]$ satisfying the equation $f(x) = \text{sgn}(g(x))$ is ________.

27. Suppose that $f(x)$ is a function of the form

$$f(x) = \frac{ax^8 + bx^6 + cx^4 + dx^2 + 15x + 1}{x}, (x \neq 0).$$

If $f(5) = 2$, then the value of $f(-5)$ is ________.

28. If $f: [2, \infty) \to [8, \infty)$ is a surjective function defined by $f(x) = x^2 - (p-2)x + 3p - 2$, $p \in R$ then number of values of p is ________.

29. Period of the function

$f(x) = \sin\left(\frac{x}{2}\right)\cos\left[\frac{x}{2}\right] - \cos\left(\frac{x}{2}\right)\sin\left[\frac{x}{2}\right]$, where $[\cdot]$ denotes the greatest integer function, is ________.

30. If the interval x satisfying the equation $[x] + [-x] = \frac{\log_3(x-2)}{|\log_3(x-2)|}$ is (a, b), then $a + b =$ ________.

[Here $[\cdot]$ denote greater integer less than or equal to x]

31. Let $f(x)$ be a polynomial of degree 5 such that $f(|x|) = 0$ has 8 real distinct. Then number of real roots of $f(x) = 0$ is ________.

Archives

JEE Main

Single Correct Answer Type

1. For real x, let $f(x) = x^3 + 5x + 1$, then
(1) f is one-one but not onto R
(2) f is onto R but not one-one
(3) f is one-one and onto R
(4) f is neither one-one nor onto R **(AIEEE 2009)**

2. Let $f(x) = (x+1)^2 - 1, x \geq -1$
Statement 1: The set $\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$.
Statement 2: f is a bijection.
(1) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.
(2) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1.
(3) Statement 1 is true, statement 2 is false.
(4) Statement 1 is false, statement 2 is true. **(AIEEE 2009)**

3. Consider the following relations:
$R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$;
$S = \{\left(\frac{m}{n}, \frac{p}{q}\right) m, n, p$, and q are integers such that $n, q \neq 0$ and $qm = pn\}$. Then
(1) R and S both are equivalence relations.
(2) R is an equivalence relation but S is not an equivalence relation.
(3) Neither R nor S is an equivalence relation.
(4) S is an equivalence relation but R is not an equivalence relation. **(AIEEE 2010)**

4. Let R be the set of real numbers.
Statement 1: $A = \{(x, y) \in R \times R : y - x$ is an integer$\}$ is an equivalence relation on R.
Statement 2: $B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation on R.
(1) Statement 1 is false, statement 2 is true.
(2) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.

(3) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1.

(4) Statement 1 is true, statement 2 is false.

(AIEEE 2011)

5. The domain of the function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is

(1) $(-\infty, \infty) \sim \{0\}$ (2) $(-\infty, \infty)$

(3) $(0, \infty)$ (4) $(-\infty, 0)$ **(AIEEE 2011)**

6. If $a \in R$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where $[x]$ denotes the greatest integer $\leq x$) has no integral solution, then all possible values of a lie in the interval

(1) $(-1, 0) \cup (0, 1)$ (2) $(1, 2)$

(3) $(-2, -1)$ (4) $(-\infty, -2) \cup (2, \infty)$

(JEE Main 2014)

7. If $f(x) + 2f\left(\dfrac{1}{x}\right) = 3x, x \neq 0$, and $S = \{x \in R : f(x) = f(-x)\}$; then S

(1) contains exactly one element

(2) contains exactly two elements

(3) contains more than two elements

(4) is an empty set **(JEE Main 2016)**

8. The function $f : R \to \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$ defined as $f(x) = \dfrac{x}{1+x^2}$, is

(1) neither injective nor surjective.

(2) invertible.

(3) injective but not surjective.

(4) surjective but not injective. **(JEE Main 2017)**

9. Let $a, b, c \in R$. If $f(x) = ax^2 + bx + c$ is such that $a + b + c = 3$ and $f(x + y) = f(x) + f(y) + xy, \forall x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to

(1) 255 (2) 330 (3) 165 (4) 190

(JEE Main 2017)

JEE Advanced

Multiple Correct Answers Type

1. Let $f : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to R$ be given by $f(x) = (\log(\sec x + \tan x))^3$. Then

(1) $f(x)$ is an odd function

(2) $f(x)$ is a one-one function

(3) $f(x)$ is an onto function

(4) $f(x)$ is an even function **(JEE Advanced 2014)**

2. Let $f(x) = \sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x) = \dfrac{\pi}{2}\sin x$ for all $x \in R$. Let $(fog)(x)$ denote $f(g(x))$ and $(gof)(x)$ denote $g(f(x))$. Then which of the following is (are) true?

(1) Range of f is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

(2) Range of fog is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

(3) $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \dfrac{\pi}{6}$

(4) There is an $x \in R$ such that $(gof)(x) = 1$

(JEE Advanced 2015)

Matrix Match Type

1. Let $E_1 = \left\{x \in R : x \neq 1 \text{ and } \dfrac{x}{x-1} > 0\right\}$ and

$E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\dfrac{x}{x-1}\right)\right) \text{ is a real number}\right\}$

(Here, the inverse trigonometric function $\sin^{-1}x$ assumes values in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.)

Let $f : E_1 \to R$ be the function defined by $f(x) = \log_e\left(\dfrac{x}{x-1}\right)$ and $g : E_2 \to R$ be the function defined by

$g(x) = \sin^{-1}\left(\log_e\left(\dfrac{x}{x-1}\right)\right)$.

List I	List II
a. The range of f is	**p.** $\left(-\infty, \dfrac{1}{1-e}\right] \cup \left[\dfrac{e}{e-1}, \infty\right)$
b. The range of g contains	**q.** $(0, 1)$
c. The domain of f contains	**r.** $[-1/2, 1/2]$
d. the domain of g is	**s.** $(-\infty, 0) \cup (0, \infty)$
	t. $\left(-\infty, \dfrac{e}{e-1}\right]$
	u. $(-\infty, 0) \cup \left(\dfrac{1}{2}, \dfrac{e}{e-1}\right]$

The correct option is

(1) a → s; b → q; c → p; d → p

(2) a → r; b → r; c → u; d → t

(3) a → s; b → q; c → p; d → u

(4) a → s; b → r; c → u; d → t

(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	2. (2)	3. (2)	4. (4)	5. (1)
6. (1)	7. (2)	8. (3)	9. (2)	10. (4)
11. (3)	12. (4)	13. (4)	14. (1)	15. (4)
16. (4)	17. (2)	18. (4)	19. (4)	20. (3)
21. (2)	22. (3)	23. (2)	24. (2)	25. (4)
26. (2)	27. (2)	28. (1)	29. (2)	30. (3)
31. (3)	32. (1)	33. (4)	34. (4)	35. (4)
36. (2)	37. (3)	38. (3)	39. (2)	40. (3)
41. (3)	42. (1)	43. (3)	44. (3)	45. (1)
46. (4)	47. (4)	48. (3)	49. (4)	50. (3)
51. (1)	52. (1)	53. (3)	54. (2)	55. (3)
56. (3)	57. (4)	58. (4)	59. (2)	60. (3)
61. (4)	62. (2)	63. (2)	64. (1)	65. (4)
66. (2)	67. (1)	68. (2)	69. (4)	70. (1)
71. (1)	72. (3)	73. (4)	74. (1)	75. (2)
76. (2)	77. (3)	78. (1)	79. (3)	80. (4)
81. (4)	82. (2)	83. (3)	84. (2)	85. (3)
86. (3)	87. (4)	88. (4)	89. (1)	90. (3)
91. (3)	92. (2)	93. (2)	94. (2)	95. (3)
96. (1)	97. (4)	98. (2)	99. (1)	100. (3)
101. (3)	102. (4)	103. (2)	104. (2)	105. (2)
106. (2)	107. (4)	108. (1)	109. (2)	110. (2)
111. (2)	112. (4)	113. (1)	114. (3)	115. (1)
116. (4)	117. (3)	118. (3)	119. (1)	120. (3)
121. (3)	122. (1)	123. (3)	124. (2)	125. (3)

Multiple Correct Answers Type

1. (1), (4)	2. (2), (4)
3. (1), (2), (3), (4)	4. (1), (2), (3)
5. (1), (2), (3), (4)	6. (1), (2), (4)
7. (2), (3), (4)	8. (1), (2), (3)
9. (1), (2), (4)	10. (2), (4)
11. (2), (3)	12. (2), (3)
13. (2), (4)	14. (1), (2), (3), (4)
15. (1), (3), (4)	16. (1), (2), (3), (4)
17. (1), (3)	18. (3), (4)
19. (1), (2)	20. (1), (2)
21. (1), (3)	22. (1), (4)
23. (1), (2), (3), (4)	24. (1), (2), (3), (4)
25. (2), (3), (4)	26. (1), (2), (3), (4)
27. (1), (2), (3)	

Linked Comprehension Type

1. (3)	2. (3)	3. (2)	4. (2)	5. (3)
6. (4)	7. (1)	8. (2)	9. (3)	10. (4)
11. (3)	12. (3)	13. (4)	14. (4)	15. (3)
16. (1)	17. (2)	18. (1)	19. (1)	20. (4)
21. (3)	22. (3)	23. (3)	24. (3)	25. (3)
26. (3)	27. (2)	28. (1)	29. (4)	30. (1)
31. (3)	32. (2)			

Matrix Match Type

1. $a \to s$; $b \to r$; $c \to p$; $d \to q$
2. $a \to p$; $b \to q$; $c \to q, s$; $d \to r, s$
3. $a \to p, r, s$; $b \to q, s$; $c \to q, s$; $d \to p, r, s$.
4. $a \to r, s$; $b \to p, q, r, s$; $c \to s$; $d \to p$
5. $a \to r, s$; $b \to r, s$; $c \to p, q$; $d \to p, s$
6. $a \to q$; $b \to q$; $c \to s$; $d \to p$
7. $a \to s$; $b \to r$; $c \to s$; $d \to p$
8. (2)
9. (3)

Numerical Value Type

1. (3)	2. (5)	3. (6)	4. (5)	5. (0)
6. (5)	7. (2)	8. (5)	9. (9)	10. (3)
11. (3)	12. (0.86)	13. (6)	14. (0)	15. (0.25)
16. (–15)	17. (1)	18. (14)	19. (1)	20. (105)
21. (3)	22. (2)	23. (4)	24. (3)	25. (3)
26. (9)	27. (28)	28. (2)	29. (2)	30. (5)
31. (5)				

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (3)	2. (3)	3. (4)	4. (4)	5. (4)
6. (1)	7. (2)	8. (4)	9. (2)	

JEE ADVANCED

Multiple Correct Answers Type

1. (1), (2), (3)	2. (1), (2), (3)

Matrix Match Type

1. (1)

2 Limits

LIMIT OF A FUNCTION

NEIGHBOURHOOD (NBD) OF REAL NUMBER

Consider real number a. Now what is the smallest interval in which a lie? The smallest interval depends upon how close we can move to a either from its left side or from its right side on real number line. Since there is no measure how close to a can be moved, we consider interval $(a - \delta, a + \delta)$, where δ is positive but infinitely small real number. The set of all real numbers lying between $a - \delta$ and $a + \delta$ is called the neighbourhood of a of radius δ and is denoted by $N_\delta(a)$.

Thus, $N_\delta(a) = (a - \delta, a + \delta) = \{x \in R \mid a - \delta < x < a + \delta\}$.

The set $(a - \delta, a)$ is called the left NBD of a and the set $(a, a + \delta)$ is known as the right NBD of a.

Now suppose x is not exactly a but very close to a. The value of x then may be slightly less than a or slightly more than a. Symbolically we write $x \to a$ read as x approaches to a. To give the directional sense like "from which side, number a is approached", we write $x \to a^-$ when a is approached from the left side of a and $x \to a^+$, when a is approached from the right side of a.

LIMITING VALUE OF A FUNCTION

Consider function $f(x) = x^2 + 3$.

The value of function at $x = 2$ is $f(2) = 2^2 + 3 = 7$.

Now when $x \to 2^-$, where the value of $f(x)$ goes?

For this we write $\lim_{x \to 2^-}(x^2 + 3)$. This is called limit of the function $f(x)$ when $x \to 2^-$.

In above we replace x by $2 - h$, where h is positive and infinitely small, say $h = 10^{-25}$ or smaller.

When $h \to 0$, $x \to 2^-$ (as we are subtracting very small positive value of from 2)

So, we have now $\lim_{x \to 2^-}(x^2 + 3) = \lim_{h \to 0}((2 - h)^2 + 3)$

$= \lim_{h \to 0}(7 + h(h - 4))$

Clearly, when $h \to 0$, $h(h - 4) \to 0^-$.

So, $\lim_{x \to 2^-}(x^2 + 3) = 7 - 0 = 7$. We say, limiting value of $f(x)$ when $x \to 2^-$ is 7. However, the limiting value 7 is not exactly 7. It is slightly less than 7, in fact 7^-. Since 2 is approached from its left–hand side, this limiting value of function is called its Left Hand Limit (LHL) at $x = 2$.

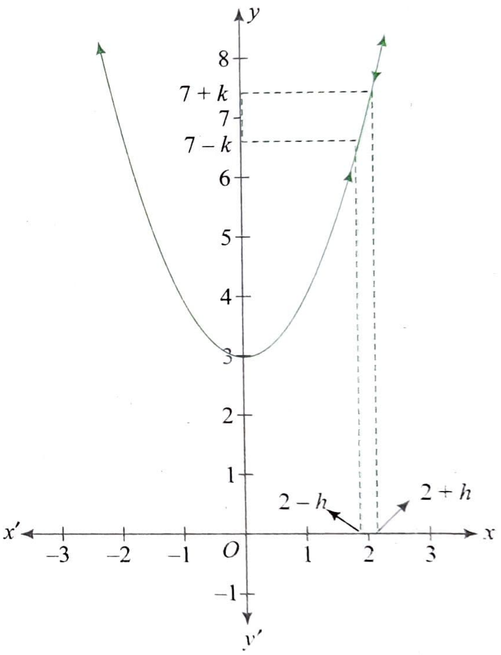
Similarly, we have limiting value of $f(x)$ when $x \to 2^+$ or Right Hand Limit (RHL) of the function.

$$\lim_{x \to 2^+}(x^2 + 3) = \lim_{h \to 0}((2 + h)^2 + 3)$$

$$= \lim_{h \to 0}(7 + h(h + 4))$$

$$= 7 + 0 = 7 \text{ (actually } 7^+)$$

Here, LHL and RHL are not exactly 7 but both these values are approximately equal to 7. That means tendency or behaviour of function $f(x)$ is same when x is approached from both sides of 2, that is to approach 7. This is the reason we say that $\lim_{x \to 2} f(x)$ exists.



In the figure we observe that when $2 - h < x < 2 + h$ or $-h < x - 2 < h$ or $0 < |x - 2| < h$, we have $7 - k < f(x) < 7 + k$ or $-k < f(x) - 7 < k$ or $0 < |f(x) - 7| < k$, where k is infinitely small positive value. More you squeeze the interval $(2 - h, 2 + h)$ more the interval $(7 - k, 7 + k)$ is squeezed or you get limiting value of $f(x)$ closer to 7 when x tends to 2.

Above, we have considered function which is defined and continuous everywhere. Let us consider the case when function is not defined everywhere.

Let $f(x) = \begin{cases} \dfrac{x^2 - 1}{x^2 - 3x + 2}, & x \neq 1 \\ 2, & x = 1 \end{cases}$. Clearly, function is not defined at $x = 1$, still we can check the behaviour of the function when x moves close to 1. Let x approach to 1 from its left hand side.

$$\therefore \quad \text{LHL} = \lim_{x\to 1^-} \frac{x^2-1}{x^2-3x+2}$$

$$= \lim_{h\to 0} \frac{(1-h)^2-1}{(1-h)^2-3(1-h)+2}$$

$$= \lim_{h\to 0} \frac{h^2-2h}{h^2+h}$$

$$= \lim_{h\to 0} \frac{h-2}{h+1} \quad \text{(here we can cancel out } h \text{ as it is not exactly zero)}$$

$$= \frac{0-2}{0+1} = -2$$

Also, $\text{RHL} = \lim_{x\to 1^+} \frac{x^2-1}{x^2-3x+2}$

$$= \lim_{h\to 0} \frac{(1+h)^2-1}{(1+h)^2-3(1+h)+2}$$

$$= \lim_{h\to 0} \frac{2h+h^2}{h^2-h}$$

$$= \lim_{h\to 0} \frac{2+h}{h-1}$$

$$= \frac{2+0}{0-1} = -2$$

Thus LHH = RHL = −2, i.e., behaviour of the function from both sides of $x = 1$ is same, though function is not defined at $x = 1$. The limiting value when x tends to '1' is '−2'. Also the value of the function at $x = 1$ is 2, i.e., $f(1) = 2$. So, here value of function is different from its limiting value at $x = 1$.

Now consider function $f(x) = \frac{|x|}{x}$. Let us check the behaviour of $f(x)$ in the neighbourhood of $x = 0$.

$$\text{LHL} = \lim_{x\to 0^-} \frac{|x|}{x}$$

$$= \lim_{h\to 0} \frac{|0-h|}{0-h}$$

$$= \lim_{h\to 0} \frac{|h|}{-h} \quad (\text{as } |-h| = |h|)$$

$$= \lim_{h\to 0} \frac{h}{-h} \quad (\text{as } h \text{ is positive})$$

$$= -1$$

$$\text{RHL} = \lim_{x\to 0^+} \frac{|x|}{x}$$

$$= \lim_{h\to 0} \frac{|0+h|}{0+h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = 1$$

Thus, here we have LHL ≠ RHL.

EXISTENCE OF LIMIT OF A FUNCTION AT $x = a$

From above discussion we define existence of limit as follows:

The limit of the function $f(x)$ at $x = a$ exists if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ or $\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+h)$, i.e., LHL = RHL at $x = a$. Also, now it is evident that for limit to exist at $x = a$, function need not to be defined at $x = a$.

If function $f(x)$ is single definition function which is defined and continuous at $x = a$, then it is not required to find both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ separately. We can find $\lim_{x\to a} f(x)$.

For example, $\lim_{x\to 0} \cos x = 1$, $\lim_{x\to 1} \frac{x+2}{x^2+1} = \frac{3}{2}$ etc.

ILLUSTRATION 2.1

Find the value of $\lim_{x\to 3^-} \frac{x-2}{x-3}$.

Sol. $L = \lim_{x\to 3^-} \frac{x-2}{x-3} = \lim_{h\to 0} \frac{(3-h)-2}{(3-h)-3} = \lim_{h\to 0} \frac{1-h}{-h}$

Now when h tends to '0', numerator $(1-h)$ moves close to 1 and denominator $(-h)$ moves close to '0' but negative.

Thus, we are dividing 1 with very small negative value. So, the value of the ratio will approach to negative infinity.

Therefore, L approaches to $-\infty$.

ILLUSTRATION 2.2

Prove that $\lim_{x\to 2} [x]$ does not exists, where $[\cdot]$ represents the greatest integer function.

Sol. We know that greatest integer function changes its value at integral values of x.

So, let us find LHL and RHL.

$$\text{LHL} = \lim_{x\to 2^-} [x] = \lim_{h\to 0} [2-h]$$

Sine h is positive and infinitely small $(2-h)$ is value very close to 2 but smaller than 2 (say 1.99999)

$\therefore$ $[2-h] = 1$, when h approaches to zero. Thus, LHL = 1

$$\text{RHL} = \lim_{x\to 2^+} [x] = \lim_{h\to 0} [2+h] = [2.00001] = 2$$

We observe that LHL ≠ RHL. So, $\lim_{x\to 2} [x]$ does not exist.

ILLUSTRATION 2.3

Let $f(x) = \begin{cases} x+1, & \text{if } x \ge 0 \\ x-1, & \text{if } x < 0 \end{cases}$. Then prove that $\lim_{x\to 0} f(x)$ does not exist.

Sol. We have

$$f(x) = \begin{cases} x+1, & \text{if } x \ge 0 \\ x-1, & \text{if } x < 0 \end{cases}$$

This function is piecewise function which changes its definition at $x = 0$.

So, to find limiting value of $f(x)$ at $x = 0$ we find LHL and RHL.

To the left of zero, we have $f(x) = (x - 1)$ and to the right of zero, we have $f(x) = (x + 1)$.

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}((-h)-1)$$
$$= -1$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}(h+1)$$
$$= 1$$

Thus, LHL $\neq$ RHL.

Therefore, $\lim_{x\to 0} f(x)$ does not exist.

ILLUSTRATION 2.4

If $f(x) = \begin{cases} x, \text{ if } x \text{ is rational} \\ 1-x, \text{ if } x \text{ is irrational} \end{cases}$, then find $\lim_{x\to 1/2} f(x)$ if exists.

Sol. $\lim_{x\to 1/2^+} f(x) = \lim_{x\to 1/2^+} x$ (if $1/2^+$ is rational)

$= 1/2$

$\lim_{x\to 1/2^-} f(x) = \lim_{x\to 1/2^+} (1-x)$ (if $1/2^+$ is irrational)

$= 1 - 1/2 = 1/2$

So, $\lim_{x\to 1/2^+} f(x) = 1/2$ in any case.

Similarly, we get $\lim_{x\to 1/2^-} f(x) = 1/2$

$\therefore \quad \lim_{x\to 1/2} f(x) = 1/2$

ILLUSTRATION 2.5

Evaluate $\lim_{x\to 1^+} 2^{-2^{\frac{1}{1-x}}}$.

Sol. $\lim_{x\to 1^+} 2^{-2^{\frac{1}{1-x}}} = \lim_{h\to 0} 2^{-2^{\frac{1}{1-(1+h)}}} = \lim_{h\to 0} 2^{-2^{-\frac{1}{h}}}$

Now when $h \to 0$, $-\dfrac{1}{h} \to -\infty$,

$\therefore \quad 2^{-\frac{1}{h}} \to 0^-$

$\Rightarrow \quad -2^{-\frac{1}{h}} \to 0^+$

$\Rightarrow \quad 2^{-2^{-\frac{1}{h}}} \to 1$

ILLUSTRATION 2.6

Evaluate $\lim_{x\to 0^-} \dfrac{x^2-3x+2}{x^3-2x^2}$.

Sol. $L = \lim_{x\to 0^-} \dfrac{x^2-3x+2}{x^3-2x^2}$

$$= \lim_{h\to 0} \frac{(-h)^2-3(-h)+2}{(-h)^3-2(-h)^2} = -\lim_{h\to 0} \frac{h^2+3h+2}{h^3+2h^2}$$

When $h \to 0$, Numerator $\to 2$ and Denominator $\to 0$

$\therefore \quad L$ tends to $-\infty$

ILLUSTRATION 2.7

Evaluate $\lim_{x\to 0} \dfrac{\sin x - 2}{\cos x - 1}$.

Sol. $L = \lim_{x\to 0} \dfrac{\sin x - 2}{\cos x - 1}$

When $x \to 0$ (either $x \to 0^+$ or $x \to 0^-$, $\cos x \to 1$ (< 1) or $\cos x \to 1^-$)

Thus, denominator tends to 0^-

Also, numerator tends to '-2'

Therefore, L tends to ∞.

ILLUSTRATION 2.8

If a and b are positive and $[x]$ denotes greatest integer less than or equal to x, then find $\lim_{x\to 0^+} \dfrac{x}{a}\left[\dfrac{b}{x}\right]$.

Sol. $L = \lim_{x\to 0^+} \dfrac{x}{a}\left[\dfrac{b}{x}\right]$

$$= \lim_{x\to 0^+} \frac{x}{a}\left(\frac{b}{x} - \left\{\frac{b}{x}\right\}\right),$$ where $\{\cdot\}$ represents the functional part function

$$= \lim_{x\to 0^+} \left(\frac{b}{a} - \frac{x}{a}\left\{\frac{b}{x}\right\}\right)$$

$$= \frac{b}{a} - \frac{1}{a} \lim_{x\to 0^+} \frac{\left\{\frac{b}{x}\right\}}{\frac{1}{x}}$$

Since $0 < \left\{\dfrac{b}{x}\right\} < 1$ and $\lim_{x\to 0^+} \dfrac{1}{x} = \infty$, we have

$$L = \frac{b}{a} - 0 = \frac{b}{a}$$

ILLUSTRATION 2.9

Evaluate the left- and right-hand limits of the function

$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$ at $x = 4$.

Sol. LHL of $f(x)$ at $x = 4$ is

$$\lim_{x\to 4^-} f(x) = \lim_{h\to 0} f(4-h)$$
$$= \lim_{h\to 0} \frac{|4-h-4|}{4-h-4} = \lim_{h\to 0} \frac{|-h|}{-h}$$
$$= \lim_{h\to 0} \frac{h}{-h} = \lim_{h\to 0} -1$$
$$= -1$$

RHL of $f(x)$ at $x = 4$ is

$$\lim_{x\to 4^+} f(x) = \lim_{h\to 0} f(4+h)$$
$$= \lim_{h\to 0} \frac{|4+h-4|}{4+h-4}$$
$$= \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1$$
$$= 1$$

ILLUSTRATION 2.10

Evaluate the left- and right-hand limits of the function defined by $f(x) = \begin{cases} 1+x^2, & \text{if } 0 \le x < 1 \\ 2-x, & \text{if } x > 1 \end{cases}$ at $x = 1$. Also, show that $\lim_{x\to 1} f(x)$ does not exist.

Sol. LHL of $f(x)$ at $x = 1$ is

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$$
$$= \lim_{h\to 0} [1 + (1-h)^2]$$
$$= \lim_{h\to 0} (2 - 2h + h^2) = 2$$

RHL of $f(x)$ at $x = 1$ is

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$
$$= \lim_{h\to 0} [2 - (1+h)]$$
$$= \lim_{h\to 0} (1-h) = 1$$

Clearly, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

So, $\lim_{x\to 1} f(x)$ does not exist.

ILLUSTRATION 2.11

Let $f(x) = \begin{cases} \cos[x], & x \ge 0 \\ |x|+a, & x < 0 \end{cases}$. Then find the value of a, so that $\lim_{x\to 0} f(x)$ exists, where $[x]$ denotes the greatest integer function less than or equal to x.

Sol. Since $\lim_{x\to 0} f(x)$ exists, we have

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x)$$

or $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h)$

or $\lim_{h\to 0} |0-h| + a = \lim_{h\to 0} \cos[0+h]$

or $a = \cos 0 = 1$

$\therefore$ $a = 1$

ILLUSTRATION 2.12

Evaluate: $\lim_{x\to \frac{5\pi}{4}} [\sin x + \cos x]$, where $[\cdot]$ denotes the greatest integer function.

Sol. We have,

$$L = \lim_{x\to \frac{5\pi}{4}} [\sin x + \cos x] = \lim_{x\to \frac{5\pi}{4}} \left[\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right]$$

Now, $\lim_{x\to \frac{5\pi}{4}^+} \left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right] = \left[\sqrt{2}\sin\left(\frac{3\pi^+}{2}\right)\right] = [-\sqrt{2}] = -2$

And $\lim_{x\to \frac{5\pi}{4}^+} \left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right] = \left[\sqrt{2}\sin\left(\frac{3\pi^-}{2}\right)\right] = [-\sqrt{2}] = -2$

So, $L = -2$

ILLUSTRATION 2.13

Let $f(x) = \begin{cases} x+1, & x > 0 \\ 2-x, & x \le 0 \end{cases}$ and $g(x) = \begin{cases} x+3, & x < 1 \\ x^2 - 2x - 2, & 1 \le x < 2 \\ x-5, & x \ge 2 \end{cases}$

Find the LHL and RHL of $g(f(x))$ at $x = 0$ and, hence, find $\lim_{x\to 0} g(f(x))$.

Sol. $x \to 0^- \Rightarrow f(x) \to f(0^-) = 2^+$ (using $f(x) = 2 - x$)

or $\lim_{x\to 0^-} g(f(x)) = g(2^+) = -3$ (using $g(x) = x - 5$)

Also, $x \to 0^+ \Rightarrow f(x) \to f(0^+) = 1^+$ (using $f(x) = x + 1$)

or $\lim_{x\to 0^+} g(f(x)) = g(1^+) = -3$ (using $g(x) = x^2 - 2x - 2$)

Hence, $\lim_{x\to 0} g(f(x))$ exists and is equal to -3. Therefore,

$$\lim_{x\to 0} g(f(x)) = -3$$

CONCEPT APPLICATION EXERCISE 2.1

1. Evaluate $\lim_{x\to -2^+} \frac{x^2-1}{2x+4}$.

2. Evaluate $\lim_{x\to 2^+} \frac{[x-2]}{\log(x-2)}$, where $[\cdot]$ represents the greatest integer function.

3. Evaluate $\lim_{x\to 0} \frac{\sin[\cos x]}{1+[\cos x]}$ ($[\cdot]$ denotes the greatest integer function).

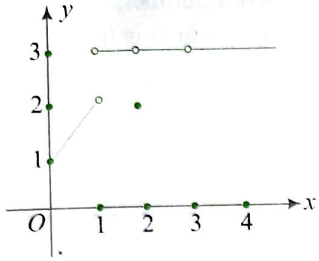
4. If $f(x) = \begin{cases} \frac{x-|x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$, show that $\lim_{x\to 0} f(x)$ does not exist.

5. Show that $\lim_{x\to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$ does not exist.

6. Evaluate $\lim_{x\to 0} \frac{3x+|x|}{7x-5|x|}$.

7. If $f(x)=\begin{cases} x, & x<0 \\ 1, & x=0 \\ x^2, & x>0 \end{cases}$, then find $\lim\limits_{x\to 0} f(x)$ if exists.

8. Consider the following graph of the function $y = f(x)$. Which of the following is/are correct?



(a) $\lim\limits_{x\to 1} f(x)$ does not exist.

(b) $\lim\limits_{x\to 2} f(x)$ does not exist.

(c) $\lim\limits_{x\to 3} f(x) = 3$.

(d) $\lim\limits_{x\to 1.99} f(x)$ exists.

9. Evaluate $\lim\limits_{x\to 0} \dfrac{\tan(\operatorname{sgn}(x))}{\operatorname{sgn}(x)}$ if exists.

10. If $f(x)=\begin{cases} \sin x, & x \ne n\pi,\ n\in I \\ 2, & \text{otherwise} \end{cases}$ and

$g(x)=\begin{cases} x^2+1, & x\ne 0, 2 \\ 4, & x=0 \\ 5, & x=2 \end{cases}$ then find $\lim\limits_{x\to 0} g\{f(x)\}$.

ANSWERS

1. ∞. **2.** 0 **3.** 0
6. Limit does not exist. **7.** 0 **8.** (a), (c), (d)
9. tan 1 **10.** 1

RULES TO FIND LIMIT

ALGEBRA OF LIMITS

Let $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m$. If l and m exist, then

(i) $\lim\limits_{x\to a} (f \pm g)(x) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = l \pm m$

(ii) $\lim\limits_{x\to a} (fg)(x) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = lm$

(iii) $\lim\limits_{x\to a} \left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{l}{m}$, provided $m \ne 0$

(iv) $\lim\limits_{x\to a} kf(x) = k.\ \lim\limits_{x\to a} f(x)$, where k is constant

(v) $\lim\limits_{x\to a} (f(x))^{g(x)} = \lim\limits_{x\to a} f(x)^{\lim\limits_{x\to a} g(x)} = l^m$

(vi) $\lim\limits_{x\to a} fog(x) = f\left(\lim\limits_{x\to a} g(x)\right) = f(m)$, only if f is continuous at $g(x) = m$

In particular,

(a) $\lim\limits_{x\to a} \log f(x) = \log\left(\lim\limits_{x\to a} f(x)\right) = \log l$

(b) $\lim\limits_{x\to a} e^{f(x)} = e^{\lim\limits_{x\to a} f(x)} = e^l$

(c) $\lim\limits_{x\to a} |f(x)| = \left|\lim\limits_{x\to a} f(x)\right| = |l|$

(vii) If $\lim\limits_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim\limits_{x\to a} \dfrac{1}{f(x)} = 0$

FINDING LIMIT USING DIRECT SUBSTITUTION

Consider the following limits:

(i) $\lim\limits_{x\to a} f(x)$ **(ii)** $\lim\limits_{x\to a} \dfrac{g(x)}{h(x)}$

If $f(a)$ and $\dfrac{g(a)}{h(a)}$ exist and are fixed real numbers and $h(a) \ne 0$ then we say that $\lim\limits_{x\to a} f(x) = f(a)$ and $\lim\limits_{x\to a} \dfrac{g(x)}{h(x)} = \dfrac{g(a)}{h(a)}$

In other words, if by direct substitution of the value of x to which the variable tends to, we obtain a fixed real number, then the number obtained is the limit of the function. In fact, if the point to which the variable tends to is a point in the domain of the function, then the value of the function at that point is its limit.

e.g., $\lim\limits_{x\to 1} (3x^2 + 4x + 5) = 3(1)^2 + 4(1) + 5 = 12$,

$\lim\limits_{x\to 2} \dfrac{x^2-4}{x+3} = \dfrac{4-4}{2+3} = \dfrac{0}{5} = 0$, etc.

ILLUSTRATION 2.14

If $\lim\limits_{x\to a}[f(x)+g(x)] = 2$ and $\lim\limits_{x\to a}[f(x)-g(x)] = 1$, then find the value of $\lim\limits_{x\to a} f(x)g(x)$.

Sol. $\lim\limits_{x\to a}[f(x)+g(x)] = 2$

or $\lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = 2$ (1)

$\lim\limits_{x\to a}[f(x)-g(x)] = 1$

or $\lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = 1$ (2)

Adding (1) and (2),

$2\lim\limits_{x\to a} f(x) = 3$ or $\lim\limits_{x\to a} f(x) = \dfrac{3}{2}$

Subtracting (2) from (1),

$2\lim\limits_{x\to a} g(x) = 1$ or $\lim\limits_{x\to a} g(x) = \dfrac{1}{2}$

or $\lim\limits_{x\to a} f(x)g(x) = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x) = \dfrac{3}{2} \times \dfrac{1}{2} = \dfrac{3}{4}$

ILLUSTRATION 2.15

Find the following limits.

(i) $\lim_{x\to 2}\frac{4x}{x^3-3}$ (ii) $\lim_{x\to 1}\frac{\log_{10}x-3}{3x-2}$ (iii) $\lim_{x\to\pi}\frac{3+\cos x}{2-\sin x}$

Sol. In each of the given limits function is defined and continuously exists in the NBD of x where we have to find the limit. So, we will find the limits using the direct substitution

(i) $\lim_{x\to 2}\frac{4x}{x^3-3}=\frac{4(2)}{(2)^3-3}=\frac{8}{5}$

(ii) $\lim_{x\to 1}\frac{\log_{10}x-3}{3x-2}=\frac{(\log_{10}1)-3}{3(1)-2}=\frac{0-3}{1}=-3$

(iii) $\lim_{x\to\pi}\frac{3+\cos x}{2-\sin x}=\frac{3+\cos\pi}{2-\sin\pi}=\frac{3-1}{2-0}=1$

ILLUSTRATION 2.16

Evaluate $\lim_{x\to 0}\left(\frac{x+4}{2-x}\right)^{\frac{x^2+2x-3}{x-1}}$.

Sol. $\lim_{x\to 0}\left(\frac{x+4}{2-x}\right)^{\frac{x^2+2x-3}{x-1}}$

$=\left(\lim_{x\to 0}\frac{x+4}{2-x}\right)^{\lim_{x\to 0}\frac{x^2+2x-3}{x-1}}$

$=\left(\frac{0+4}{2-0}\right)^{\frac{0+0-3}{0-1}}$

$=2^{-3}$

$=1/8$

ILLUSTRATION 2.17

If $a_1=1$ and $a_{n+1}=\frac{4+3a_n}{3+2a_n}$, $n\ge 1$ and if $\lim_{n\to\infty}a_n=a$, then find the value of a.

Sol. We have $a_{n+1}=\frac{4+3a_n}{3+2a_n}$

$\therefore\quad \lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}\frac{4+3a_n}{3+2a_n}$

$\Rightarrow\quad \lim_{n\to\infty}a_{n+1}=\frac{4+3\lim_{n\to\infty}a_n}{3+2\lim_{n\to\infty}a_n}$

$\Rightarrow\quad a=\frac{4+3a}{3+2a}$ $\left(\because \lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}a_n=a\right)$

$\Rightarrow\quad 2a^2=4$

$\Rightarrow\quad a=\sqrt{2}$ $(a\ne-\sqrt{2}$ because $a_n>0)$

ILLUSTRATION 2.18

Evaluate $\lim_{x\to\infty}\frac{\sin x}{x}$.

Sol. We have $\lim_{x\to\infty}\frac{\sin x}{x}$

Since infinity is not a fixed number, when $x\to\infty$, $\sin(\infty)$ takes any value between -1 to 1, which is finite.

So, we have finite value divided by infinity. So, ratio approaches to zero.

$\therefore\quad \lim_{x\to\infty}\frac{\sin x}{x}=0$

IMPORTANT CONCEPTS

- If $\lim_{x\to c}f(x)\cdot g(x)$ exists then we can have following cases:

  (i) Both $\lim_{x\to c}f(x)$ and $\lim_{x\to c}g(x)$ exist.

  Obviously then $\lim_{x\to c}f(x)\cdot g(x)$ exists.

  (ii) $\lim_{x\to c}f(x)$ exists and $\lim_{x\to c}g(x)$ does not exist.

  Consider $f(x)=x^2-4$, and $g(x)=\frac{1}{x^2+x-6}$.

  Now $\lim_{x\to 2}f(x)\cdot g(x)=\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}=\frac{4}{5}$,

  though $\lim_{x\to 2}\frac{1}{x^2+x-6}$ does not exist.

  (iii) Both $\lim_{x\to c}f(x)$ and $\lim_{x\to c}g(x)$ do not exist.

  Consider functions $f(x)=\begin{cases}1, & x\le 0\\ 2, & x>0\end{cases}$ and $g(x)=\begin{cases}2, & x\le 0\\ 1, & x>0\end{cases}$.

  Then, $f(x)\times g(x)=2\ \forall\ x\in R$, and so $\lim_{x\to 0}f(x)\cdot g(x)=2$, though $\lim_{x\to 0}f(x)$ and $\lim_{x\to 0}g(x)$ do not exist.

- If $\lim_{x\to c}(f(x)+g(x))$ exist then we can have following cases:

  (i) If $\lim_{x\to c}f(x)$ exists, then $\lim_{x\to c}g(x)$ must exist.

  Proof : This is true as $g=(f+g)-f$.

  Therefore, by the limit theorem,

  $\lim_{x\to c}g(x)=\lim_{x\to c}(f(x)+g(x))-\lim_{x\to 0}f(x)$, which exists.

  (ii) Both $\lim_{x\to c}f(x)$ and $\lim_{x\to c}g(x)$ do not exist.

  Consider $\lim_{x\to 1}[x]$ and $\lim_{x\to 1}\{x\}$, where $[\cdot]$ and $\{\cdot\}$ represent greatest integer and fractional part function, respectively. Here both the limits do not exist but $\lim_{x\to 1}([x]+\{x\})=\lim_{x\to 1}x=1$.

ILLUSTRATION 2.19

Find $\lim_{x\to 0}[x]\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$, (where $[\cdot]$ represents the greatest integer function).

Sol. $\lim_{x\to 0^+}[x]\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right) = \lim_{h\to 0}[h]\left(\frac{e^{1/h}-1}{e^{1/h}+1}\right)$

$$= \lim_{h\to 0}[h]\left(\frac{1-e^{-1/h}}{1+e^{-1/h}}\right)$$

$$= 0\times\frac{1-0}{1+0} = 0$$

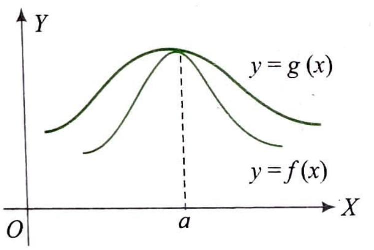
$$\lim_{x\to 0^-}[x]\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right) = \lim_{h\to 0}[-h]\left(\frac{e^{-1/h}-1}{e^{-1/h}+1}\right)$$

$$= (-1)\times\frac{0-1}{0+1} = 1$$

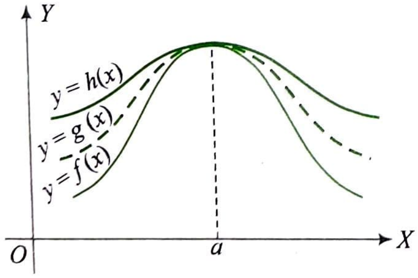
SANDWICH RULE

Sandwich theorem also known as squeeze theorem is a theorem regarding the limit of a function. It is typically used to confirm the limit of a function via comparison with two other functions whose limits are known or easily computed. It was first used geometrically by the mathematicians Archimedes and Eudoxus in an effort to compute π, and was formulated in modern terms by Carl Friedrich Gauss.

Let f and g be two real valued functions with the same domain such that $f(x) \le g(x)$ for all x in the domain of definition. For some a, if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.



Let f, g and h be real functions such that $f(x) \le g(x) \le h(x)$ for all x in the common domain of definition. For some real number a, if $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$.



It should be noted that In **Sandwich Theorem**, we assume that $f(x) \le g(x) \le h(x)$ for all x near a, "except possibly at a". This means that it is not required that when $x = a$, we have the inequality for the functions. That is, it is not required that $f(a) \le h(a)$. The reason is that we are dealing with limits as x approaches a. So, none of the following claims can be guaranteed by the assumptions in the Sandwich Theorem:

(a) $f(a) = g(a) = h(a)$. [Well, not even $f(a) \le g(a) \le h(a)$]

(b) $g(a) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

(c) $\lim_{x\to a} g(x) = g(a)$ etc.

Consider $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

This limit cannot be evaluated through limit law

$$\lim_{x\to\infty} f(x)\cdot g(x) = \lim_{x\to\infty} f(x)\cdot \lim_{x\to\infty} g(x)$$

as $\quad \lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

But $\quad -1 \le \sin\frac{1}{x} \le 1$

$\Rightarrow \quad -x^2 \le x^2 \sin\frac{1}{x} \le x^2$

Now $\quad \lim_{x\to 0} x^2 = \lim_{x\to 0}(-x^2) = 0$.

So, by the Sandwich theorem, $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

ILLUSTRATION 2.20

In the neighbourhood of $x = 0$ it is known that $1+|x| < \frac{e^x-1}{x} < 1-|x|$ then find $\lim_{x\to 0}\frac{e^x-1}{x}$.

Sol. We have $1+|x| < \frac{e^x-1}{x} < 1-|x|$ for $x \in (0-\delta, 0+\delta)$.

$\therefore \quad$ Using Sandwich rule,

$$\lim_{x\to 0}(1+|x|) < \lim_{x\to 0}\frac{e^x-1}{x} < \lim_{x\to 0}(1-|x|)$$

$$\Rightarrow \quad 1 < \lim_{x\to 0}\frac{e^x-1}{x} < 1$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{e^x-1}{x} = 1$$

ILLUSTRATION 2.21

If $0 < \log_e x < \sqrt{x}$ for all $x > 1$, then find the value of $\lim_{x\to\infty}\frac{\log_e x}{x}$.

Sol. As $0 < \log_e x < \sqrt{x}$ $(x > 1)$

$$\Rightarrow \quad 0 < \frac{\log_e x}{x} < \frac{1}{\sqrt{x}}$$

$$\Rightarrow \quad 0 < \lim_{x\to\infty}\frac{\log_e x}{x} < \lim_{x\to\infty}\frac{1}{\sqrt{x}}$$

$$\Rightarrow \quad 0 < \lim_{x\to\infty}\frac{\log_e x}{x} < 0$$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{\log_e x}{x} = 0$$

(Using Sandwich Rule

ILLUSTRATION 2.22

If $3-\left(\frac{x^2}{12}\right)\le f(x)\le 3+\left(\frac{x^3}{9}\right)$ in the neighborhood of $x=0$, then find the value of $\lim_{x\to 0} f(x)$.

Sol. According to question,

$$\lim_{x\to 0}\left(3-\frac{x^2}{12}\right)\le \lim_{x\to 0} f(x)\le \lim_{x\to 0}\left(3+\frac{x^3}{9}\right)$$

or $\quad (3-0)\le \lim_{x\to 0} f(x)\le (3+0)$

Hence, $\lim_{x\to 0} f(x)=3$ (Using sandwich theorem)

CONCEPT APPLICATION EXERCISE 2.2

1. If $|f(x)|\le x^2$, then prove that $\lim_{x\to 0}\frac{f(x)}{x}=0$.
2. If $f(x)=\text{sgn}(x)$ and $g(x)=x^3$, then prove that $\lim_{x\to 0} f(x)\cdot g(x)$ exists though $\lim_{x\to 0} f(x)$ does not exist.
3. If $f(x)=\begin{cases}\frac{\sin[x]}{[x]}, & \text{for } [x]\ne 0\\ 0, & \text{for } [x]=0\end{cases}$, where $[x]$ denotes the greatest integer less than or equal to x. Then find $\lim_{x\to 0} f(x)$.
4. Find the value of $\lim_{x\to 0^+}(\sin x)^{\frac{1}{x}}$.
5. Let the sequence $<b_n>$ of real numbers satisfy the recurrence relation $b_{n+1}=\frac{1}{3}\left(2b_n+\frac{125}{b_n^2}\right)$, $b_n\ne 0$. Then find $\lim_{n\to\infty} b_n$.
6. Let $f: (1,2)\to R$ satisfy the inequality $\frac{\cos(2x-4)-33}{2}<f(x)<\frac{x^2|4x-8|}{x-2}\ \forall x\in(1,2)$. Then find $\lim_{x\to 2^-} f(x)$.
7. If $\frac{x^2+x-2}{x+3}\le\frac{f(x)}{x^2}\le\frac{x^2+2x-1}{x+3}$ holds for a certain interval containing the point $x=-1$ and $\lim_{x\to -1} f(x)$ exists, then find the value of $\lim_{x\to -1} f(x)$.

ANSWERS

3. Limit does not exist. 4. 0 5. 5
6. −16 7. −1

LIMIT OF INDETERMINATE FORMS

INDETERMINATE FORMS

Consider function $f(x)=\frac{x^2-1}{x^2-3x+2}$.

Clearly $f(x)$ is not defined at $x=1$.
But we can find limiting value of $f(x)$ at $x=1$.

In $\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2}$, we cannot substitute $x=1$ as denominator and numerator becomes zero.

We may say that when x approaches to zero, numerator and denominator both approaches to zero and the function takes form $\frac{0}{0}$. This form is called indeterminate form as we cannot judge the value of such ratio (certainly such ratio is not '1' always as degree of closeness to zero of numerator and denominator may be different). To find the limiting value of such ratio we first try to eliminate the factor which causes such indeterminate form.

So, $$\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x-2)}$$
$$=\lim_{x\to 1}\frac{x+1}{x-2}$$
$$=\frac{0+1}{0-2}=-\frac{1}{2}.$$

Here we are cancelling factor $(x-1)$ from numerator and denominator to get rid of indeterminate form. Then we have used direct substitution to get the limiting value.

We have other indeterminate forms like $0\times\infty$, $\infty-\infty$, $\frac{\infty}{\infty}$, ∞^0, 0^0, 1^∞. Here '0' is not exact zero, it is very close to zero and '1' is not exactly 1.

We can convert one indeterminate form in the other indeterminate form.

i.e., $$0\times\infty=\frac{0}{1/\infty}=\frac{0}{0}.$$
$$\frac{\infty_1}{\infty_2}=\frac{1/\infty_2}{1/\infty_1}=\frac{0}{0}.$$

If $\quad y=\infty^0$, then $\log y=0\times\log\infty=0\times\infty$
If $\quad y=0^0$, then $\log y=0\times\log 0=0\times(-\infty)$
If $\quad y=1^\infty$, then $\log y=\infty\times\log 1=\infty\times 0$

ILLUSTRATION 2.23

Evaluate $\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}$.

Sol. When $x=2$, the expression $\frac{x^2-5x+6}{x^2-4}$ assumes the indeterminate form $\frac{0}{0}$. Here, $(x-2)$ is a common factor in numerator and denominator. Factorizing the numerator and denominator, we have

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x-3)}{(x+2)(x-2)}$$

$$=\lim_{x\to 2}\frac{x-3}{x+2}=\frac{2-3}{2+2}=-\frac{1}{4}$$

ILLUSTRATION 2.24

Evaluate $\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right)$.

Sol. We have

$$\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right) \qquad (\infty-\infty \text{ form})$$

$$=\lim_{x\to 1}\left(\frac{2}{1-x^2}-\frac{1}{1-x}\right)$$

When $x = 1$, the expression $\frac{2}{1-x^2}-\frac{1}{1-x}$ assumes the form $\infty-\infty$.
So, we need some simplification to express it in the form $\frac{0}{0}$.
Then,

$$\lim_{x\to 1}\left(\frac{2}{1-x^2}-\frac{1}{1-x}\right)=\lim_{x\to 1}\frac{2-(1+x)}{1-x^2}=\lim_{x\to 1}\frac{1-x}{1-x^2}=\lim_{x\to 1}\frac{1}{1+x}=\frac{1}{2}$$

ILLUSTRATION 2.25

Evaluate $\lim_{x\to 1}\frac{x^2+x\log_e x-\log_e x-1}{(x^2-1)}$.

Sol. $\lim_{x\to 1}\frac{x^2+x\log_e x-\log_e x-1}{(x^2-1)} \quad \left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 1}\frac{(x-1)(\log_e x+x+1)}{(x+1)(x-1)} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 1}\frac{\log_e x+x+1}{x+1}$$

$$=\frac{\log_e 1+1+1}{1+1}=\frac{0+2}{2}=1$$

ILLUSTRATION 2.26

Evaluate $\lim_{x\to \frac{3\pi}{4}}\frac{1+\sqrt[3]{\tan x}}{1-2\cos^2 x}$.

Sol. $\lim_{x\to \frac{3\pi}{4}}\frac{1+\sqrt[3]{\tan x}}{1-2\cos^2 x}$

$$=\lim_{x\to \frac{3\pi}{4}}\frac{1+(\tan x)^{1/3}}{-\cos 2x}\cdot\frac{1-(\tan x)^{1/3}+(\tan x)^{2/3}}{1-(\tan x)^{1/3}+(\tan x)^{2/3}}$$

$$=-\lim_{x\to \frac{3\pi}{4}}\frac{1+\tan x}{\frac{1-\tan^2 x}{1+\tan^2 x}}\cdot\frac{1}{1-(-1)^{1/3}+(-1)^{2/3}}$$

$$=-\lim_{x\to \frac{3\pi}{4}}\frac{1+\tan^2 x}{1-\tan x}\cdot\frac{1}{3}=-\frac{1+1}{1-(-1)}\cdot\frac{1}{3}=-\frac{1}{3}$$

ILLUSTRATION 2.27

Evaluate $\lim_{x\to\infty}\frac{\log_e x}{x}$.

Sol. We have $\lim_{x\to\infty}\frac{\log_e x}{x}$

When $x\to\infty$, $\log_e x\to\infty$. So, we have indeterminate form $\frac{\infty}{\infty}$.

But $\log_e x$ is very sluggish function, i.e., increases very slowly compared to x.

So, at infinity though both $\log_e x$ and x approaches to infinity, but the difference between these infinity values is also infinity, i.e., x is far more infinite than $\log_e x$.

Therefore $\lim_{x\to\infty}\frac{\log_e x}{x}=0$

ILLUSTRATION 2.28

Evaluate $\lim_{n\to\infty}(4^n+5^n)^{1/n}$.

Sol. We have,

$$\lim_{n\to\infty}(4^n+5^n)^{1/n} \qquad (\infty^0 \text{ from})$$

$$=\lim_{n\to\infty}5\left(1+\left(\frac{4}{5}\right)^n\right)^{1/n}$$

$$=5 \qquad \left(\because \left(\frac{4}{5}\right)^n\to 0 \text{ as } n\to\infty\right)$$

FINDING LIMITS USING RATIONALIZATION METHOD

This is particularly used when either the numerator or denominator or both involve expression consisting of square roots and on substituting the value of x, expression takes the form $\frac{0}{0}, \frac{\infty}{\infty}$.

ILLUSTRATION 2.29

Evaluate $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$.

Sol. When $x\to 0$, the expression $\frac{\sqrt{2+x}-\sqrt{2}}{x}$ takes the form $\frac{0}{0}$. Rationalizing the numerator, we have

$$\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}=\lim_{x\to 0}\frac{\left(\sqrt{2+x}-\sqrt{2}\right)\left(\sqrt{2+x}+\sqrt{2}\right)}{x\left(\sqrt{2+x}+\sqrt{2}\right)}$$

$$=\lim_{x\to 0}\frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}}=\frac{1}{2\sqrt{2}}$$

ILLUSTRATION 2.30

Evaluate $\lim_{x\to a} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}, (a \neq 0)$.

Sol. We have,

$$\lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to a} \frac{\left(\sqrt{a+2x}-\sqrt{3x}\right)\left(\sqrt{a+2x}+\sqrt{3x}\right)}{\left(\sqrt{3a+x}-2\sqrt{x}\right)\left(\sqrt{3a+x}+2\sqrt{x}\right)} \frac{\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\left(\sqrt{a+2x}+\sqrt{3x}\right)} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to a} \frac{(a+2x-3x)}{(3a+x-4x)} \frac{\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x\to a} \frac{\sqrt{3a+x}+2\sqrt{x}}{3\left(\sqrt{a+2x}+\sqrt{3x}\right)}$$

$$= \frac{\sqrt{3a+a}+2\sqrt{a}}{3\left(\sqrt{a+2a}+\sqrt{3a}\right)} = \frac{1}{3}\cdot\frac{4\sqrt{a}}{2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

ILLUSTRATION 2.31

Evaluate

$\lim_{x\to\pi/2} \tan^2 x(\sqrt{2\sin^2 x+3\sin x+4}-\sqrt{\sin^2 x+6\sin x+2})$.

Sol. Rationalizing we get

$$\lim_{x\to\pi/2} \tan^2 x \frac{(2\sin^2 x+3\sin x+4-\sin^2 x-6\sin x-2)}{\sqrt{2\sin^2 x+3\sin x+4}+\sqrt{\sin^2 x+6\sin x+2}}$$

$$= \lim_{x\to\pi/2} \frac{\sin^2 x(\sin x-1)(\sin x-2)}{(1-\sin^2 x)(\sqrt{9}+\sqrt{9})}$$

$$= \lim_{x\to\pi/2} \frac{-\sin^2 x(\sin x-2)}{6(1+\sin x)}$$

$$= \frac{-1(1-2)}{6(1+1)} = \frac{1}{12}$$

FINDING LIMITS AT INFINITY

We know that $\lim_{x\to\pm\infty} \dfrac{c}{x}, \lim_{x\to\pm\infty} \dfrac{c}{x^2}, \lim_{x\to\pm\infty} \dfrac{c}{x^3} = 0$.

Now consider the limit of the form

$\lim_{x\to\infty} \dfrac{a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n}{b_0x^n+b_1x^{n-1}+b_2x^{n-2}+\ldots+b_{n-1}x+b_n}$. This is $\dfrac{\infty}{\infty}$ form.

Divide numerator and denominator by highest power of x occurring that is x^n, we get

$$\lim_{x\to\infty} \frac{a_0+\dfrac{a_1}{x}+\dfrac{a_2}{x^2}+\ldots+\dfrac{a_{n-1}}{x^{n-1}}+\dfrac{a_n}{x^n}}{b_0+\dfrac{b_1}{x}+\dfrac{b_2}{x^2}+\ldots+\dfrac{b_{n-1}}{x^{n-1}}+\dfrac{b_n}{x^n}}$$

$$= \frac{a_0+0+0+\ldots+0}{b_0+0+0+\ldots+0} = \frac{a_0}{b_0}$$

ILLUSTRATION 2.32

Evaluate $\lim_{n\to\infty} \dfrac{1^3+2^3+3^3+\ldots+n^3}{\sqrt{4n^8+1}}$.

Sol. $$\lim_{n\to\infty} \frac{1^3+2^3+3^3+\ldots+n^3}{\sqrt{4n^8+1}} = \frac{1}{4}\lim_{n\to\infty} \frac{n^2(n+1)^2}{\sqrt{4n^8+1}}$$

$$= \frac{1}{4}\lim_{n\to\infty} \frac{\dfrac{(n+1)^2}{n^2}}{\dfrac{1}{n^4}\sqrt{4n^8+1}}$$

$$= \frac{1}{4}\lim_{n\to\infty} \frac{\left(1+\dfrac{1}{n}\right)^2}{\sqrt{4+\dfrac{1}{n^8}}} = \frac{1}{8}$$

ILLUSTRATION 2.33

If $f(x) = \dfrac{x^2-3x+2}{x^2-7x+12}$, then which of the following limits exists?

(i) $\lim_{x\to-\infty} \sin^{-1} f(x)$ **(ii)** $\lim_{x\to\infty} \cos^{-1} f(x)$

Sol. $$\lim_{x\to\infty} \frac{x^2-3x+2}{x^2-7x+12}$$

$$= \lim_{x\to\infty} \frac{1-\dfrac{3}{x}+\dfrac{2}{x^2}}{1-\dfrac{7}{x}+\dfrac{12}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1-\dfrac{3}{x}}{1-\dfrac{7}{x}} \quad \text{(Ignoring division with higher power)}$$

$$= 1^+ \quad \text{(as } 3/x < 7/x\text{, when } x > 0\text{)}$$

So, $\lim_{x\to\infty} \cos^{-1} f(x)$ does not exist as $\cos^{-1}x$ is defined for $x \in [-1, 1]$.

Also, $$\lim_{x\to-\infty} \frac{1-\dfrac{3}{x}+\dfrac{2}{x^2}}{1-\dfrac{7}{x}+\dfrac{12}{x^2}} = \lim_{x\to\infty} \frac{1-\dfrac{3}{x}}{1-\dfrac{7}{x}}$$

$$= 1^- \quad \text{(as } 3/x > 7/x\text{, when } x < 0\text{)}$$

So, $\lim_{x\to-\infty} \sin^{-1} f(x)$ exists.

ILLUSTRATION 2.34

Evaluate $\lim_{x\to\infty} \dfrac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$.

Sol. Dividing each term in the numerator and denominator by x, we get

$$\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$$

$$= \lim_{x\to\infty} \frac{\sqrt{3-1/x^2}-\sqrt{2-1/x^2}}{4+3/x}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{4}$$

ILLUSTRATION 2.35

Evaluate $\lim_{x\to\infty} \sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)$.

Sol. $\lim_{x\to\infty} \sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)$ $(\infty - \infty \text{ form})$

$$= \lim_{x\to\infty} \frac{\sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)\left(\sqrt{x+c}+\sqrt{x}\right)}{\left(\sqrt{x+c}+\sqrt{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}(x+c-x)}{\sqrt{x+c}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{c\sqrt{x}}{\sqrt{x+c}+\sqrt{x}} \quad \left(\text{Form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{c}{\sqrt{1+\frac{c}{x}}+1} \quad [\text{Dividing } N^r \text{ and } D^r \text{ by } \sqrt{x}]$$

$$= \frac{c}{\sqrt{1+0}+1} = \frac{c}{2}$$

ILLUSTRATION 2.36

Find the value of $\lim_{x\to 0^+} \dfrac{3(\log_e x)^2+5\log_e x+6}{1+(\log_e x)^2}$.

Sol. We know that $\lim_{x\to 0^+} \log_e x = -\infty$

$$\therefore \quad \lim_{x\to 0^+} \frac{3(\log_e x)^2+5\log_e x+6}{1+(\log_e x)^2} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{3+\dfrac{5}{\log_e x}+\dfrac{6}{(\log_e x)^2}}{1+\dfrac{1}{(\log_e x)^2}} \right)$$

$$= 3$$

ILLUSTRATION 2.37

Evaluate $\lim_{x\to-\infty} \dfrac{3^{\sin x}+2x+1}{\sin x-\sqrt{x^2+1}}$.

Sol.
$$\lim_{x\to-\infty} \frac{3^{\sin x}+2x+1}{\sin x-\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{\dfrac{3^{\sin x}}{x}+2+\dfrac{1}{x}}{\dfrac{\sin x}{x}-\dfrac{1}{x}\sqrt{x^2+1}}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{3^{\sin x}}{x}+2+\dfrac{1}{x}}{\dfrac{\sin x}{x}+\dfrac{1}{|x|}\sqrt{x^2+1}}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{3^{\sin x}}{x}+2+\dfrac{1}{x}}{\dfrac{\sin x}{x}+\sqrt{1+\dfrac{1}{x^2}}} = 2$$

ILLUSTRATION 2.38

Evaluate $\lim_{x\to\infty} \dfrac{x+7\sin x}{-2x+13}$ using sandwich theorem.

Sol. We know that $-1 \le \sin x \le 1$ for all x. So,

$$-7 \le 7\sin x \le 7$$

or $\quad x-7 \le x+7\sin x \le x+7$

Dividing throughout by $-2x + 13$, we get

$$\frac{x-7}{-2x+13} \ge \frac{x+7\sin x}{-2x+13} \ge \frac{x+7}{-2x+13}$$

for all x that are large (Why did we switch the inequality signs?)

Now, $\lim_{x\to\infty} \dfrac{x-7}{-2x+13} = \lim_{x\to\infty} \dfrac{1-\left(\dfrac{7}{x}\right)}{-2+\left(\dfrac{13}{x}\right)} = \dfrac{1-0}{-2+0} = -\dfrac{1}{2}$

and $\lim_{x\to\infty} \dfrac{x+7}{-2x+13} = \lim_{x\to\infty} \dfrac{1+\left(\dfrac{7}{x}\right)}{-2+\left(\dfrac{13}{x}\right)} = \dfrac{1+0}{-2+0} = -\dfrac{1}{2}$

$$\lim_{x\to\infty} \frac{x+7\sin x}{-2x+13} = -\frac{1}{2}$$

ILLUSTRATION 2.39

Evaluate $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$.

Sol. We have $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\dfrac{1}{x^2}}-\sqrt[3]{1+\dfrac{1}{x^3}}}{\sqrt[4]{1+\dfrac{1}{x^4}}-\sqrt[5]{\dfrac{1}{x}+\dfrac{1}{x^5}}} = \frac{1-1}{1-0} = 0$$

ILLUSTRATION 2.40

Evaluate $\lim_{x\to-\infty}\left(\sqrt{25x^2-3x}+5x\right)$.

Sol. We have

$$\lim_{x\to-\infty}\left(\sqrt{25x^2-3x}+5x\right) \quad (\infty-\infty \text{ form})$$

$$=\lim_{y\to\infty}\left(\sqrt{25y^2+3y}-5y\right), \text{ where } y=-x$$

$$=\lim_{y\to\infty}\frac{25y^2+3y-25y^2}{\sqrt{25y^2+3y}+5y}$$

$$=\lim_{y\to\infty}\frac{3y}{\sqrt{25y^2+3y}+5y}$$

$$=\lim_{y\to\infty}\frac{3}{\sqrt{25+\frac{3}{y}}+5}=\frac{3}{5+5}=\frac{3}{10}$$

ILLUSTRATION 2.41

Evaluate $\lim_{x\to\infty}\left(\frac{x^2+x-1}{3x^2+2x+4}\right)^{\frac{3x^2+x}{x-2}}$.

Sol. $L=\lim_{x\to\infty}\left(\frac{x^2+x-1}{3x^2+2x+4}\right)^{\frac{3x^2+x}{x-2}}$

$$=\lim_{x\to\infty}\left(\frac{x^2+x-1}{3x^2+2x+4}\right)^{\lim_{x\to\infty}\frac{3x^2+x}{x-2}}$$

Now, $\lim_{x\to\infty}\frac{x^2+x-1}{3x^2+2x+4}=\lim_{x\to\infty}\frac{1+\frac{1}{x}-\frac{1}{x^2}}{3+\frac{2}{x}+\frac{4}{x^2}}=\frac{1}{3}$

Also, $\lim_{x\to\infty}\frac{3x^2+x}{x-2}=\lim_{x\to\infty}\frac{3x+\frac{1}{x}}{1-\frac{2}{x}}\to\infty$

Thus, $L=\left(\frac{1}{3}\right)^{\infty}=0$

ILLUSTRATION 2.42

Evaluate $\lim_{n\to\infty}\sin^n\left(\frac{2\pi n}{3n+1}\right)$, $n\in N$.

Sol. $\lim_{n\to\infty}\left\{\sin\left(\frac{2\pi n}{3n+1}\right)\right\}^n=\lim_{n\to\infty}\left\{\sin\left(\frac{2\pi}{3+\frac{1}{n}}\right)\right\}^n$

$$=\lim_{n\to\infty}\left\{\frac{\sqrt{3}}{2}\right\}^n=0$$

ILLUSTRATION 2.43

Evaluate $\lim_{x\to\infty}\left(\sqrt[3]{(x+1)(x+2)(x+3)}-x\right)$.

Sol. $L=\lim_{x\to\infty}\left(\sqrt[3]{(x+1)(x+2)(x+3)}-x\right)$

Using $a-b=\frac{a^3-b^3}{a^2+ab+b^2}$, we get

$$L=\lim_{x\to\infty}\frac{(x+1)(x+2)(x+3)-x^3}{[(x+1)(x+2)(x+3)]^{2/3}+x^2+x[(x+1)(x+2)(x+3)]^{1/3}}$$

$$=\lim_{x\to\infty}\frac{x^2\left[6+\frac{11}{x}+\frac{6}{x^2}\right]}{x^2\left[\left(1+\frac{6}{x}+\frac{11}{x^2}+\frac{6}{x^2}\right)^{2/3}+1+\left(1+\frac{6}{x}+\frac{11}{x^2}+\frac{6}{x^3}\right)^{1/3}\right]}$$

$$=\frac{6}{3}=2$$

ILLUSTRATION 2.44

Evaluate $\lim_{n\to\infty}\frac{1}{1+n^2}+\frac{1}{2+n^2}+\cdots+\frac{n}{n+n^2}$.

Sol. $P_n=\frac{1}{1+n^2}+\frac{2}{2+n^2}+\cdots+\frac{n}{n+n^2}$

Now, $P_n<\frac{1}{1+n^2}+\frac{2}{1+n^2}+\cdots+\frac{n}{1+n^2}$

$$=\frac{1}{1+n^2}(1+2+3+\cdots+n)$$

$$=\frac{n(n+1)}{2(1+n^2)}$$

Also, $P_n>\frac{1}{n+n^2}+\frac{2}{n+n^2}+\frac{3}{n+n^2}+\cdots+\frac{n}{n+n^2}$

$$=\frac{n(n+1)}{2(n+n^2)}$$

Thus, $\frac{n(n+1)}{2(n+n^2)}<P_n<\frac{n(n+1)}{2(1+n^2)}$

or $\lim_{n\to\infty}\frac{n(n+1)}{2(n+n^2)}<\lim_{n\to\infty}P_n<\lim_{n\to\infty}\frac{n(n+1)}{2(1+n^2)}$

or $\lim_{n\to\infty}\frac{1\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n}+1\right)}<\lim_{n\to\infty}P_n<\lim_{n\to\infty}\frac{1\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n^2}+1\right)}$

or $\frac{1}{2}<\lim_{n\to\infty}P_n<\frac{1}{2}$

or $\lim_{n\to\infty}P_n=\frac{1}{2}$

ILLUSTRATION 2.45

If $[x]$ denotes the greatest integer less than or equal to x, then evaluate

$$\lim_{n\to\infty} \frac{1}{n^3}\left([1^2 x] + [2^2 x] + [3^2 x] + \ldots + [n^2 x]\right).$$

Sol. We have, $\lim_{n\to\infty} \frac{\sum_{r=1}^{n}[r^2 x]}{n^3} = \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^2 x - \{r^2 x\}}{n^3}$,

where $\{\cdot\}$ denotes the fractional part function

$$= \lim_{n\to\infty}\left(\frac{x\cdot\frac{n(n+1)(2n+1)}{6}}{n^3} - \sum_{r=1}^{n}\frac{\{r^2x\}}{n^3}\right)$$

$$= \frac{x}{6}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\{r^2x\}}{n^3}$$

$$= \frac{x}{6} - 0 \qquad (\because\ 0 \le \{r^2x\} < 1)$$

$$= \frac{x}{6}$$

ILLUSTRATION 2.46

If $\lim_{x\to\infty}\left\{\frac{x^2+1}{x+1} - (ax+b)\right\} = 0$, then find the values of a and b.

Sol. We have $\lim_{x\to\infty}\left\{\frac{x^2+1}{x+1} - (ax+b)\right\} = 0$

or $\lim_{x\to\infty}\left\{\frac{x^2(1-a) - x(a+b) + 1 - b}{x+1}\right\} = 0$

Since the limit of the given expression is zero, the degree of numerator is less than that of denominator. Denominator on L.H.S. is a polynomial of degree one. So, numerator must be a constant. For this, we must have

coeff. of $x^2 = 0$ and coeff. of $x = 0$

or $1 - a = 0$ and $a + b = 0$

or $a = 1, b = -1$

ILLUSTRATION 2.47

Evaluate $\lim_{n\to\infty}\frac{n^p\sin^2(n!)}{n+1}$, where $0 < p < 1$.

Sol. We have

$$\lim_{n\to\infty}\frac{n^p\sin^2(n!)}{n+1} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{n\to\infty}\frac{\sin^2(n!)}{n^{1-p}\left(1+\frac{1}{n}\right)}$$

$$= \frac{\text{some number between 0 and 1}}{\infty\times 1} = 0 \qquad (\text{as } 0 < p < 1)$$

ILLUSTRATION 2.48

Evaluate $\lim_{n\to\infty}(-1)^{n-1}\sin\left(\pi\sqrt{n^2+0.5n+1}\right)$, where $n \in N$

Sol. $L = (-1)^{n-1}\lim_{n\to\infty}\sin\left(\pi\sqrt{n^2+0.5n+1}\right)$

$$= \lim_{n\to\infty}(-1)^{n-1}(-1)^{n-1}\sin\left(n\pi - \pi\sqrt{n^2+\frac{n}{2}+1}\right)$$

$$= \lim_{n\to\infty}\sin\pi\left\{\frac{\left(n-\sqrt{n^2+\frac{n}{2}+1}\right)\left(n+\sqrt{n^2+\frac{n}{2}+1}\right)}{n+\sqrt{n^2+\frac{n}{2}+1}}\right\}$$

$$= \lim_{n\to\infty}\sin\pi\left\{\frac{n^2-n^2-\frac{n}{2}-1}{n\left(1+\sqrt{1+\frac{1}{2n}+\frac{1}{n^2}}\right)}\right\}$$

$$= \lim_{n\to\infty}\sin\pi\left\{\frac{-\frac{n}{2}-1}{n\left(1+\sqrt{1+\frac{1}{2n}+\frac{1}{n^2}}\right)}\right\}$$

$$= \lim_{n\to\infty}\sin\pi\left(\frac{-\frac{1}{2}-\frac{1}{n}}{1+\sqrt{1+\frac{1}{2n}+\frac{1}{n^2}}}\right) = \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

STANDARD FORMULA FOR LIMITS

Recall the binomial expansion for any rational index

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots,$$ where $|x| < 1$

When x is infinitely small (approaching to zero), we can ignore higher powers of x,

So, $(1+x)^n = 1 + nx$ (approximately)

Let us use this fact to find the value of $\lim_{x\to a}\frac{x^n - a^n}{x-a}$

Let $x = a + h$

$$\therefore \quad \lim_{x\to a}\frac{x^n - a^n}{x-a} = \lim_{h\to 0}\frac{(a+h)^n - a^n}{a+h-a}$$

$$= \lim_{h\to 0}\frac{a^n\left\{\left(1+\frac{h}{a}\right)^n - 1\right\}}{h}$$

$$= a^n\lim_{h\to 0}\frac{\left\{1+n.\frac{h}{a}\right\}-1}{h}$$

$$= a^n.\frac{n}{a}$$

$\therefore \quad \lim_{x\to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

Also, $\lim_{x\to 0} \dfrac{(1+x)^n - 1}{x} = n$

ILLUSTRATION 2.49

Evaluate $\lim_{x\to 2} \dfrac{x^{10}-1024}{x^5-32}$.

Sol. $\lim_{x\to 2} \dfrac{x^{10}-1024}{x^5-32} = \lim_{x\to 2} \dfrac{x^{10}-2^{10}}{x^5-2^5} = \lim_{x\to 2} \dfrac{\dfrac{x^{10}-2^{10}}{x-2}}{\dfrac{x^5-2^5}{x-2}}$

$= \dfrac{\lim_{x\to 2} \dfrac{x^{10}-2^{10}}{x-2}}{\lim_{x\to 2} \dfrac{x^5-2^5}{x-2}} = \dfrac{10\times 2^{10-1}}{5\times 2^{5-1}}$

$= 64$

ILLUSTRATION 2.50

If $\lim_{x\to 2} \dfrac{x^n-2^n}{x-2} = 80$ and $n \in N$, then find the value of n.

Sol. We have $\lim_{x\to 2} \dfrac{x^n-2^n}{x-2} = 80$

or $\quad n2^{n-1} = 80$

or $\quad n2^{n-1} = 5\times 2^{5-1}$

or $\quad n = 5$

ILLUSTRATION 2.51

Evaluate $\lim_{x\to 1} \dfrac{\sqrt{x}+\sqrt{\sqrt{x}}+\sqrt{\sqrt{\sqrt{x}}}+\sqrt{\sqrt{\sqrt{\sqrt{x}}}}-4}{x-1}$.

Sol. $\lim_{x\to 1} \dfrac{\sqrt{x}+\sqrt{\sqrt{x}}+\sqrt{\sqrt{\sqrt{x}}}+\sqrt{\sqrt{\sqrt{\sqrt{x}}}}-4}{x-1}$

$= \lim_{x\to 1} \dfrac{x^{1/2}+x^{1/4}+x^{1/8}+x^{1/16}-4}{x-1}$

$= \lim_{x\to 1} \left(\dfrac{x^{1/2}-1}{x-1}+\dfrac{x^{1/4}-1}{x-1}+\dfrac{x^{1/8}-1}{x-1}+\dfrac{x^{1/16}-1}{x-1}\right)$

$= \dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16} = \dfrac{8+4+2+1}{16} = \dfrac{15}{16}$

ILLUSTRATION 2.52

Evaluate $\lim_{x\to a} \dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$.

Sol. $\lim_{x\to a} \dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$

$= \lim_{x\to a} \dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{(x+2)-(a+2)}$

$= \lim_{y\to b} \dfrac{y^{5/3}-b^{5/3}}{y-b}$, where $x+2=y$ and $a+2=b$

$= \dfrac{5}{3} b^{5/3-1} = \dfrac{5}{3} b^{2/3} = \dfrac{5}{3} (a+2)^{2/3}$

ILLUSTRATION 2.53

Evaluate $\lim_{x\to 2} \dfrac{\sqrt{(x+7)}-3\sqrt{(2x-3)}}{\sqrt[3]{(x+6)}-2\sqrt[3]{(3x-5)}}$.

Sol. We have $L = \lim_{x\to 2} \dfrac{\sqrt{(x+7)}-3\sqrt{(2x-3)}}{\sqrt[3]{(x+6)}-2\sqrt[3]{(3x-5)}} \quad \left(\dfrac{0}{0} \text{ form}\right)$

Let $x-2=t$ such that when $x\to 2$, $t\to 0$. Then

$L = \lim_{t\to 0} \dfrac{(t+9)^{\frac{1}{2}}-3(2t+1)^{\frac{1}{2}}}{(t+8)^{\frac{1}{3}}-2(3t+1)^{\frac{1}{3}}} \quad \left(\dfrac{0}{0} \text{ form}\right)$

$= \dfrac{3}{2} \lim_{t\to 0} \dfrac{\left(1+\dfrac{t}{9}\right)^{\frac{1}{2}}-(2t+1)^{\frac{1}{2}}}{\left(1+\dfrac{t}{8}\right)^{\frac{1}{3}}-(3t+1)^{\frac{1}{3}}} \quad \left(\dfrac{0}{0} \text{ form}\right)$

$= \dfrac{3}{2} \lim_{t\to 0} \dfrac{\dfrac{1}{2}\dfrac{t}{9}-(2t)\dfrac{1}{2}}{\dfrac{t}{8}\dfrac{1}{3}-(3t)\dfrac{1}{3}} = \dfrac{3}{2} \dfrac{\dfrac{1}{18}-1}{\dfrac{1}{24}-1} = \dfrac{34}{23}$

ILLUSTRATION 2.54

If $\lim_{x\to 0} \dfrac{(4x-1)^{\frac{1}{3}}+a+bx}{x} = \dfrac{1}{3}$ then find the values of a and b.

Sol. We have $\lim_{x\to 0} \dfrac{-(1-4x)^{\frac{1}{3}}+a+bx}{x} = \dfrac{1}{3}$

$\Rightarrow \quad \lim_{x\to 0} \dfrac{-\left[1-\dfrac{4}{3}x\right]+a+bx}{x} = \dfrac{1}{3}$

$\Rightarrow \quad \lim_{x\to 0} \dfrac{(a-1)+\left(\dfrac{4}{3}+b\right)x}{x} = \dfrac{1}{3}$

For limit to exist, $a-1=0$ or $a=1$

$\Rightarrow \quad \lim_{x\to 0} \dfrac{\left(\dfrac{4}{3}+b\right)x}{x} = \dfrac{1}{3}$

$\Rightarrow \quad \dfrac{4}{3}+b = \dfrac{1}{3}$

$\Rightarrow \quad b = -1$

CONCEPT APPLICATION EXERCISE 2.3

1. Evaluate $\lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$.

2. Evaluate $\lim_{x\to 1} \frac{x^4-3x^3+2}{x^3-5x^2+3x+1}$.

3. Evaluate $\lim_{x\to \frac{\pi}{4}} \frac{1-\sin 2x}{1+\cos 4x}$.

4. Evaluate $\lim_{x\to \pi/4} \frac{1-\cot^3 x}{2-\cot x-\cot^3 x}$.

5. Evaluate $\lim_{x\to a} \frac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}$.

6. Evaluate $\lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$.

7. Evaluate $\lim_{x\to \sqrt{10}} \frac{\sqrt{7+2x}-(\sqrt{5}+\sqrt{2})}{x^2-10}$.

8. Evaluate $\lim_{n\to \infty} \frac{(1^2-2^2+3^2-4^2+5^2+\cdots n \text{ terms})}{n^2}$.

9. Evaluate $\lim_{x\to \infty} \left[\sqrt{a^2x^2+ax+1}-\sqrt{a^2x^2+1}\right]$.

10. If $[x]$ denotes the greatest integer less than or equal to x, then evaluate $\lim_{n\to\infty} \frac{1}{n^2}([1\cdot x]+[2\cdot x]+[3\cdot x]+\cdots+[n\cdot x])$.

11. Evaluate $\lim_{x\to \infty} x^3\left\{\sqrt{x^2+\sqrt{1+x^4}}-x\sqrt{2}\right\}$.

12. Evaluate $\lim_{x\to \infty} \left(\frac{7x^2+1}{5x^2-1}\right)^{\frac{x^5}{1-x^3}}$.

13. Evaluate $\lim_{n\to \infty} \cos\left(\pi\sqrt{n^2+n}\right)$ when n is an integer.

14. Evaluate $\lim_{x\to 1} \frac{\sum_{k=1}^{100} x^k-100}{x-1}$.

15. Evaluate $\lim_{h\to 0} \left[\frac{1}{h\sqrt[3]{8+h}}-\frac{1}{2h}\right]$.

ANSWERS

1. –1/10	2. 5/4	3. 1/4
4. 3/4	5. $1/\sqrt{2a}$	6. $1/4\sqrt{2}$
7. $\frac{1}{2\sqrt{10}(\sqrt{5}+\sqrt{2})}$	8. 1/2 or –1/2	9. 1/2
10. $x/2$	11. $1/4\sqrt{2}$	12. 0
13. 0	14. 5050	15. –1/48

LIMITS USING EXPANSION

One of the easier methods of finding the limits is using expansion formulas of function. We use some standard expansion formulas which are known as Taylor Series. These series are listed as follows:

(i) $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\ldots\ (-1<x\le 1)$

(ii) $\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}\ldots\ (-1\le x<1)$

(iii) $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\ldots$

(iv) $e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}\ldots$

(v) $a^x = 1+x(\log_e a)+\frac{x^2}{2!}(\log_e a)^2+\ldots$

(vi) $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\ldots$

(vii) $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\ldots$

(viii) $\tan x = x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots$

ILLUSTRATION 2.55

Evaluate $\lim_{x\to 0} \frac{\sin x-x}{x^3}$.

Sol. $\lim_{x\to 0} \frac{\sin x-x}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)-x}{x^3}$$

$$= \lim_{x\to 0}\left[-\frac{1}{3!}+\frac{x^2}{5!}-\cdots\right] = \frac{-1}{3!} = \frac{-1}{6}$$

ILLUSTRATION 2.56

Evaluate $\lim_{x\to 0} \frac{5\sin x-7\sin 2x+3\sin 3x}{x^2\sin x}$.

Sol. $\lim_{x\to 0} \frac{5\sin x-7\sin 2x+3\sin 3x}{x^2\sin x}$

$$= \lim_{x\to 0} \frac{5\left(x-\frac{x^3}{3!}+\cdots\right)-7\left(2x-\frac{(2x)^3}{3!}+\cdots\right)+3\left(3x-\frac{(3x)^3}{3!}+\cdots\right)}{x^2\left(x-\frac{x^3}{3!}+\cdots\right)}$$

$$= \lim_{x\to 0} \frac{-\frac{5x^3}{3!}+\frac{56x^3}{3!}-\frac{81x^3}{3!}}{x^3\left(1-\frac{x^2}{3!}+\cdots\right)}$$

$$= \frac{-5+56-81}{3!}$$

$$= -5$$

ILLUSTRATION 2.57

Evaluate $\lim_{x\to 0} \frac{\sin x + \log(1-x)}{x^2}$.

Sol. $\lim_{x\to 0} \frac{\sin x + \log(1-x)}{x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)+\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{-x^2}{2}-x^3\left(\frac{1}{3!}+\frac{1}{3}\right)-\frac{x^4}{4}\cdots}{x^2} = -\frac{1}{2}$$

ILLUSTRATION 2.58

Evaluate $\lim_{x\to 0} \frac{e^{\sin x}-(1+\sin x)}{(\tan(\sin x))^2}$.

Sol. $\lim_{x\to 0} \frac{e^{\sin x}-(1+\sin x)}{(\tan(\sin x))^2}$

$$= \lim_{h\to 0} \frac{e^h-(1+h)}{(\tan h)^2} \quad \text{(where } \sin x = h\text{)}$$

$$= \lim_{h\to 0} \frac{\left(1+h+\frac{h^2}{2!}+\ldots\right)-(1+h)}{\left(h+\frac{h^3}{3}+\ldots\right)^2}$$

$$= \lim_{h\to 0} \frac{\frac{h^2}{2!}}{h^2}$$

$$= \frac{1}{2}$$

ILLUSTRATION 2.59

Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$.

Sol. $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x - x^2}{x^2\sin^2 x}\right)$$

$$= \lim_{x\to 0} \frac{(\sin x + x)(\sin x - x)}{x^2 \sin^2 x}$$

$$= \lim_{x\to 0} \frac{\left(\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right)+x\right)\left(\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right)-x\right)}{x^2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right)^2}$$

$$= \lim_{x\to 0} \frac{-\left(2-\frac{x^2}{3!}+\frac{x^4}{5!}\cdots\right)\left(\frac{1}{3!}-\frac{x^2}{5!}\cdots\right)}{x^2} = -\frac{1}{3}$$

ILLUSTRATION 2.60

If $\lim_{x\to 0} \frac{\cos 4x + a\cos 2x + b}{x^4}$ is finite, find a and b using expansion formula. Also, find the limit.

Sol. $\lim_{x\to 0} \frac{\cos 4x + a\cos 2x + b}{x^4} = \text{Finite}$

Using expansion formula for $\cos 4x$ and $\cos 2x$, we get

$$\lim_{x\to 0} \frac{\left(1-\frac{(4x)^2}{2!}+\frac{(4x)^4}{4!}\right)+a\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}\right)+b}{x^4}$$

or $$\lim_{x\to 0} \frac{(1+a+b)+(-8-2a)x^2+\left(\frac{32}{3}+\frac{2}{3}a\right)x^4+\cdots}{x^4}$$

or $1+a+b=0$ (1)

$-8-2a=0$ (2)

Solving (1) and (2) for a and b, we get

$a = -4$ and $b = 3$

Also, Limit $= \frac{32}{3}+\frac{2}{3}a = \frac{32-8}{3} = 8$

ILLUSTRATION 2.61

Find the integral value of n for which

$$\lim_{x\to 0} \frac{\cos^2 x - \cos x - e^x \cos x + e^x - \frac{x^3}{2}}{x^n}$$

is a finite nonzero number.

Sol. Given that

$$\lim_{x\to 0} \frac{\cos^2 x - \cos x - e^x \cos x + e^x - \frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to 0} \frac{(\cos x - 1)(\cos x - e^x) - \frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to0}\frac{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots-1\right)\left[\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}-\ldots\right)\right]-\frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to0}\frac{\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)\left[\left(-x-x^2-\frac{x^3}{3!}-\frac{x^5}{5!}-\ldots\right)\right]-\frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to0}\frac{\left(\frac{x^3}{2}+\frac{x^4}{2}+\frac{x^5}{12}-\frac{x^5}{24}+\ldots\right)-\frac{x^3}{2}}{x^n}$$

= nonzero if $n = 4$

CONCEPT APPLICATION EXERCISE 2.4

1. Evaluate $\lim_{x\to0}\left\{\frac{\sin x - x + \frac{x^3}{6}}{x^5}\right\}$.
2. Evaluate $\lim_{x\to0}\frac{e^x-1-x}{x^2}$.
3. Evaluate $\lim_{x\to0}\frac{e^x-e^{-x}-2x}{x-\sin x}$.
4. If $\lim_{x\to0}\frac{1-\cos x}{e^{ax}-bx-1}=1$ then find the values of a and b.
5. Find the values of a and b in order that $\lim_{x\to0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1.$

ANSWERS

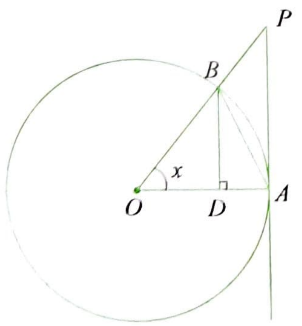
1. 1/120 2. 1/2 3. 2

4. $a = 1, b = 1$ and $a = -1, b = -1$ 5. $a=\frac{-5}{2}, b=\frac{-3}{2}$

TRIGONOMETRIC LIMITS

One of the most fundamental formula in trigonometric limits is $\lim_{x\to0}\frac{\sin x}{x}=1$. Let us prove this using Sandwich Rule.

Consider a circle of radius r.



Let O be the centre of the circle such that $\angle AOB = x$ where x is measured in radians and it is very small.

Suppose the tangent at A meets OB produced at P.

From the figure, we have

Area of ΔOAB < area of sector OAB < Area of ΔOAP

$\Rightarrow \frac{1}{2}OA\cdot OB\sin x < \frac{1}{2}(OA)^2x < \frac{1}{2}OA\cdot AP$

$\Rightarrow \frac{1}{2}r^2\sin x < \frac{1}{2}r^2x < \frac{1}{2}r^2\tan x$ [In ΔOAP, $AP = OA\tan x$]

$\Rightarrow \sin x < x < \tan x$

$\Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ [$\because \sin x > 0$]

$\Rightarrow 1 > \frac{\sin x}{x} > \cos x$

$\Rightarrow 1 > \lim_{x\to0}\frac{\sin x}{x} > \lim_{x\to0}\cos x$

$\Rightarrow 1 > \lim_{x\to0}\frac{\sin x}{x} > 1$

$\Rightarrow \lim_{x\to0}\frac{\sin x}{x} = 1$ (by Sandwich Rule)

Other standard formulas are as follows:

(i) $\lim_{x\to0}\frac{\tan x}{x}=1$

As $\lim_{x\to0}\frac{\tan x}{x} = \lim_{x\to0}\frac{\sin x}{x}\times\frac{1}{\cos x}$

$= \lim_{x\to0}\frac{\sin x}{x}\times\lim_{x\to0}\frac{1}{\cos x}=1\times1=1$

(ii) $\lim_{x\to a}\frac{\sin(x-a)}{x-a}=1$

As $\lim_{x\to a}\frac{\sin(x-a)}{x-a} = \lim_{h\to0}\frac{\sin(a+h-a)}{a+h-a}$

$= \lim_{h\to0}\frac{\sin h}{h}$

$= 1$

(iii) $\lim_{x\to a}\frac{\tan(x-a)}{x-a}=1$

(iv) $\lim_{x\to0}\frac{\sin^{-1}x}{x}=1$

As $\lim_{x\to0}\frac{\sin^{-1}x}{x} = \lim_{y\to0}\frac{y}{\sin y}$ (Putting $\sin^{-1}x = y$)

$= 1$

(v) $\lim_{x\to0}\frac{\tan^{-1}x}{x}=1$

(vi) $\lim_{x\to a}\frac{\sin(f(x))}{f(x)}=1$, if $\lim_{x\to a}f(x)=0$.

Similarly, other formulas follow.

ILLUSTRATION 2.62

Find the limits of the following:

(i) $\lim_{x\to0}\frac{\sin 3x}{x}$ (ii) $\lim_{x\to0}\frac{\sin 7x}{\sin 4x}$ (iii) $\lim_{x\to0}\frac{1-\cos 2x}{x^2}$

Sol.

(i) $\lim_{x\to 0}\frac{\sin 3x}{x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x}$

$=3\times 1\left[as\ \lim_{x\to a}\frac{\sin(f(x))}{f(x)}=1,\ if\ \lim_{x\to a}f(x)=0\right]$

$= 3$

(ii) $\lim_{x\to 0}\frac{\sin 7x}{\sin 4x} = \lim_{x\to 0}\frac{\frac{\sin 7x}{7x}}{\frac{\sin 4x}{4x}}\times\frac{7x}{4x}$

$=\frac{\lim_{x\to 0}\frac{\sin 7x}{7x}}{\lim_{x\to 0}\frac{\sin 4x}{4x}}\times\frac{7}{4}$

$=\frac{1}{1}\times\frac{7}{4}=\frac{7}{4}$

(iii) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2}$

$=2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2=2$

ILLUSTRATION 2.63

Find the following limits:

(i) $\lim_{x\to 0}\frac{1}{x}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ (ii) $\lim_{x\to 0}\frac{1}{x}\sin^{-1}(3x-4x^3)$

Sol.

(i) We know that $\sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$, for $-1\le x\le 1$

$\Rightarrow\ \lim_{x\to 0}\frac{1}{x}\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\lim_{x\to 0}\frac{2\tan^{-1}x}{x}=2$

(ii) We know that $\sin^{-1}(3x-4x^3)=3\sin^{-1}x$, for $-1/2\le x\le 1/2$

$\Rightarrow\ \lim_{x\to 0}\frac{1}{x}\sin^{-1}(3x-4x^3)=\lim_{x\to 0}\frac{3\sin^{-1}x}{x}=3$

ILLUSTRATION 2.64

Evaluate $\lim_{x\to\infty}2^{x-1}\tan\left(\frac{a}{2^x}\right)$.

Sol. Since $\lim_{x\to\infty}\frac{a}{2^x}=0$, we have

$\lim_{x\to\infty}2^{x-1}\tan\left(\frac{a}{2^x}\right)$ $(\infty\times 0\text{ form})$

$=\lim_{x\to\infty}\frac{a}{2}\cdot\frac{\tan\left(\frac{a}{2^x}\right)}{\left(\frac{a}{2^x}\right)}$ $\left(\frac{0}{0}\text{ form}\right)$

$=\frac{a}{2}\times 1=\frac{a}{2}$

ILLUSTRATION 2.65

Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}$.

Sol. $\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$=\lim_{h\to 0}\frac{1-\cos(\pi-2x)}{(\pi-2x)^2}=\lim_{x\to\frac{\pi}{2}}\frac{2\sin^2\left(\frac{\pi}{2}-x\right)}{4\left(\frac{\pi}{2}-x\right)^2}$

$=\frac{1}{2}\left(\lim_{x\to\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)}\right)^2$

$=\frac{1}{2}$

ILLUSTRATION 2.66

Evaluate $\lim_{x\to 2}\frac{x^2-x-2}{x^2-2x-\sin(x-2)}$.

Sol. $\lim_{x\to 2}\frac{x^2-x-2}{x^2-2x-\sin(x-2)}$ $\left(\frac{0}{0}\text{ form}\right)$

$=\lim_{x\to 2}\frac{(x-2)(x+1)}{x(x-2)-\sin(x-2)}$

$=\lim_{x\to 2}\frac{(x+1)}{x-\frac{\sin(x-2)}{x-2}}$

$=\frac{2+1}{2-1}=3$

ILLUSTRATION 2.67

Find $\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}$ if exists.

Sol. $\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}=\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}\cdot 2\sin^2 x}}{x}=\lim_{x\to 0}\frac{|\sin x|}{x}$

Now $\lim_{x\to 0^+}\frac{|\sin x|}{x}=\lim_{x\to 0^+}\frac{\sin x}{x}=1$

$\lim_{x\to 0^-}\frac{|\sin x|}{x}=\lim_{x\to 0^-}\frac{-\sin x}{x}=-1$

Thus $\lim_{x\to 0^+}\frac{|\sin x|}{x}\ne\lim_{x\to 0^-}\frac{|\sin x|}{x}$.

So, $\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}$ does not exist.

ILLUSTRATION 2.68

Evaluate $\lim_{x\to 0} \dfrac{\sin(\pi\cos^2 x)}{x^2}$.

Sol. $\lim_{x\to 0} \dfrac{\sin(\pi\cos^2 x)}{x^2} = \lim_{x\to 0} \dfrac{\sin(\pi - \pi\cos^2 x)}{x^2}$

$$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x} \times \frac{\pi\sin^2 x}{x^2} = \pi$$

ILLUSTRATION 2.69

Evaluate $\lim_{x\to \frac{\pi}{6}} \dfrac{2-\sqrt{3}\cos x - \sin x}{(6x-\pi)^2}$.

Sol. $\lim_{x\to \frac{\pi}{6}} \dfrac{2-\sqrt{3}\cos x - \sin x}{(6x-\pi)^2}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$$= \lim_{x\to \frac{\pi}{6}} 2\times \frac{1-\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right)}{(6x-\pi)^2}$$

$$= \lim_{x\to \frac{\pi}{6}} 2\times \frac{1-\cos\left(x-\frac{\pi}{6}\right)}{36\left(x-\frac{\pi}{6}\right)^2}$$

$$= \lim_{x\to \frac{\pi}{6}} \frac{2\sin^2\left(\frac{x}{2}-\frac{\pi}{12}\right)}{18\times 4\left(\frac{x}{2}-\frac{\pi}{12}\right)^2}$$

$$= \frac{1}{36}\left(\lim_{x\to \frac{\pi}{6}} \frac{\sin\left(\frac{x}{2}-\frac{\pi}{12}\right)}{\left(\frac{x}{2}-\frac{\pi}{12}\right)}\right)^2 = \frac{1}{36}$$

ILLUSTRATION 2.70

Evaluate $\lim_{x\to \pi} \dfrac{\sin^{-1}(1+\cos x)\cdot \sec\left(\frac{x}{2}\right)}{(x-\pi)}$.

Sol. $\lim_{x\to \pi} \dfrac{\sin^{-1}(1+\cos x)\cdot \sec\left(\frac{x}{2}\right)}{(x-\pi)}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$$= \lim_{x\to \pi} \frac{\sin^{-1}(1+\cos x)}{(1+\cos x)} \cdot \frac{1+\cos x}{\cos\left(\frac{x}{2}\right)} \cdot \frac{1}{(x-\pi)}$$

$$= 1\times \lim_{x\to \pi} \frac{2\cos^2\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} \cdot \frac{1}{(x-\pi)}$$

$$= \lim_{x\to \pi} \frac{2\cos\left(\frac{x}{2}\right)}{(x-\pi)}$$

$$= \lim_{x\to \pi} \frac{\sin\left(\frac{\pi}{2}-\frac{x}{2}\right)}{\frac{x}{2}-\frac{\pi}{2}} = -1$$

ILLUSTRATION 2.71

Evaluate $\lim_{x\to -\infty} \left[\dfrac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{(1+|x|^3)}\right]$.

Sol. $\lim_{x\to -\infty} \left[\dfrac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{(1-x^3)}\right]$ (as x is negative)

$$= \lim_{x\to -\infty} \left[\frac{x\sin\left(\frac{1}{x}\right) + \frac{1}{x}}{\frac{1}{x^3}-1}\right]$$

$$= \lim_{x\to -\infty} \left[\frac{\dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} + \frac{1}{x}}{\frac{1}{x^3}-1}\right]$$

$$= \frac{1+0}{0-1} = -1$$

ILLUSTRATION 2.72

Evaluate $\lim_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$.

Sol. We have

$\lim_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \left(\frac{\sin x - \sin x\cos x}{x^3 \cos x}\right)$$

$$= \lim_{x\to 0} \left\{\frac{\sin x(1-\cos x)}{x^3\cos x}\right\} = \lim_{x\to 0} \left\{\frac{\sin x}{x}\frac{1-\cos x}{x^2}\frac{1}{\cos x}\right\}$$

$$= \left\{\lim_{x\to 0} \frac{\sin x}{x}\right\}\left\{\lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2 \times 4}\right\} \lim_{x\to 0} \frac{1}{\cos x}$$

$$= \left\{\lim_{x\to 0}\frac{\sin x}{x}\right\}\frac{1}{2}\left\{\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\right\}\lim_{x\to 0}\frac{1}{\cos x}$$

$$= 1\times\frac{1}{2}(1)^2\times\frac{1}{1}=\frac{1}{2}$$

ILLUSTRATION 2.73

Evaluate $\lim_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+4}-\frac{\pi}{4}\right)$.

Sol. We have

$$\lim_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+4}-\frac{\pi}{4}\right)$$

$$= \lim_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+4}-\tan^{-1}1\right)$$

$$= \lim_{x\to\infty} x\tan^{-1}\left(\frac{\frac{x+1}{x+4}-1}{1+\frac{x+1}{x+4}}\right)$$

$$= \lim_{x\to\infty} x\tan^{-1}\left(\frac{-3}{2x+5}\right)$$

$$= \lim_{x\to\infty}\left\{\frac{\tan^{-1}\left(\frac{-3}{2x+5}\right)}{\frac{-3}{2x+5}}\right\}\left(\frac{-3x}{2x+5}\right)$$

$$= \lim_{x\to\infty}\left\{\frac{\tan^{-1}\left(\frac{-3}{2x+5}\right)}{\frac{-3}{2x+5}}\right\}\lim_{x\to\infty}\left(\frac{-3x}{2x+5}\right)$$

$$= 1\times\lim_{x\to\infty}\left(\frac{-3}{2+\frac{5}{x}}\right)=1\times\left(\frac{-3}{2}\right)=-\frac{3}{2}$$

ILLUSTRATION 2.74

Evaluate $\lim_{n\to\infty} n\sin\left(2\pi\sqrt{1+n^2}\right)$, $(n\in N)$.

Sol. $L=\lim_{n\to\infty} n\sin\left(2\pi\sqrt{1+n^2}\right)$

$$= \lim_{n\to\infty} n\sin\left(2\pi\sqrt{1+n^2}-2n\pi\right)$$

$$= \lim_{n\to\infty} n\sin\left\{\frac{2\pi\left(\sqrt{1+n^2}-n\right)}{\left(\sqrt{1+n^2}+n\right)}\left(\sqrt{1+n^2}+n\right)\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{n\sin\left(\frac{2\pi}{\sqrt{1+n^2}+n}\right)}{\left(\frac{2\pi}{\sqrt{1+n^2}+n}\right)}\left(\frac{2\pi}{\sqrt{1+n^2}+n}\right)\right\}$$

$$= \lim_{n\to\infty}\frac{2n\pi}{n\left(\sqrt{1+\frac{1}{n^2}}+1\right)}$$

$$= \frac{2\pi}{2}=\pi$$

ILLUSTRATION 2.75

Evaluate $\lim_{x\to -1^+}\frac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{1+x}}$.

Sol. $\lim_{x\to -1^+}\frac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{1+x}}$

$$= \lim_{x\to -1^+}\frac{\pi-\cos^{-1}x}{\sqrt{1+x}}\cdot\frac{1}{\sqrt{\pi}+\sqrt{\cos^{-1}x}}$$

$$= \lim_{x\to -1^+}\frac{\cos^{-1}(-x)}{\sqrt{1+x}}\cdot\frac{1}{\sqrt{\pi}+\sqrt{\pi}}$$

$$= \frac{1}{2\sqrt{\pi}}\lim_{\theta\to 0^+}\frac{\theta}{\sqrt{1-\cos\theta}} \qquad \text{(Putting } \cos^{-1}(-x)=\theta\text{)}$$

$$= \frac{1}{2\sqrt{\pi}}\lim_{\theta\to 0^+}\frac{\theta}{\sqrt{2\sin^2\frac{\theta}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\theta\to 0^+}\frac{\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{1}{\sqrt{2\pi}}$$

ILLUSTRATION 2.76

Evaluate $\lim_{x\to\infty} x\left[\tan^{-1}\left(\frac{x+1}{x+2}\right)-\tan^{-1}\left(\frac{x}{x+2}\right)\right]$.

Sol. $\lim_{x\to\infty} x\left[\tan^{-1}\frac{x+1}{x+2}-\tan^{-1}\frac{x}{x+2}\right]$

$$= \lim_{x\to\infty} x\tan^{-1}\left(\frac{\frac{x+1}{x+2}-\frac{x}{x+2}}{1+\frac{x+1}{x+2}\cdot\frac{x}{x+2}}\right)$$

$$= \lim_{x\to\infty} x\tan^{-1}\left(\frac{x+2}{2x^2+5x+4}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\tan^{-1}\left(\frac{x+2}{2x^2+5x+4}\right)}{\frac{x+2}{2x^2+5x+4}}\right)\times\frac{x(x+2)}{2x^2+5x+4}$$

$$= 1\times\frac{1}{2}=\frac{1}{2}$$

ILLUSTRATION 2.77

Evaluate $\lim_{x\to 0}\frac{1-\cos(1-\cos x)}{x^4}$.

Sol. $1-\cos(1-\cos x) = 2\sin^2\left(\frac{1-\cos x}{2}\right)$

$$= 2\sin^2\left(\sin^2\frac{x}{2}\right)$$

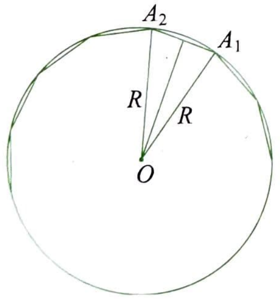
$$\therefore \lim_{x\to 0}\frac{1-\cos(1-\cos x)}{x^4} = \lim_{x\to 0}\frac{2\sin^2\left(\sin^2\frac{x}{2}\right)}{x^4}$$

$$= \lim_{x\to 0}\frac{2\sin^2\left(\sin^2\frac{x}{2}\right)}{\left(\sin^2\frac{x}{2}\right)^2}\cdot\frac{\sin^4\frac{x}{2}}{\left(\frac{x}{2}\right)^4.16} = \frac{1}{8}$$

ILLUSTRATION 2.78

Using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, prove that area of circle of radius R is πR^2.

Sol. In the following figure regular polygon of n sides is inscribed in the circle of radius R.



In the figure, $OA_1 = OA_2 = R$ and $\angle A_1OA_2 = 2\pi/n$

Clearly area of polygon

$$\Delta = n\times(\text{Area of the } \Delta OA_1A_2)$$

$$= n\times\frac{1}{2}R\times R\times\sin\frac{2\pi}{n}$$

$$\therefore \quad \Delta = \frac{nR^2}{2}\sin\frac{2\pi}{n}$$

We know that circle is regular polygon having infinite number of sides.

So, from above formula

$$\lim_{n\to\infty}\frac{nR^2}{2}\sin\frac{2\pi}{n} = \lim_{n\to\infty}\pi R^2\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}$$

$$= \pi R^2\lim_{n\to\infty}\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}$$

$$= \pi R^2\times 1 \qquad \left(\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right)$$

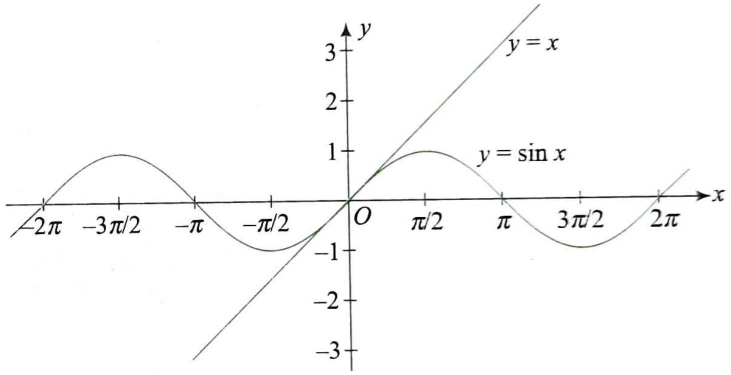
$$= \pi R^2$$

Thus, area of circle having radius R is πR^2.

ILLUSTRATION 2.79

Prove that $\left[\lim_{x\to 0}\frac{\sin x}{x}\right] = 0$, where [·] represents the greatest integer function.

Sol. See the graphs of $y = x$ and $\sin x$ in the following figure.



From the figure when $x\to 0^+$, graph of $y = x$ is above the graph of $y = \sin x$

i.e., $\quad \sin x < x$ or $\frac{\sin x}{x} < 1$

$$\Rightarrow \quad \lim_{x\to 0^+}\frac{\sin x}{x} = 1^-$$

$$\Rightarrow \quad \left[\lim_{x\to 0^+}\frac{\sin x}{x}\right] = 0$$

When $x\to 0^-$, graph of $y = x$ is below the graph of $y = \sin x$

i.e., $\sin x > x$ or $\frac{\sin x}{x} < 1$ (as x is negative)

$$\Rightarrow \quad \lim_{x\to 0^-}\frac{\sin x}{x} = 1^-$$

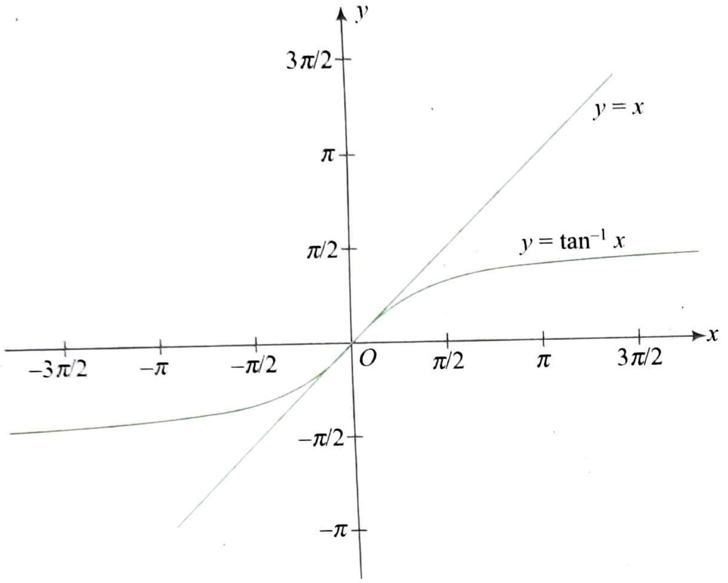
$$\Rightarrow \quad \left[\lim_{x\to 0^-}\frac{\sin x}{x}\right] = 0$$

Thus, $\left[\lim_{x\to 0}\frac{\sin x}{x}\right] = 0$

ILLUSTRATION 2.80

Prove that $\left[\lim_{x\to 0}\frac{\tan^{-1}x}{x}\right]=0$, where $[\cdot]$ represents the greatest integer function.

Sol. See the graph of $y=x$ and $\tan^{-1}x$ in the figure:



From the figure, when $x\to 0^+$, graph of $y=x$ is above the graph of $y=\tan^{-1}x$.

So, $\tan^{-1}x<x$ or $\frac{\tan^{-1}x}{x}<1$

$$\Rightarrow \lim_{x\to 0^+}\frac{\tan^{-1}x}{x}=1^-$$

$$\Rightarrow \left[\lim_{x\to 0^+}\frac{\tan^{-1}x}{x}\right]=0$$

When $x\to 0^-$, graph of $y=\tan^{-1}x$ is above the graph of $y=x$.

So, $\tan^{-1}x>x$ or $\frac{\tan^{-1}x}{x}<1$ (as x is negative)

$$\Rightarrow \lim_{x\to 0^-}\frac{\tan^{-1}x}{x}=1^-$$

$$\Rightarrow \left[\lim_{x\to 0^-}\frac{\tan x}{x}\right]=0$$

Thus $\left[\lim_{x\to 0}\frac{\tan^{-1}x}{x}\right]=0$

ILLUSTRATION 2.81

If $L=\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}$ is finite, then find the value of a and L.

Sol. $L=\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}$

$$=\lim_{x\to 0}\frac{2\sin x\cos x+a\sin x}{x^3}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\frac{(2\cos x+a)}{x^2}$$

$$=\lim_{x\to 0}\frac{2\cos x+a}{x^2}$$

Now, D^r tends to 0 when $x\to 0$. Then N^r also must tend to zero for which $\lim_{x\to 0}(2\cos x+a)=0\Rightarrow a=-2$. Now,

$$L=\lim_{x\to 0}\frac{2\cos x-2}{x^2}=-2\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=-1$$

ILLUSTRATION 2.82

If $m, n\in I_0$ and $\lim_{x\to 0}\frac{\tan 2x-n\sin x}{x^3}=$ some integer, then find the value of n and also the value of limit.

Sol. $L=\lim_{x\to 0}\frac{\tan 2x-n\sin x}{x^3}$

$$=\lim_{x\to 0}\frac{\sin 2x-n\sin x\cos 2x}{x^3\cos 2x}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\frac{(2\cos x-n\cos 2x)}{x^2}\frac{1}{\cos 2x}$$

$$=\lim_{x\to 0}\frac{(2\cos x-n\cos 2x)}{x^2}$$

Now, for $x\to 0$, $x^2\to 0$.
Therefore, for $x\to 0$, $2\cos x-n\cos 2x\to 0$. So, $n=2$.
For, $n=2$.

$$L=\lim_{x\to 0}\frac{(2\cos x-2\cos 2x)}{x^2}$$

$$=4\lim_{x\to 0}\frac{\sin\frac{x}{2}\sin\frac{3x}{2}}{x^2}$$

$$=3$$

CONCEPT APPLICATION EXERCISE 2.5

1. Evaluate $\lim_{x\to 0}\frac{\sin x^0}{x}$.
2. Evaluate $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}$.
3. Evaluate $\lim_{x\to 0}\frac{\cot 2x-\text{cosec } 2x}{x}$.
4. Evaluate $\lim_{x\to 0}\frac{\tan 2x-x}{3x-\sin x}$.
5. Evaluate $\lim_{n\to\infty} n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$.
6. Evaluate $\lim_{x\to 0}\frac{\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)}{\sin^{-1}x}$.

7. Evaluate $\lim_{h\to 0} \dfrac{2\left[\sqrt{3}\sin\left(\dfrac{\pi}{6}+h\right)-\cos\left(\dfrac{\pi}{6}+h\right)\right]}{\sqrt{3}h\,(\sqrt{3}\cos h-\sin h)}$.

8. Evaluate $\lim_{x\to 0} \dfrac{8}{x^8}\left\{1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right\}$.

9. Evaluate $\lim_{\substack{x\to 0\\ y\to 0}} \dfrac{y^2+\sin x}{x^2+\sin y^2}$, where $(x, y)\to(0, 0)$ along the curve $x=y^2$.

10. Evaluate $\lim_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$.

11. Evaluate $\lim_{x\to 0}\dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$.

12. If $\lim_{x\to 2}\dfrac{\tan(x-2)\cdot(x^2+(k-2)x-2k)}{(x^2-4x+4)}=5$, then find the value of k.

ANSWERS

1. $\pi/180$	2. m^2/n^2	3. -1	4. $1/2$	5. $\pi/4$
6. 2	7. 4/3	8. 1/32	9. 2	10. $2/\pi$
11. 1/2	12. 3			

EXPONENTIAL AND LOGARITHMIC LIMITS

In order to evaluate the limits of the function involving exponential and logarithmic functions, we use the following standard results:

(i) $\lim_{x\to 0}\dfrac{a^x-1}{x}=\log_e a$

Proof :

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{x\to 0}\frac{\left(1+\dfrac{x(\log a)}{1!}+\dfrac{x^2(\log a)^2}{2!}+\ldots\right)-1}{x}$$

$$=\lim_{x\to 0}\left(\frac{\log a}{1!}+\frac{x(\log a)^2}{2!}+\ldots\right)$$

$$=\log_e a$$

(ii) $\lim_{x\to 0}\dfrac{e^x-1}{x}=1$ (replace 'a' by e in above formula)

(iii) $\lim_{x\to 0}\dfrac{\log_e(1+x)}{x}=1$

Proof:

$$\lim_{x\to 0}\frac{\log_e(1+x)}{x}=\lim_{x\to 0}\frac{x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots}{x}$$

$$=\lim_{x\to 0}\left(1-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)$$

$$=1$$

In general, if $\lim_{x\to a} f(x)=0$, then we have

(a) $\lim_{x\to a}\dfrac{a^{f(x)}-1}{f(x)}=\log_e a$

(b) $\lim_{x\to a}\dfrac{e^{f(x)}-1}{f(x)}=1$

(c) $\lim_{x\to a}\dfrac{\log_e(1+f(x))}{f(x)}=1$

ILLUSTRATION 2.83

Evaluate $\lim_{x\to 0}\dfrac{3^{2x}-2^{3x}}{x}$.

Sol. We have

$$\lim_{x\to 0}\frac{3^{2x}-2^{3x}}{x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\left\{\left(\frac{3^{2x}-1}{x}\right)-\left(\frac{2^{3x}-1}{x}\right)\right\}$$

$$=\lim_{x\to 0}\left(\frac{3^{2x}-1}{2x}\cdot 2\right)-\lim_{x\to 0}\left(\frac{2^{3x}-1}{3x}\cdot 3\right)$$

$$=2\log 3-3\log 2=\log 9-\log 8=\log\left(\frac{9}{8}\right)$$

ILLUSTRATION 2.84

Evaluate $\lim_{x\to 0}\dfrac{10^x-2^x-5^x+1}{x\tan x}$.

Sol. We have

$$\lim_{x\to 0}\frac{10^x-2^x-5^x+1}{x\tan x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{5^x\cdot 2^x-2^x-5^x+1}{x\tan x}$$

$$=\lim_{x\to 0}\frac{(5^x-1)\,(2^x-1)}{x\tan x}$$

$$=\lim_{x\to 0}\frac{5^x-1}{x}\,\frac{2^x-1}{x}\,\frac{x}{\tan x}$$

$$=\lim_{x\to 0}\frac{5^x-1}{x}\lim_{x\to 0}\frac{2^x-1}{x}\lim_{x\to 0}\frac{x}{\tan x}$$

$$=(\log 5)(\log 2)(1)$$

$$=(\log 5)(\log 2)$$

ILLUSTRATION 2.85

Evaluate $\lim_{x\to 0}\dfrac{2^x-1}{\sqrt{1+x}-1}$.

Sol. We have

$$\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2^x - 1}{\sqrt{1+x}-1} \frac{\left(\sqrt{1+x}+1\right)}{\left(\sqrt{1+x}+1\right)}$$

$$= \lim_{x\to 0} \frac{2^x-1}{x} \lim_{x\to 0}\left(\sqrt{1+x}+1\right)$$

$$= (\log 2)\, 2$$

$$= \log 4$$

ILLUSTRATION 2.86

Evaluate $\lim_{x\to 1} \frac{a^{x-1}-1}{\sin \pi x}$.

Sol. We have, $\lim_{x\to 1} \frac{a^{x-1}-1}{\sin \pi x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 1} \frac{a^{x-1}-1}{\sin(\pi - \pi x)}$$

$$= \lim_{x\to 1} \frac{a^{x-1}-1}{x-1} \cdot \lim_{x\to 1} \frac{-\pi(1-x)}{\pi \sin(\pi(1-x))}$$

$$= \log_e a \times \left(-\frac{1}{\pi}\right) = -\frac{\log_e a}{\pi}$$

ILLUSTRATION 2.87

Evaluate $\lim_{x\to 0} \frac{e^x - e^{x\cos x}}{x + \sin x}$.

Sol. $\lim_{x\to 0} \frac{e^x - e^{x\cos x}}{x+\sin x} = \lim_{x\to 0} e^x \frac{e^{x - x\cos x}-1}{x + \sin x}$

$$= e^0 \lim_{x\to 0} \frac{e^{x-x\cos x}-1}{x - x\cos x} \cdot \lim_{x\to 0} \frac{x - x\cos x}{x + \sin x}$$

$$= 1\times 1\times \lim_{x\to 0} \frac{1-\cos x}{1+\frac{\sin x}{x}}$$

$$= \frac{1-1}{1+1}$$

$$= 0$$

ILLUSTRATION 2.88

Evaluate $\lim_{x\to 2} \frac{x-2}{\log_a (x-1)}$.

Sol. $\lim_{x\to 2} \frac{x-2}{\log_a (x-1)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 2} \frac{x-2}{\log_a \{1+(x-2)\}}$$

$$= \lim_{h\to 0} \frac{h}{\log_e (1+h)} \log_e a \qquad \text{(Substituting } x-2=h\text{)}$$

$$= \log_e a$$

ILLUSTRATION 2.89

Evaluate $\lim_{x\to a} \frac{\log x - \log a}{x-a}$.

Sol. Let $x - a = h$. Then if $x \to a$, $h \to 0$. Therefore,

$$\lim_{x\to a} \frac{\log x - \log a}{x-a} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{h\to 0} \frac{\log(a+h) - \log a}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(1+\frac{h}{a}\right)}{\frac{h}{a}a} = \frac{1}{a}$$

ILLUSTRATION 2.90

Evaluate $\lim_{x\to 0} \frac{\log(5+x) - \log(5-x)}{x}$.

Sol. We have $\lim_{x\to 0} \frac{\log(5+x)-\log(5-x)}{x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{\log\left\{5\left(1+\frac{x}{5}\right)\right\} - \log\left\{5\left(1-\frac{x}{5}\right)\right\}}{x}$$

$$= \lim_{x\to 0} \frac{\left\{\log 5 + \log\left(1+\frac{x}{5}\right)\right\} - \left\{\log 5 + \log\left(1-\frac{x}{5}\right)\right\}}{x}$$

$$= \lim_{x\to 0} \frac{\log\left(1+\frac{x}{5}\right) - \log\left(1-\frac{x}{5}\right)}{x}$$

$$= \lim_{x\to 0} \frac{1}{5} \frac{\log\left(1+\frac{x}{5}\right)}{x/5} + \lim_{x\to 0} \frac{\log\left(1-\frac{x}{5}\right)}{-x/5} \frac{1}{5} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

ILLUSTRATION 2.91

Evaluate $\lim_{h\to 0} \frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2}$.

Sol. $\lim_{h\to 0} \frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2}$

$$= \lim_{h\to 0} \frac{\log_e\left[\frac{1+2h}{1+2h+h^2}\right]}{h^2}$$

$$= \lim_{h\to 0} \frac{\log_e\left[1 + \frac{-h^2}{1+2h+h^2}\right]}{\frac{-h^2}{1+2h+h^2}} \times \frac{-1}{1+2h+h^2}$$

$$= 1\times \lim_{h\to 0} \frac{-1}{1+2h+h^2}$$

$$= -1$$

ILLUSTRATION 2.92

Let $P_n = a^{P_{n-1}} - 1$, $\forall\, n = 2, 3, \ldots$, and let $P_1 = a^x - 1$, where $a \in R^+$. Then evaluate $\lim\limits_{x\to 0} \frac{P_n}{x}$.

Sol. Clearly, if $P_k \to 0$, then $P_{k+1} \to 0$.
Now, as $x \to 0$, we get $P_1 \to 0$ or $P_2, P_3, P_4, \ldots, P_n \to 0$. Therefore,

$$\lim_{x\to 0}\frac{P_n}{x} = \lim_{x\to 0}\frac{P_n}{P_{n-1}}\frac{P_{n-1}}{P_{n-2}}\cdots\frac{P_1}{x}$$

Now, $\lim\limits_{x\to 0}\frac{P_k}{P_{k-1}} = \lim\limits_{x\to 0}\frac{a^{P_{k-1}}-1}{P_{k-1}} = \ln a$

$\therefore$ Required limit $= (\ln a)^n$

ILLUSTRATION 2.93

If $\lim\limits_{x\to 0}\frac{ae^x - b}{x}$, then find the values of a and b.

Sol. $\lim\limits_{x\to 0}\frac{ae^x - b}{x} = 2$

or $\lim\limits_{x\to 0}\frac{ae^x - a + a - b}{x} = 2$

or $a\lim\limits_{x\to 0}\frac{e^x - 1}{x} + \lim\limits_{x\to 0}\frac{a-b}{x} = 2$

or $a + \lim\limits_{x\to 0}\frac{a-b}{x} = 2$

or $a - b = 0$ and $a = 2$

or $a = 2, b = 2$

CONCEPT APPLICATION EXERCISE 2.6

Evaluate the following limits:

1. $\lim\limits_{x\to\infty} [x(a^{1/x} - 1)], a > 1$
2. $\lim\limits_{x\to 0}\frac{x2^x - x}{1-\cos x}$
3. $\lim\limits_{x\to 2}\frac{\sin(e^{x-2}-1)}{\log(x-1)}$
4. $\lim\limits_{x\to 0}\frac{e^{x^2} - \cos x}{x^2}$
5. $\lim\limits_{x\to 0}\frac{e^x + e^{-x} - 2}{x^2}$
6. $\lim\limits_{x\to a}\frac{\log(x-a)}{\log(e^x - e^a)}$
7. $\lim\limits_{x\to 0}\frac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x}, a > 0$
8. $\lim\limits_{x\to 0}\frac{(1 - 3^x - 4^x + 12^x)}{\sqrt{(2\cos x + 7)} - 3}$
9. $\lim\limits_{x\to 0}\frac{(729)^x - (243)^x - (81)^x + 9^x + 3^x - 1}{x^3}$

ANSWERS

1. $\log_e a$	2. log 4	3. 1	4. 3/2
5. 1	6. 1	7. ln a	8. −12 ln 2 × ln 3
9. $6(\ln 3)^3$			

LIMIT OF INDETERMINATE FORM 1^∞

We know that 1^∞ is one of the indeterminate forms. To find the limit of such form we use following standard formula.

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \text{ or } \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

Proof:

We know that for any rational index binomial expansion is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots, \text{where } |x| < 1$$

$$\therefore \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left(1 + \frac{1}{x}x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2 + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3 + \ldots\right)$$

$$= \lim_{x\to 0}\left(1 + 1 + \frac{1(1-x)}{2!} + \frac{1(1-x)(1-2x)}{3!} + \ldots\right)$$

$$= \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right)$$

$$= e$$

Also $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim\limits_{y\to 0}(1+y)^{\frac{1}{y}}$, where $y = \frac{1}{x}$

In general, $\lim\limits_{x\to a}(1+f(x))^{\frac{1}{f(x)}} = e$ if $\lim\limits_{x\to a} f(x) = 0$

ILLUSTRATION 2.94

Find the following limits:

(i) $\lim\limits_{x\to 0}(1-x)^{\frac{1}{x}}$ **(ii)** $\lim\limits_{x\to 1}(1+\log_e x)^{\frac{1}{\log_e x}}$

(iii) $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$

Sol.

(i) $\lim\limits_{x\to 0}(1-x)^{\frac{1}{x}} = \left(\lim\limits_{x\to 0}(1-x)^{\frac{1}{-x}}\right)^{-1} = e^{-1}$

(ii) Since $\lim\limits_{x\to 1}\log_e x = 0$, $\lim\limits_{x\to 1}(1+\log_e x)^{\frac{1}{\log_e x}} = 1$

(iii) $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}} = \lim\limits_{x\to 0}\left((1+\sin x)^{\frac{1}{\sin x}}\right)^{\frac{\sin x}{x}}$

$$= \left(\lim_{x\to 0}(1+\sin x)^{\frac{1}{\sin x}}\right)^{\lim\limits_{x\to 0}\frac{\sin x}{x}}$$

$$= e^1 = e$$

ILLUSTRATION 2.95

Evaluate $\lim\limits_{x\to 0}(\cos x)^{\cot x}$.

Sol. $\lim\limits_{x\to 0}(\cos x)^{\cot x} = \lim\limits_{x\to 0}\left[(1+(\cos x-1))^{\frac{1}{\cos x-1}}\right]^{\frac{\cos x-1}{\tan x}}$

$$= \left[\lim_{x\to 0}(1+(\cos x-1))^{\frac{1}{\cos x-1}}\right]^{\lim\limits_{x\to 0}\frac{\cos x-1}{\tan x}}$$

$$= e^{\lim\limits_{x\to 0}\frac{\cos x-1}{\sin x}\cos x}$$

$$= e^{\lim\limits_{x\to 0}\frac{\cos x-1}{\sin^2 x}\cos x\sin x}$$

$$= e^{\lim\limits_{x\to 0}\frac{\cos x-1}{1-\cos^2 x}\cos x\sin x}$$

$$= e^{\lim\limits_{x\to 0}-\frac{\sin x\cos x}{1+\cos x}} = e^0 = 1$$

ILLUSTRATION 2.96

The population of a country increases by 2% every year. If it increases k times in a century, then prove that $[k] = 7$, where $[\cdot]$ represents the greatest integer function.

Sol. If the initial number of inhabitants of the given country is A, then after a year, the total population will amount to

$$A + \frac{A}{100}2 = \left(1+\frac{1}{50}\right)A$$

After two years, the population will amount to $\left(1+\frac{1}{50}\right)^2 A$.

After 100 years, it will reach the total of $\left(1+\frac{1}{50}\right)^{100} A$, i.e., it will have increased $\left\{\left(1+\frac{1}{50}\right)^{50}\right\}^2$ times.

Taking into account that $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, we can approximately consider that $\left(1+\frac{1}{50}\right)^{50} \approx e$.

Hence, after 100 years, the population of the country will have increased $e^2 \approx 7.39$ times.

Hence, $[k] = [7.39] = 7$.

ILLUSTRATION 2.97

If $\lim\limits_{x\to 0}(1+ax+bx^2)^{2/x} = e^3$, then find the values of a and b.

Sol. We have

$$\lim_{x\to 0}(1+ax+bx^2)^{2/x} = e^3 \qquad (1^\infty \text{ form})$$

$$\Rightarrow \left(\lim_{x\to 0}(1+ax+bx^2)^{\frac{1}{ax+bx^2}}\right)^{\frac{2(ax+bx^2)}{x}} = e^3$$

$$\Rightarrow e^{2a} = e^3$$

$$\Rightarrow 2a = 3$$

$$\Rightarrow a = 3/2 \text{ and } b \in R.$$

EASIER METHOD TO FIND LIMIT OF THE FORM 1^∞

Let $L = \lim\limits_{x\to a} f(x)^{g(x)}$, where $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$

$$\therefore \quad L = \lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a}\left(1+(f(x)-1)\right)^{\frac{1}{f(x)-1}(f(x)-1)\times g(x)}$$

$$= \left[\lim_{x\to a}\left(\left(1+(f(x)-1)\right)^{\frac{1}{f(x)-1}}\right)\right]^{\lim\limits_{x\to a}(f(x)-1)\times g(x)}.$$

$$\therefore \quad L = e^{\lim\limits_{x\to a}(f(x)-1)\times g(x)}$$

ILLUSTRATION 2.98

Evaluate $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{\sin x}{x-\sin x}\right)}$.

Sol. Since $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

and $\lim\limits_{x\to 0}\frac{\sin x}{x-\sin x} = \lim\limits_{x\to 0}\frac{1}{\left(\frac{x}{\sin x}-1\right)} = \frac{1}{1-1} = \infty$

we have $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{\sin x}{x-\sin x}\right)} = e^{\lim\limits_{x\to 0}\left(\frac{\sin x}{x}-1\right)\left(\frac{\sin x}{(x-\sin x)}\right)}$

$$= e^{\lim\limits_{x\to 0}-\frac{\sin x}{x}} = e^{-1} = \frac{1}{e}$$

ILLUSTRATION 2.99

Evaluate $\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$; $(a, b, c > 0)$

Sol. We have

$$\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x} = e^{\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}-1\right)\frac{2}{x}}$$

$$= e^{\frac{2}{3}\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)}$$

$$= e^{\frac{2}{3}\lim\limits_{x\to 0}\left(\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}\right)}$$

$$= e^{\frac{2}{3}\left\{\lim_{x\to 0}\frac{a^x-1}{x}+\lim_{x\to 0}\frac{b^x-1}{x}+\lim_{x\to 0}\frac{c^x-1}{x}\right\}}$$

$$= e^{(2/3)\,\{\ln a+\ln b+\ln c\}}$$

$$= e^{(2/3)\ln(abc)}$$

$$= e^{\ln(abc)^{2/3}}$$

$$= (abc)^{2/3}$$

ILLUSTRATION 2.100

If $f(n) = \lim_{x\to 0}\left\{\left(1+\sin\frac{x}{2}\right)\left(1+\sin\frac{x}{2^2}\right)\ldots\left(1+\sin\frac{x}{2^n}\right)\right\}^{\frac{1}{x}}$ then find $\lim_{n\to\infty} f(n)$.

Sol. $f(n) = \lim_{x\to 0} e^{\frac{1}{x}\left\{\left(1+\sin\frac{x}{2}\right)\left(1+\sin\frac{x}{2^2}\right)\ldots\left(1+\sin\frac{x}{2^n}\right)-1\right\}}$

$$= \lim_{x\to 0} e^{\frac{\left\{1+\left(\sin\frac{x}{2}+\sin\frac{x}{2^2}+\cdots+\sin\frac{x}{2^n}\right)+\left(\sin\frac{x}{2}\sin\frac{x}{2^2}+\cdots\right)+\cdots-1\right\}}{x}}$$

$$= \lim_{x\to 0} e^{\left\{\frac{\sin\frac{x}{2}}{x}+\frac{\sin\left(\frac{x}{2^2}\right)}{x}+\cdots+\frac{\sin\left(\frac{x}{2^n}\right)}{x}\right\}}$$

$$= e^{\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)}$$

$$\therefore \quad \lim_{n\to\infty} f(n) = e^{\frac{1/2}{1-\frac{1}{2}}} = e$$

CONCEPT APPLICATION EXERCISE 2.7

1. Evaluate $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$.
2. Evaluate $\lim_{x\to 1}(\log_3 3x)^{\log_x 3}$.
3. Evaluate $\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x+3}$.
4. Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$, where a, b, c, and d are positive.
5. Evaluate $\lim_{x\to 7/2}(2x^2-9x+8)^{\cot(2x-7)}$.
6. If x_1 and x_2 are the real and distinct roots of ax^2+bx+c $=0$ then prove that $\lim_{x\to x_1}(1+\sin(ax^2+bx+c))^{\frac{1}{x-x_1}}$ $= e^{a(x_1-x_2)}$.
7. Evaluate $\lim_{x\to 0}\left\{\sin^2\left(\frac{\pi}{2-px}\right)\right\}^{\sec^2\left(\frac{\pi}{2-qx}\right)}$.

ANSWERS

1. e^2 2. e 3. e 4. $e^{d/b}$ 5. $e^{5/2}$
7. e^{-p^2/q^2}

L'HOSPITAL'S RULE

L'Hospital's rule uses derivatives to help evaluate limits involving indeterminate forms. Application (or repeated application) of this rule often converts an indeterminate form to an expression that can be evaluated by direct substitution, allowing easier evaluation of the limit. Fair knowledge of differentiation is required to use this method.

L'Hospital's rule states that for functions f and g which are differentiable on an open interval I except possibly at a point 'a' contained in I, if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ on $\pm\infty$, $g'(x)\neq 0$, for all x in I with $x\neq a$, and $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$.

The differentiation of the numerator and denominator often simplifies the quotient or converts it to a limit that can be evaluated directly.

In special case let $\lim_{x\to c}\frac{f(x)}{g(x)}$ take $\frac{0}{0}$ form. Since many common functions have continuous derivatives (e.g., polynomials, sine, cosine, exponential, logarithmic functions), in simple case let us consider function $f(x)$ and $g(x)$ continuously differentiable at the point c and where finite limit is found after the first derivative.

Let $f(c) = g(c) = 0$ and $g'(c)\neq 0$.

$$\text{Then}\quad \lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f(x)-0}{g(x)-0}$$

$$= \lim_{x\to c}\frac{f(x)-f(c)}{g(x)-f(c)}$$

$$= \lim_{x\to c}\frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-f(c)}{x-c}}$$

$$= \frac{\lim_{x\to c}\frac{f(x)-f(c)}{x-c}}{\lim_{x\to c}\frac{g(x)-f(c)}{x-c}}$$

$$= \frac{f'(c)}{g'(c)}$$

$$= \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

($\because f(x)$ and $g(x)$ are continuously differentiable at $x = c$)

The general proof of L' Hospital's rule is given by Cauchy's Mean Value Theorem which we will discuss in Application of Derivatives.

Some of the examples of application L'Hospital's rule are

(i) $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$

(ii) $\lim_{x\to\infty}\frac{\log_e x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0$

(iii) $\lim_{x\to 0}\frac{\log_e(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = 1$ etc.

In some cases, L'Hospital's rule fails to evaluate limit, e.g.,

$$\lim_{x\to\infty}\frac{x-\sin x}{x+\cos x} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to\infty}\frac{1-\cos x}{1-\sin x}\text{, which cannot be evaluated.}$$

However, $\lim_{x\to\infty}\frac{x-\sin x}{x+\cos x}=\lim_{x\to\infty}\frac{1-\frac{\sin x}{x}}{1+\frac{\cos x}{x}}=\frac{1-0}{1+0}=1$

ILLUSTRATION 2.101

Find the following using L'Hospital's rule

(i) $\lim_{x\to 0}\frac{(16+5x)^{1/4}-2}{(32+3x)^{1/5}-2}$ (ii) $\lim_{x\to\pi/2}\left[x\tan x-(\pi/2)\sec x\right]$

Sol.

(i) $\lim_{x\to 0}\frac{(16+5x)^{1/4}-2}{(32+3x)^{1/5}-2}$ $\left(\frac{0}{0}\text{form}\right)$

$$=\lim_{x\to 0}\frac{\frac{1}{4}(16+5x)^{\frac{1}{4}-1}\times 5}{\frac{1}{5}(32+3x)^{\frac{1}{5}-1}\times 3} \quad \text{(Using L'Hospital rule)}$$

$$=\frac{\frac{1}{4}(16)^{-\frac{3}{4}}\times 5}{\frac{1}{5}(32)^{-\frac{4}{5}}\times 3}=\frac{\frac{1}{4}\times\frac{1}{8}\times 5}{\frac{1}{5}\times\frac{1}{16}\times 3}=\frac{25}{6}$$

(ii) $\lim_{x\to\pi/2}\left[x\tan x-(\pi/2)\sec x\right]$

$$=\lim_{x\to\pi/2}\left[\frac{x\sin x}{\cos x}-\frac{\pi}{2}\times\frac{1}{\cos x}\right]$$

$$=\lim_{x\to\pi/2}\frac{x\sin x-\pi/2}{\cos x} \quad \left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to\pi/2}\frac{x.\cos x+\sin x.1-0}{-\sin x} \quad \text{(Using L'Hospital rule)}$$

$$=\frac{\pi/2.\cos(\pi/2)+\sin\pi/2}{-\sin(\pi/2)}=\frac{0+1}{-1}=-1$$

ILLUSTRATION 2.102

Let $f(x)$ be a twice-differentiable function and $f''(0)=2$. Then evaluate $\lim_{x\to 0}\frac{2f(x)-3f(2x)+f(4x)}{x^2}$.

Sol. The given limit has $\frac{0}{0}$ form.

Using L' Hospital's rule, we have

$$\text{Limit}=\lim_{x\to 0}\frac{2f'(x)-6f'(2x)+4f'(4x)}{2x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{2f''(x)-12f''(2x)+16f''(4x)}{2}$$

(Using L'Hospital's rule)

$$=\frac{6f''(0)}{2}=6$$

ILLUSTRATION 2.103

Let $f(a)=g(a)=k$ and their nth derivatives exist and be not equal for some n.

If $\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4$ then find the value of k.

Sol. $\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4$

Clearly the LHS is of $\frac{0}{0}$ form.

$\therefore$ $\lim_{x\to a}\frac{f(a)g'(x)-g(a)f'(x)}{g'(x)-f'(x)}=4$ (Using L'Hospital's rule)

or $\frac{f(a)g'(a)-g(a)f'(a)}{g'(a)-f'(a)}=4$

or $\frac{kg'(a)-kf'(a)}{g'(a)-f'(a)}=4$

or $k=4$

ILLUSTRATION 2.104

Evaluate $\lim_{x\to 0}\log_{\tan^2 x}(\tan^2 2x)$.

Sol. $L=\lim_{x\to 0}\frac{\log(\tan^2 2x)}{\log(\tan^2 x)}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

Using L'Hospital's rule, we have

$$L=\lim_{x\to 0}\frac{\left(\frac{1}{\tan^2 2x}2\tan 2x\sec^2 2x\right)\times 2}{\frac{1}{\tan^2 x}2\tan x\cdot\sec^2 x}$$

$$=\lim_{x\to 0}\frac{2\left(\frac{1}{\sin 2x\cos 2x}\right)}{\left(\frac{1}{\sin x\cos x}\right)}=\lim_{x\to 0}\frac{\left(\frac{1}{\sin 2x\cos 2x}\right)}{\left(\frac{1}{\sin 2x}\right)}$$

$$=\lim_{x\to 0}\frac{1}{\cos 2x}=1$$

ILLUSTRATION 2.105

Evaluate $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$.

Sol. $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}=\lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}-\frac{1}{1+x^2}}{3\times 2}$

$$= \lim_{x\to 0} \frac{(1+x^2)-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}(1+x^2)}$$

(Using L' Hospital's rule)

$$= \lim_{x\to 0} \frac{(1+x^2)^2-(1-x^2)}{3x^2\sqrt{1-x^2}(1+x^2)} \times \frac{1}{(1+x^2)+\sqrt{1-x^2}}$$

(Rationalizing)

$$= \lim_{x\to 0} \frac{x^4+3x^2}{3x^2\sqrt{1-x^2}(1+x^2)} \times \frac{1}{(1+x^2)+\sqrt{1-x^2}}$$

$$= \lim_{x\to 0} \frac{x^2+3}{3\sqrt{1-x^2}(1+x^2)} \times \frac{1}{(1+x^2)+\sqrt{1-x^2}} = 1/2$$

ILLUSTRATION 2.106

The graph of the function $y = f(x)$ has a unique tangent at the point $(a, 0)$ through which the graph passes. Then evaluate $\lim_{x\to a} \frac{\log_e\{1+6f(x)\}}{3f(x)}$.

Sol. From the given information, $f(a) = 0$ and $f(x)$ is differentiable at $x = a$.

$$\lim_{x\to a} \frac{\log_e\{1+6f(x)\}}{3f(x)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to a} \frac{\frac{1}{1+6f(x)}.6f'(x)}{3f'(x)} \quad \text{(Using L'Hospital Rule)}$$

$$= 2\times\frac{1}{1+6f(a)} = 2$$

ILLUSTRATION 2.107

Evaluate $\lim_{x\to\infty} x\log_e\left\{\frac{\sin\left(a+\frac{1}{x}\right)}{\sin a}\right\}, 0 < a < \frac{\pi}{2}$.

Sol. $\lim_{x\to\infty} x\cdot\log_e\left\{\frac{\sin\left(a+\frac{1}{x}\right)}{\sin a}\right\}$

$$= \lim_{x\to\infty} \frac{\log_e\left(\frac{\sin\left(a+\frac{1}{x}\right)}{\sin a}\right)}{\frac{1}{x}} \quad \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{\sin a}{\sin(a+1/x)} \times \frac{\cos(a+1/x)}{\sin a} \times \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

(Using L'Hospital's rule)

$$= \lim_{x\to\infty} \cot\left(a+\frac{1}{x}\right)$$

$$= \cot a$$

ILLUSTRATION 2.108

Find the value of α so that $\lim_{x\to 0} \frac{1}{x^2}(e^{\alpha x} - e^x - x) = \frac{3}{2}$.

Sol. $\lim_{x\to 0} \frac{1}{x^2}(e^{\alpha x} - e^x - x) \quad \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{\alpha e^{\alpha x} - e^x - 1}{2x} \quad \text{(Using L'Hospital's Rule)}$$

Since Dr $\to 0$ for $x \to 0$,

So, Nr $\to 0$ for $x \to 0$

or $\alpha e^0 - e^0 - 1 = 0$

or $\alpha = 2$

ILLUSTRATION 2.109

Find the value of $\lim_{x\to 0} \frac{\sin x + \log_e\left(\sqrt{1+\sin^2 x} - \sin x\right)}{\sin^3 x}$.

Sol. Let $\sin x = t$. Therefore,

$$\lim_{x\to 0} \frac{\sin x + \log_e\left(\sqrt{1+\sin^2 x} - \sin x\right)}{\sin^3 x}$$

$$= \lim_{t\to 0} \frac{t + \log_e\left(\sqrt{1+t^2} - t\right)}{t^3}$$

$$= \lim_{t\to 0} \frac{1 + \frac{1}{\sqrt{1+t^2}-t}\left(\frac{t}{\sqrt{1+t^2}} - 1\right)}{3t^2}$$

(Using L'Hospital's Rule)

$$= \lim_{t\to 0} \frac{1-\frac{1}{\sqrt{1+t^2}}}{3t^2} = \lim_{t\to 0} \frac{\sqrt{1+t^2}-1}{3t^2\sqrt{1+t^2}}$$

$$= \lim_{t\to 0} \frac{(1+t^2)-1}{3t^2} \times \frac{1}{\left(\sqrt{1+t^2}+1\right)}$$

$$= \lim_{t\to 0} \frac{1}{3} \frac{1}{(\sqrt{1+t^2}+1)} = \frac{1}{6}$$

INDETERMINATE FORMS 0^0 AND ∞^0

To, find the limit of the forms 0^0 and ∞^0, we generally let the limit be y and then use logarithm. Also, L'Hospital rule often helps to evaluate limit.

ILLUSTRATION 2.110

Evaluate $\lim_{x\to\infty} x^{\frac{1}{x}}$.

Sol. We have $y = \lim_{x\to\infty} x^{\frac{1}{x}} \quad (\infty^0)$

$$\therefore \quad \log_e y = \log_e\left(\lim_{x\to\infty} x^{\frac{1}{x}}\right)$$

$$= \lim_{x\to\infty}\left(\log_e x^{\frac{1}{x}}\right)$$

$$= \lim_{x\to\infty}\frac{\log_e x}{x} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to\infty}\frac{1/x}{1} \quad \text{(Using L'Hospital Rule)}$$

$$= 0$$

$$\therefore \quad y = 1$$

ILLUSTRATION 2.111

Evaluate $\lim_{x\to\frac{\pi}{2}^-} (\cos x)^{\cos x}$.

Sol. Let $y = \lim_{x\to\frac{\pi}{2}^-} (\cos x)^{\cos x}$ $\quad (0^0$ form)

$$\therefore \quad \log y = \log \lim_{x\to\frac{\pi}{2}^-} (\cos x)^{\cos x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \log\left[(\cos x)^{\cos x}\right]$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\log(\cos x)}{\sec x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{-\sin x}{\cos x}}{\sec x \tan x} \quad \text{(Using L'Hospital's rule)}$$

$$= \lim_{x\to\frac{\pi}{2}^-} (-\cos x) = 0$$

$$\therefore \quad y = 1$$

ILLUSTRATION 2.112

Evaluate $\lim_{x\to 0^+} (x)^{\frac{1}{\log_e \sin x}}$.

Sol. Let $y = \lim_{x\to 0^+} (x)^{\frac{1}{\log_e \sin x}}$ $\quad (0^0$ form)

$$\therefore \quad \log_e y = \log_e\left(\lim_{x\to 0^+} (x)^{\frac{1}{\log_e \sin x}}\right)$$

$$= \lim_{x\to 0^+} \log_e (x)^{\frac{1}{\log_e \sin x}}$$

$$= \lim_{x\to 0^+} \frac{1}{\log_e \sin x}\log_e x$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{\cos x}{\sin x}} \quad \text{(Using L'Hospital Rule)}$$

$$= \lim_{x\to 0^+} \frac{\tan x}{x} = 1$$

$$\therefore \quad y = e$$

CONCEPT APPLICATION EXERCISE 2.8

Evaluate the following using L' Hospital rule:

1. $\lim_{x\to 1} \frac{\cos\frac{\pi}{2}x}{1-\sqrt{x}}$
2. $\lim_{x\to 1} \frac{1+\log x - x}{1-2x+x^2}$
3. $\lim_{x\to\pi/2} \tan x \log \sin x$
4. $\lim_{x\to 0} \frac{\log \cos x}{x}$
5. $\lim_{x\to 0} \frac{2^x-1}{(1+x)^{1/2}-1}$
6. $\lim_{x\to\pi/4} (2-\tan x)^{1/\ln(\tan x)}$
7. $\lim_{x\to 0^+} x^x$ and $\lim_{x\to 0^+} x^{x^x}$
8. If $\lim_{x\to a} \frac{a^x - x^a}{x^x - a^a} = -1$ and $a > 0$, then find the value of a.

ANSWERS

1. π 2. $-1/2$ 3. 0 4. 0 5. $\log 4$
6. e^{-1} 7. 1 and 0 8. $a = 1$

Solved Examples

EXAMPLE 2.1

Evaluate $\lim_{n\to\infty} n^2\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\cdots}}}\right\}$.

Sol. Putting $\frac{1}{n} = x$, we have

$$\lim_{x\to 0} \frac{\sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{(1-\cos x)\ldots}}}}{x^2}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots\infty}}{x^2}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)^{\frac{1/2}{1-1/2}}}{x^2}$$

$$= \lim_{x\to 0}\frac{(1-\cos x)}{x^2}\cdot\frac{(1+\cos x)}{(1+\cos x)}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 . \lim_{x\to 0}\frac{1}{(1+\cos x)}$$

$$= (1)^2 \cdot \frac{1}{1+1} = \frac{1}{2}$$

Example 2.2

Evaluate $\lim_{n\to\infty}\left\{\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\ldots\cos\left(\frac{x}{2^n}\right)\right\}$.

Sol. We know that

$$\cos\theta\cos 2\theta\cos 4\theta\ldots\cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n\sin\theta}$$

$$\therefore \quad \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\ldots\cos\left(\frac{x}{2^n}\right) = \frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$$

$\therefore$ Given limits is $\lim_{n\to\infty}\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)} = \lim_{n\to\infty}\frac{\frac{\sin x}{x}\left(\frac{x}{2^n}\right)}{\sin\left(\frac{x}{2^n}\right)}$

$$= \frac{\sin x}{x}$$

Example 2.3

Evaluate $\lim_{x\to 1}\sec\frac{\pi}{2^x}\cdot\log_e x$.

Sol. We have,

$$\lim_{x\to 1}\frac{\log_e x}{\cos\frac{\pi}{2^x}} = \lim_{x\to 1}\frac{\log(1+(x-1))}{\sin\left(\frac{\pi}{2}-\frac{\pi}{2^x}\right)}$$

$$= \frac{\lim_{x\to 1}\frac{\log(1+(x-1))}{x-1}}{\lim_{x\to 1}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2^x}\right)}{\left(\frac{\pi}{2}-\frac{\pi}{2^x}\right)}}\cdot\lim_{x\to 1}\frac{x-1}{\frac{\pi}{2}-\frac{\pi}{2^x}}$$

$$= \lim_{x\to 1}\frac{x-1}{\pi\left(\frac{2^{x-1}-1}{2^x}\right)}$$

$$= \lim_{x\to 1}\frac{2^x}{\pi}\cdot\frac{x-1}{2^{x-1}-1}$$

$$= \frac{2}{\pi\log 2}$$

Example 2.4

Evaluate $\lim_{x\to 0^+}\frac{1}{x}\cos^{-1}\left(\frac{\sin x}{x}\right)$.

Sol. $\lim_{x\to 0^+}\frac{1}{x}\cos^{-1}\left(\frac{\sin x}{x}\right)$

$$= \lim_{x\to 0^+}\frac{\sin^{-1}\left(\sqrt{1-\frac{\sin^2 x}{x^2}}\right)}{x}$$

$$= \lim_{x\to 0^+}\frac{\sin^{-1}\left(\sqrt{1-\frac{\sin^2 x}{x^2}}\right)}{\sqrt{1-\frac{\sin^2 x}{x^2}}}\;\frac{\sqrt{1-\frac{\sin^2 x}{x^2}}}{x}$$

$$= \lim_{x\to 0^+}\frac{\sqrt{x^2-\sin^2 x}}{x^2}$$

$$= \lim_{x\to 0^+}\sqrt{\frac{x+\sin x}{x}\cdot\frac{x-\sin x}{x^3}}$$

$$= \sqrt{2}\lim_{x\to 0^+}\sqrt{\frac{x-\left(x-\frac{x^3}{3!}\right)}{x^3}}$$

$$= \sqrt{2}\sqrt{\frac{1}{6}}$$

$$= \sqrt{\frac{1}{3}}$$

Example 2.5

If $f(x) = \frac{\tan x}{x}$, then find $\lim_{x\to 0}([f(x)]+x^2)^{\frac{1}{\{f(x)\}}}$, where $[\cdot]$ and $\{.\}$ denotes greatest integer and fractional part function respectively.

Sol. We know that $\lim_{x\to 0}\frac{\tan x}{x} = 1^+$

$$\therefore \quad \lim_{x\to 0}\left[\frac{\tan x}{x}\right] = 1$$

and $\lim_{x\to 0}\left\{\frac{\tan x}{x}\right\} = \lim_{x\to 0}\left(\frac{\tan x}{x}-\left[\frac{\tan x}{x}\right]\right)$

$$= \lim_{x\to 0}\left(\frac{\tan x}{x}-1\right)$$

$$\therefore \quad \lim_{x\to 0}([f(x)]+x^2)^{\frac{1}{\{f(x)\}}} = \lim_{x\to 0}(1+x^2)^{\lim_{x\to 0}\frac{1}{\frac{\tan x}{x}-1}}$$

$$= e^{\lim_{x\to 0}\frac{x^2}{\frac{\tan x}{x}-1}}$$

$$= e^{\lim_{x\to 0}\frac{x^3\cos x}{\sin x - x\cos x}}$$

$$= e^{\lim_{x\to 0}\frac{x^3}{\left(x-\frac{x^3}{3!}+\ldots\right)-x\left(1-\frac{x^2}{2!}+\ldots\right)}}$$

$$= e^{\lim_{x\to 0}\frac{x^3}{-\frac{x^3}{3!}+\frac{x^3}{2!}}} = e^3$$

EXAMPLE 2.6

Evaluate $\lim_{n\to\infty}\frac{1}{n^2(\log_e n - \log_e(n+1)) + n}$.

Sol. $\lim_{n\to\infty}\frac{1}{n^2(\log_e n - \log_e(n+1)) + n}$

$$= \lim_{n\to\infty}\frac{1}{n^2\left(\log_e\frac{n}{n+1}\right)+n}$$

$$= \lim_{n\to\infty}\frac{1}{n^2\left(\log_e\frac{1}{1+\frac{1}{n}}\right)+n}$$

$$= \lim_{x\to 0}\frac{1}{\frac{1}{x^2}\left(\log_e\frac{1}{1+x}\right)+\frac{1}{x}}$$

$$= \lim_{x\to 0}\frac{1}{-\frac{\log_e(1+x)}{x^2}+\frac{1}{x}}$$

$$= \lim_{x\to 0}\frac{x^2}{x-\log_e(1+x)}$$

$$= \lim_{x\to 0}\frac{x^2}{x-\left(x-\frac{x^2}{2}\right)} = 2$$

EXAMPLE 2.7

Evaluate $\lim_{x\to 0}\frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$.

Sol. $(1+x)^{1/x} = e^{\frac{1}{x}\log(1+x)} = e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)}$

$$= e^{1-\frac{x}{2}+\frac{x^2}{3}-\ldots} = e\cdot e^{-\frac{x}{2}+\frac{x^2}{3}-\ldots}$$

$$= e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}\ldots\right)^2+\ldots\right]$$

$$= e\left[1-\frac{x}{2}+\frac{11}{24}x^2-\ldots\right]$$

Hence, $\lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2} = \frac{11e}{24}$

EXAMPLE 2.8

Evaluate $\lim_{x\to 0^+} x^m(\log x)^n$, $m, n \in N$.

Sol. $\lim_{x\to 0+} x^m(\log x)^n$ $\quad (0\times\infty \text{ form})$

$$= \lim_{x\to 0^+}\frac{(\log x)^n}{x^{-m}} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 0+}\frac{n(\log x)^{(n-1)}\frac{1}{x}}{-mx^{-m-1}} \quad \text{(Using L'Hospital's rule)}$$

$$= \lim_{x\to 0+}\frac{n(\log x)^{n-1}}{-mx^{-m}} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$\lim_{x\to 0+}\frac{n(n-1)(\log x)^{(n-2)}\frac{1}{x}}{(-m)^2x^{-m-1}} \quad \text{(Using L'Hospital's rule)}$$

$$= \lim_{x\to 0+}\frac{n(n-1)(\log x)^{n-2}}{m^2x^{-m}} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$\vdots$

$$= \lim_{x\to 0+}\frac{n!}{(-m)^n x^{-m}} = 0$$

(Differentiating N^r and D^r n times)

EXAMPLE 2.9

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of equation $x^n + nax - b = 0$, show that $(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_n) = n(\alpha_1^{n-1}+a)$.

Sol. Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of equation $x^n + nax - b = 0$, we have

$x^n + nax - b = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$

or $\quad \frac{x^n+nax-b}{x-\alpha_1} = (x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)$

or $\quad \lim_{x\to\alpha_1}\frac{x^n+nax-b}{x-\alpha_1} = \lim_{x\to\alpha_1}[(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)]$

or $\quad \lim_{x\to\alpha_1}\frac{nx^{n-1}+na}{1} = (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_n)$

(Using L'Hospital's rule on LHS)

or $\quad (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_n) = n\alpha_1^{n-1}+na$

EXAMPLE 2.10

Evaluate $\lim_{x\to 0}\{1^{1/\sin^2 x}+2^{1/\sin^2 x}+\ldots+n^{1/\sin^2 x}\}^{\sin^2 x}$.

Sol. Putting $\frac{1}{\sin^2 x} = t\ (\geq 1)$, we have

$\lim_{t\to\infty}(1^t+2^t+\ldots+n^t)^{1/t}$

$$= \lim_{t\to\infty}(n^t)^{1/t}\left[\left(\frac{1}{n}\right)^t+\left(\frac{2}{n}\right)^t+\ldots+1\right]^{1/t}$$

$$= n\lim_{t\to\infty}\left[\left(\frac{1}{n}\right)^t + \left(\frac{2}{n}\right)^t + \ldots + 1\right]^{1/t}$$

$$= n[0 + 0 + \ldots + 1]^0$$

$$= n$$

EXAMPLE 2.11

Evaluate $\lim\limits_{x\to\pi/2}\dfrac{\sin x - (\sin x)^{\sin x}}{1 - \sin x + \log_e \sin x}$.

Sol. Putting $t = \sin x$, we have

$$\lim_{t\to 1}\frac{t - t^t}{1 - t + \log_e t} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{t\to 1}\frac{1 - t^t(1 + \log_e t)}{0 - 1 + \dfrac{1}{t}} \qquad \text{(Using L' Hospital's rule)}$$

$$= \lim_{t\to 1}\frac{0 - \left\{t^t\left(\dfrac{1}{t}\right) + (1 + \log_e t)\,.\,t^t\,.\,(1 + \log_e t)\right\}}{0 - \dfrac{1}{t^2}}$$

(Using L' Hospital's rule)

$$= -\frac{1+1}{-1} = 2$$

EXAMPLE 2.12

Evaluate $\lim\limits_{x\to 0}\dfrac{e - (1+x)^{1/x}}{x}$.

Sol. $\lim\limits_{x\to 0}\dfrac{e - (1+x)^{1/x}}{x} = \lim\limits_{x\to 0}\dfrac{e - e^{\frac{1}{x}\log_e(1+x)}}{x}$

$$= \lim_{x\to 0}\frac{e - e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)}}{x}$$

$$= \lim_{x\to 0}\frac{e - e.e^{-\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} - \ldots\right)}}{x}$$

$$= -e\,.\lim_{x\to 0}\frac{e^{-\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4}\ldots\right)} - 1}{-\left(\dfrac{x}{2} - \dfrac{x^2}{3} + \dfrac{x^3}{4}\ldots\right)}\cdot\frac{-\left(\dfrac{x}{2} - \dfrac{x^2}{3} + \dfrac{x^3}{4}\ldots\right)}{x}$$

$$= \frac{e}{2}$$

EXAMPLE 2.13

Evaluate $\lim\limits_{n\to\infty} n^{-n^2}[(n + 2^0)(n + 2^{-1})(n + 2^{-2})\ldots(n + 2^{-n+1})]^n$.

Sol. We have $\lim\limits_{n\to\infty} n^{-n^2}\left[(n+1)\left(n + \dfrac{1}{2}\right)\ldots\left(n + \dfrac{1}{2^{n-1}}\right)\right]^n$

$$= \lim_{n\to\infty}\left[\frac{(n+1)\left(n+\frac{1}{2}\right)\ldots\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right]^n$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{n+\frac{1}{2}}{n}\right)^n\ldots\left(\frac{n+\frac{1}{2^{n-1}}}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{2n}\right)^{\frac{2n}{2}}\ldots\left(1+\frac{1}{2^{n-1}n}\right)^{\frac{2^{n-1}.n}{2^{n-1}}}$$

$$= \lim_{n\to\infty} e^1.e^{\frac{1}{2}}.e^{\frac{1}{2^2}}\ldots e^{\frac{1}{2^{n-1}}}$$

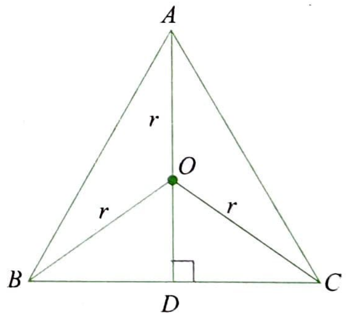
$$= e^1.e^{\frac{1}{2}}.e^{\frac{1}{2^2}}\ldots \text{ (infinite factors)}$$

$$= e^{\frac{1}{1-\frac{1}{2}}} = e^2$$

EXAMPLE 2.14

ABC is an isosceles triangle inscribed in a circle of radius r. If $AB = AC$ and h is the altitude from A to BC. If P is perimeter and A is the area of the triangle then find the value of $\lim\limits_{h\to 0}\dfrac{A}{P^3}$.

Sol. In ΔABC, $AB = AC$, $AD \perp BC$ (D is midpoint of BC)



Let r = radius of circumcircle

$\therefore$ $OA = OB = OC = r$

Now $BD = \sqrt{BO^2 - OD^2} = \sqrt{r^2 - (h-r)^2} = \sqrt{2rh - h^2}$

$\therefore$ $BC = 2\sqrt{2rh - h^2}$

Also $AB^2 = BD^2 + AD^2 = 2hr - h^2 + h^2 = 2hr.$

$\therefore$ $AB = AC = \sqrt{2hr}$

$\therefore$ Perimeter, $P = 2\sqrt{2rh - h^2} + 2\sqrt{2hr}$

$\therefore$ Area of $\Delta ABC = \dfrac{1}{2}\times BC \times AD = h\sqrt{2rh - h^2}$

So, $\lim\limits_{h\to 0}\dfrac{A}{P^3} = \dfrac{h\sqrt{2rh - h^2}}{8\left(\sqrt{2rh - h^2} + \sqrt{2hr}\right)^3}$

$$= \lim_{h\to 0} \frac{h^{3/2}\sqrt{2r-h}}{8h^{3/2}\left(\sqrt{2r-h}+\sqrt{2r}\right)^3}$$

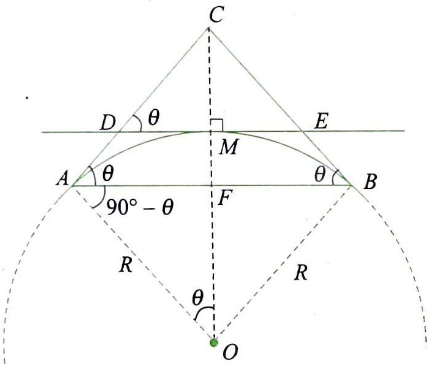
$$= \lim_{h\to 0} \frac{\sqrt{2r-h}}{8\left[\sqrt{2r-h}+\sqrt{2r}\right]^3}$$

$$= \frac{\sqrt{2r}}{8\left(\sqrt{2r}+\sqrt{2r}\right)^3} = \frac{1}{128r}$$

EXAMPLE 2.15

At the end point and the midpoint of a circular arc AB tangent lines are drawn, and the points A and B are joined with a chord. Prove that the ratio of the areas of the triangles thus formed tends to 4 as the arc AB decreases infinitely.

Sol.



Let Δ_1 = Area of triangle ABC

and Δ_2 = Area of triangle CDE

In AFO, $AF = R\sin\theta$ and $OF = R\cos\theta$

$MF = OM - OF = R - R\cos\theta$

In triangle OAC, $OC = R\sec\theta$

$\therefore \quad CM = OC - OM = R\sec\theta - R$

$\therefore \quad CF = CM + MF = R\sec\theta - R\cos\theta$

So, $\Delta_1 = \frac{1}{2}AB\cdot CF$

$$= \frac{1}{2}2R\sin\theta\cdot R(\sec\theta - \cos\theta)$$

$$= R^2\tan\theta(1-\cos^2\theta)$$

In triangle CMD, $DM = CM\cot\theta$

$$= (R\sec\theta - R)\cot\theta$$

$\therefore \quad \Delta_2 = \frac{1}{2}DE\cdot CM$

$$= \frac{1}{2}2(R\sec\theta - R)\cot\theta\cdot(R\sec\theta - R)$$

$$= R^2\frac{(1-\cos\theta)^2}{\cos^2\theta\cdot\tan\theta}$$

$$\therefore \quad \frac{\Delta_1}{\Delta_2} = \frac{\tan\theta\,(1-\cos^2\theta)\cos^2\theta\tan\theta}{(1-\cos\theta)^2}$$

$$= \frac{\sin^2\theta(1+\cos\theta)}{(1-\cos\theta)}$$

$$\therefore \quad \lim_{\theta\to 0}\frac{\Delta_1}{\Delta_2} = \lim_{\theta\to 0}\frac{\sin^2\theta(1+\cos\theta)}{(1-\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{\sin^2\theta(1+\cos\theta)^2}{1-\cos^2\theta}$$

$$= \lim_{\theta\to 0}(1+\cos\theta)^2 = 4$$

Exercises

Single Correct Answer Type

1. $\lim_{x\to 0}\left[\frac{\sin(\text{sgn}(x))}{(\text{sgn}(x))}\right]$, where [·] denotes the greatest integer function, is equal to
(1) 0 (2) 1
(3) −1 (4) does not exist

2. Let $\lim_{x\to 0}\frac{[x]^2}{x^2} = l$ and $\lim_{x\to 0}\frac{[x^2]}{x^2} = m$, where [·] denotes greatest integer. Then,
(1) l exists but m does not (2) m exists but l does not
(3) both l and m exist (4) neither l nor m exists

3. $\lim_{x\to 1}\left[\text{cosec}\,\frac{\pi x}{2}\right]^{1/(1-x)}$ (where [.] represents the greatest integer function) is equal to
(1) 0 (2) 1
(3) ∞ (4) does not exist

4. The value of the limit $\lim_{x\to 0}\frac{a^{\sqrt{x}}-a^{1/\sqrt{x}}}{a^{\sqrt{x}}+a^{1/\sqrt{x}}}$, $a > 1$, is
(1) 4 (2) 2 (3) −1 (4) 0

5. If $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\}$ exists, then
(1) both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ must exist
(2) $\lim_{x\to a} f(x)$ need not exist but $\lim_{x\to a} g(x)$ exists
(3) neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ may exist
(4) $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ need not exist

6. $\lim_{x\to -1}\frac{1}{\sqrt{|x|-\{-x\}}}$ (where $\{x\}$ denotes the fractional part of x) is equal to
(1) does not exist (2) 1
(3) ∞ (4) $\frac{1}{2}$

7. If $x_1 = 3$ and $x_{n+1} = \sqrt{2+x_n}$, $n \geq 1$, then $\lim_{n\to\infty} x_n$ is
(1) −1 (2) 2 (3) $\sqrt{5}$ (4) 3

8. $\lim_{x\to 0^-}\frac{\sum_{r=1}^{2n+1}[x^r]+(n+1)}{1+[x]+|x|+2x}$, where $n \in N$ and [.] denotes the greatest integer function, equals
(1) −1 (2) 1
(3) 0 (4) does not exist

9. $\lim_{x\to\infty}\frac{\sin^4 x-\sin^2 x+1}{\cos^4 x-\cos^2 x+1}$ is equal to
(1) 0 (2) 1 (3) $\frac{1}{3}$ (4) $\frac{1}{2}$

10. If $f(x) = \frac{2}{x-3}$, $g(x) = \frac{x-3}{x+4}$, and $h(x) = -\frac{2(2x+1)}{x^2+x-12}$, then $\lim_{x\to 3}[f(x)+g(x)+h(x)]$ is
(1) −2 (2) −1 (3) $-\frac{2}{7}$ (4) 0

11. The value of $\lim_{x\to\pi}\frac{1+\cos^3 x}{\sin^2 x}$ is
(1) 1/3 (2) 2/3 (3) −1/4 (4) 3/2

12. The value of $\lim_{x\to 2}\frac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$ is
(1) $\frac{1}{8\sqrt{3}}$ (2) $\frac{1}{4\sqrt{3}}$
(3) 0 (4) none of these

13. The value of $\lim_{x\to 2}\frac{2^x+2^{3-x}-6}{\sqrt{2^{-x}-2^{1-x}}}$ is
(1) 16 (2) 8 (3) 4 (4) 2

14. The value of $\lim_{x\to 2}\left(\left(\frac{x^3-4x}{x^3-8}\right)^{-1}-\left(\frac{x+\sqrt{2x}}{x-2}-\frac{\sqrt{2}}{\sqrt{x}-\sqrt{2}}\right)^{-1}\right)$ is
(1) 1/2 (2) 2
(3) 1 (4) None of these

15. If $\lim_{x\to -2^-}\frac{ae^{1/|x+2|}-1}{2-e^{1/|x+2|}} = \lim_{x\to -2^+}\sin\left(\frac{x^4-16}{x^5+32}\right)$, then a is
(1) $\sin\frac{3}{5}$ (2) 2 (3) $\sin\frac{2}{5}$ (4) $\sin\frac{1}{5}$

16. $\lim_{x\to 1}\frac{(1-x)(1-x^2)\cdots(1-x^{2n})}{\{(1-x)(1-x^2)\cdots(1-x^n)\}^2}$, $n \in N$, equals
(1) ${}^{2n}P_n$ (2) ${}^{2n}C_n$
(3) $(2n)!$ (4) none of these

17. The value of $\lim_{x\to\frac{1}{\sqrt{2}}}\frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)}$ is
(1) $-\frac{1}{\sqrt{2}}$ (2) $\frac{1}{\sqrt{2}}$ (3) $\sqrt{2}$ (4) $-\sqrt{2}$

18. Among **(i)** $\lim_{x\to\infty} \sec^{-1}\left(\frac{x}{\sin x}\right)$ and **(ii)** $\lim_{x\to\infty} \sec^{-1}\left(\frac{\sin x}{x}\right)$,

(1) (i) exists, (ii) does not exist
(2) (i) does not exist, (ii) exists
(3) both (i) and (ii) exist
(4) neither (i) nor (ii) exists

19. $\lim_{x\to\infty}\left(\frac{x^3}{3x^2-4} - \frac{x^2}{3x+2}\right)$ is equal to

(1) does not exist (2) 1/3
(3) 0 (4) 2/9

20. $\lim_{n\to\infty} \frac{n(2n+1)^2}{(n+2)(n^2+3n-1)}$ is equal to

(1) 0 (2) 2 (3) 4 (4) ∞

21. $\lim_{x\to\infty} \frac{(2x+1)^{40}(4x-1)^5}{(2x+3)^{45}}$ is equal to

(1) 16 (2) 24 (3) 32 (4) 8

22. $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\right]$ is equal to

(1) 0 (2) $\frac{1}{2}$ (3) log 2 (4) e^4

23. $\lim_{x\to\infty} \frac{2+2x+\sin 2x}{(2x+\sin 2x)e^{\sin x}}$ is equal to

(1) 0 (2) 1
(3) −1 (4) does not exist

24. $\lim_{x\to\infty} \frac{(x+1)^{10}+(x+2)^{10}+\cdots+(x+100)^{10}}{x^{10}+10^{10}}$ is equal to

(1) 0 (2) 1 (3) 10 (4) 100

25. $\lim_{x\to\infty} \frac{2\sqrt{x}+3\sqrt[3]{x}+4\sqrt[4]{x}+\cdots+n\sqrt[n]{x}}{\sqrt{(2x-3)}+\sqrt[3]{(2x-3)}+\cdots+\sqrt[n]{(2x-3)}}$ is equal to

(1) 1 (2) ∞
(3) $\sqrt{2}$ (4) none of these

26. If $\lim_{n\to\infty} \frac{n\cdot 3^n}{n(x-2)^n + n\cdot 3^{n+1} - 3^n} = \frac{1}{3}$, then the range of x is (where $n \in N$)

(1) $[2, 5)$ (2) $(1, 5)$ (3) $(-1, 5)$ (4) $(-\infty, \infty)$

27. $\lim_{n\to\infty} n^2\left(x^{1/n} - x^{1/(n+1)}\right), x > 0,$ is equal to

(1) 0 (2) e^x
(3) $\log_e x$ (4) none of these

28. Let $f(x) = \lim_{n\to\infty} \frac{1}{\left(\frac{3}{\pi}\tan^{-1}2x\right)^{2n}+5}$. Then the set of values of x for which $f(x) = 0$ is

(1) $|2x| > \sqrt{3}$ (2) $|2x| < \sqrt{3}$
(3) $|2x| \ge \sqrt{3}$ (4) $|2x| \le \sqrt{3}$

29. $f(x) = \frac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}$. Then $\lim_{x\to\infty} f(x)$ is equal to

(1) 1 (2) 1/2
(3) 2 (4) none of these

30. The value of $\lim_{n\to\infty}\left[\frac{2n}{2n^2-1}\cos\frac{n+1}{2n-1} - \frac{n}{1-2n}\cdot\frac{n}{n^2+1}\right]$ is

(1) 1 (2) −1
(3) 0 (4) none of these

31. If $f(x) = 0$ is a quadratic equation such that $f(-\pi) = f(\pi) = 0$ and $f\left(\frac{\pi}{2}\right) = -\frac{3\pi^2}{4}$, then $\lim_{x\to-\pi} \frac{f(x)}{\sin(\sin x)}$ is equal to

(1) 0 (2) π
(3) 2π (4) none of these

32. $\lim_{x\to 1} \frac{x\sin(x-[x])}{x-1}$, where $[\cdot]$ denotes the greatest integer function, is equal to

(1) 0 (2) −1
(3) non-existent (4) none of these

33. $\lim_{x\to-\infty} \frac{x^2 \tan\frac{1}{x}}{\sqrt{8x^2+7x+1}}$ is equal to

(1) $-\frac{1}{2\sqrt{2}}$ (2) $\frac{1}{2\sqrt{2}}$
(3) $\frac{1}{\sqrt{2}}$ (4) does not exist

34. $\lim_{x\to 0} \frac{x^a \sin^b x}{\sin(x^c)}$, where $a, b, c \in R \sim \{0\}$, exists and has non-zero value. Then,

(1) $a + c = b$ (2) $b + c = a$
(3) $a + b = c$ (4) none of these

35. $\lim_{x\to 0} \frac{x^4(\cot^4 x - \cot^2 x + 1)}{(\tan^4 x - \tan^2 x + 1)}$ is equal to

(1) 1 (2) 0
(3) 2 (4) none of these

36. $\lim_{x\to 1} \frac{1-x^2}{\sin 2\pi x}$ is equal to

(1) $\frac{1}{2\pi}$ (2) $\frac{-1}{\pi}$
(3) $\frac{-2}{\pi}$ (4) none of these

37. $\lim_{x\to 0} \frac{1}{x}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is equal to

(1) 1 (2) 0
(3) 2 (4) none of these

38. $\lim_{y\to 0} \frac{(x+y)\sec(x+y) - x\sec x}{y}$ is equal to
(1) $\sec x(x\tan x + 1)$ (2) $x\tan x + \sec x$
(3) $x\sec x + \tan x$ (4) none of these

39. $\lim_{x\to 1} \frac{1+\sin\pi\left(\frac{3x}{1+x^2}\right)}{1+\cos\pi x}$ is equal to
(1) 0 (2) 1 (3) 2 (4) 4

40. $\lim_{n\to\infty} \sum_{x=1}^{20} \cos^{2n}(x-10)$ is equal to
(1) 0 (2) 1 (3) 19 (4) 20

41. $\lim_{x\to -1} \left(\frac{x^4+x^2+x+1}{x^2-x+1}\right)^{\frac{1-\cos(x+1)}{(x+1)^2}}$ is equal to
(1) 1 (2) $(2/3)^{1/2}$ (3) $(3/2)^{1/2}$ (4) $e^{1/2}$

42. $\lim_{x\to\infty} \{(x+5)\tan^{-1}(x+5) - (x+1)\tan^{-1}(x+1)\}$ is equal to
(1) π (2) 2π
(3) $\pi/2$ (4) none of these

43. The value of $\lim_{x\to 0}\left(\left[\frac{100x}{\sin x}\right] + \left[\frac{99\sin x}{x}\right]\right)$ (where [.] represents the greatest integral function) is
(1) 199 (2) 198
(3) 0 (4) none of these

44. The value of $\lim_{x\to a} \sqrt{a^2-x^2}\, \cot\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}$ is
(1) $\frac{2a}{\pi}$ (2) $-\frac{2a}{\pi}$ (3) $\frac{4a}{\pi}$ (4) $-\frac{4a}{\pi}$

45. $\lim_{x\to 0}\left[\min(y^2-4y+11)\frac{\sin x}{x}\right]$ (where [.] denotes the greatest integer function) is
(1) 5 (2) 6
(3) 7 (4) does not exist

46. The value of $\lim_{x\to 0} \frac{1-(\cos x)\sqrt{\cos 2x}}{x^2}$ is
(1) $-\frac{3}{2}$ (2) $-\frac{1}{2}$ (3) $\frac{1}{2}$ (4) $\frac{3}{2}$

47. $\lim_{x\to\infty} \frac{1}{x+1}\tan\left(\frac{\pi x+1}{2x+2}\right)$ is equal to
(1) $\frac{2}{\pi-1}$ (2) $\frac{\pi+1}{2}$ (3) $\frac{2}{\pi+1}$ (4) $\frac{2(\pi+1)}{\pi-1}$

48. The value of $\lim_{x\to 1^-} \frac{1-\sqrt{x}}{\left(\cos^{-1}x\right)^2}$ is
(1) 4 (2) 1/2 (3) 2 (4) 1/4

49. $\lim_{x\to\pi/2} \frac{\sin(x\cos x)}{\cos(x\sin x)}$ is equal to
(1) 0 (2) $\pi/2$ (3) π (4) 2π

50. $\lim_{x\to 0}\left[\left(1-e^x\right)\frac{\sin x}{|x|}\right]$ is (where [·] represents the greatest integer function)
(1) -1 (2) 1
(3) 0 (4) does not exist

51. $\lim_{x\to 0} \frac{x(e^x-1)}{1-\cos x}$ is equal to
(1) 0 (2) ∞ (3) -2 (4) 2

52. If $f(x) = \lim_{n\to\infty} n(x^{1/n}-1)$, then for $x > 0$, $y > 0$, $f(xy)$ is equal to
(1) $f(x)f(y)$ (2) $f(x)+f(y)$
(3) $f(x)-f(y)$ (4) none of these

53. $\lim_{x\to 0}\left\{(1+x)^{\frac{2}{x}}\right\}$ (where {·} denotes the fractional part of x) is equal to
(1) e^2-7 (2) e^2-8
(3) e^2-6 (4) none of these

54. The value of $\lim_{x\to\infty} \frac{(2^{x^n})^{\frac{1}{e^x}} - (3^{x^n})^{\frac{1}{e^x}}}{x^n}$ (where $n \in N$) is
(1) $\log n\left(\frac{2}{3}\right)$ (2) 0
(3) $n\log n\left(\frac{2}{3}\right)$ (4) not defined

55. $\lim_{x\to 0} \frac{\sin(x^2)}{\ln(\cos(2x^2-x))}$ is equal to
(1) 2 (2) -2 (3) 1 (4) -1

56. $\lim_{x\to\infty} \frac{e^{1/x^2}-1}{2\tan^{-1}(x^2)-\pi}$ is equal to
(1) 1 (2) -1 (3) $\frac{1}{2}$ (4) $-\frac{1}{2}$

57. $\lim_{x\to 0} \frac{(2^m+x)^{1/m} - (2^n+x)^{1/n}}{x}$ is equal to
(1) $\frac{1}{m2^m} - \frac{1}{n2^n}$ (2) $\frac{1}{m2^m} + \frac{1}{n2^n}$
(3) $\frac{1}{m2^{m-1}} - \frac{1}{n2^{n-1}}$ (4) $\frac{1}{m2^{m-1}} + \frac{1}{n2^{n-1}}$

58. The value of $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{e^{1/n}}{n} + \frac{e^{2/n}}{n} + \dots + \frac{e^{(n-1)/n}}{n}\right]$ is
(1) 1 (2) 0
(3) $e-1$ (4) $e+1$

59. $\lim_{x\to 1}\frac{nx^{n+1}-(n+1)x^n+1}{(e^x-e)\sin \pi x}$, where $n = 100$, is equal to

(1) $\frac{5050}{\pi e}$ (2) $\frac{100}{\pi e}$ (3) $-\frac{5050}{\pi e}$ (4) $-\frac{4950}{\pi e}$

60. $\lim_{x\to 0}\frac{\log(1+x+x^2)+\log(1-x+x^2)}{\sec x-\cos x}=$

(1) -1 (2) 1 (3) 0 (4) 2

61. The value of $\lim_{x\to\infty}\left(\sqrt[3]{x^3+2x^2}-\sqrt{x^2+x}\right)$ is

(1) $\frac{1}{3}$ (2) $\frac{1}{2}$ (3) $\frac{1}{6}$ (4) $\frac{5}{12}$

62. The value of $\lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}$ is

(1) $\frac{1}{2}$ (2) $-\frac{1}{2}$
(3) 0 (4) none of these

63. $\lim_{x\to 0}\frac{\cos(\tan x)-\cos x}{x^4}$ is equal to

(1) 1/6 (2) $-1/3$ (3) 1/2 (4) 1

64. If $\lim_{x\to 0}(x^{-3}\sin 3x + ax^{-2} + b)$ exists and is equal to 0, then

(1) $a = -3$ and $b = 9/2$ (2) $a = 3$ and $b = 9/2$
(3) $a = -3$ and $b = -9/2$ (4) $a = 3$ and $b = -9/2$

65. If $\lim_{x\to 0}\frac{x^n\sin^n x}{x^n-\sin^n x}$ is non-zero finite, then n is equal to

(1) 1 (2) 2
(3) 3 (4) none of these

66. $\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$ is equal to

(1) e (2) $\frac{1}{e}$
(3) 1 (4) None of these

67. The value of $\lim_{x\to 1}(2-x)^{\tan\left(\frac{\pi x}{2}\right)}$ is

(1) $e^{-2/\pi}$ (2) $e^{1/\pi}$ (3) $e^{2/\pi}$ (4) $e^{-1/\pi}$

68. The value of $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m$ is

(1) 1 (2) e
(3) e^{-1} (4) none of these

69. $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}$ is equal to

(1) e (2) e^2 (3) e^{-1} (4) 1

70. $\lim_{n\to\infty}\left\{\left(\frac{n}{n+1}\right)^{\alpha}+\sin\frac{1}{n}\right\}^n$ (when $\alpha \in Q$) is equal to

(1) $e^{-\alpha}$ (2) $-\alpha$ (3) $e^{1-\alpha}$ (4) $e^{1+\alpha}$

71. $\lim_{x\to\infty}\left[\left(\frac{e}{1-e}\right)\left(\frac{1}{e}-\frac{x}{1+x}\right)\right]^x$ is

(1) $e^{(1-e)}$ (2) $e^{\left(\frac{1-e}{e}\right)}$ (3) $e^{\left(\frac{e}{1-e}\right)}$ (4) $e^{\frac{1+e}{e}}$

72. $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+\dots+n^x}{n}\right)^{1/x}$ is equal to

(1) $(n!)^n$ (2) $(n!)^{1/n}$ (3) $n!$ (4) $ln\,(n!)$

73. The value of $\lim_{x\to 1}\left(\frac{p}{1-x^p}-\frac{q}{1-x^q}\right)$, $p, q, \in N$, equals

(1) $\frac{p+q}{2}$ (2) $\frac{pq}{2}$ (3) $\frac{p-q}{2}$ (4) $\sqrt{\frac{p}{q}}$

74. $\lim_{x\to\infty}\frac{x(\log x)^3}{1+x+x^2}$ equals

(1) 0 (2) -1
(3) 1 (4) does not exist

75. $\lim_{x\to\infty}\frac{\cot^{-1}(x^{-a}\log_a x)}{\sec^{-1}(a^x\log_x a)}$, $(a > 1)$, is equal to

(1) 2 (2) 1 (3) $\log_a 2$ (4) 0

76. The value of $\lim_{n\to\infty}\frac{e^n}{\left(1+\frac{1}{n}\right)^{n^2}}$ is

(1) e (2) e^2 (3) $\sqrt{e}$ (4) e^{-1}

Multiple Correct Answers Type

1. Let $f(x)=\begin{cases}1+\frac{2x}{a}, & 0\le x<1\\ ax, & 1\le x<2\end{cases}$. If $\lim_{x\to 1}f(x)$ exists, then a is

(1) 1 (2) -1 (3) 2 (4) -2

2. If $f(x) = |x - 1| - [x]$, where $[x]$ is the greatest integer less than or equal to x, then

(1) $f(1 + 0) = -1, f(1 - 0) = 0$
(2) $f(1 + 0) = 0 = f(1 - 0)$
(3) $\lim_{x\to 1}f(x)$ exists
(4) $\lim_{x\to 1}f(x)$ does not exist

3. If $\lim_{n\to\infty}\left(an-\frac{1+n^2}{1+n}\right)=b$, where a is a finite number, then

(1) $a = 1$ (2) $a = 0$ (3) $b = 1$ (4) $b = -1$

4. If $m, n \in N$, $\lim_{x\to 0}\frac{\sin x^n}{(\sin x)^m}$ is

(1) 1, if $n = m$ (2) 0, if $n > m$
(3) ∞, if $n < m$ (4) n/m, if $n < m$

5. Let $f(x)=\begin{cases} x^n \sin(1/x^2), x \neq 0 \\ 0, \qquad x=0 \end{cases}$, ($n \in I$). Then

(1) $\lim_{x\to 0} f(x)$ exists for $n>0$

(2) $\lim_{x\to 0} f(x)$ does not exists for $n<0$

(3) $\lim_{x\to 0} f(x)$ does not exists for any value of n

(4) $\lim_{x\to 0} f(x)$ exists for any value of n

6. If $L=\lim_{x\to 0}\frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}}-\frac{1+ax}{1+bx}\right)$ exists, then

(1) $a=1/4$ (2) $b=3/4$

(3) $L=-1/32$ (4) $L=1/32$

7. Which of the following is true ({.} denotes the fractional part of the function)?

(1) $\lim_{x\to\infty}\frac{\log_e x}{\{x\}}=\infty$ (2) $\lim_{x\to 2^+}\frac{x}{x^2-x-2}=\infty$

(3) $\lim_{x\to -1^-}\frac{x}{x^2-x-2}=-\infty$ (4) $\lim_{x\to\infty}\frac{\log_{0.5} x}{\{x\}}=\infty$

8. Which of the following is/are correct?

(1) $\lim_{x\to 0}\frac{[x+|x|]}{x}=0$, where $[x]$ denotes the greatest integer functions.

(2) $\lim_{x\to 0}\frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}=0$

(3) $\lim_{x\to 3}(x-3)^{\frac{1}{5}}\,\text{sgn}(x-3)=0$, where sgn stands for signum function.

(4) $\lim_{x\to 0}\frac{\tan^{-1}|x|}{x}=0$

9. If $\lim_{x\to 1}(2-x+a[x-1]+b[1+x])$ exists, then a and b can take the values (where $[\cdot]$ denotes the greatest integer function)

(1) $a=1/3, b=1$ (2) $a=1, b=-1$

(3) $a=9, b=-9$ (4) $a=2, b=2/3$

10. Let $L=\lim_{x\to a}\frac{|2\sin x-1|}{2\sin x-1}$. Then

(1) limit does not exist when $a=\pi/6$

(2) $L=-1$ when $a=\pi$

(3) $L=1$ when $a=\pi/2$

(4) $L=1$ when $a=0$

11. Let $f(x)=\lim_{n\to\infty}\frac{x}{x^{2n}+1}$. Then

(1) $f(1^+)+f(1^-)=0$

(2) $f(1^+)+f(1^-)+f(1)=3/2$

(3) $f(-1^+)+f(-1^-)=-1$

(4) $f(1^+)+f(-1^-)=0$

12. $\lim_{n\to\infty}\frac{-3n+(-1)^n}{4n-(-1)^n}$ is equal to ($n \in N$)

(1) $-\frac{3}{4}$ (2) 0 if n is even

(3) $-\frac{3}{4}$ if n is odd (4) none of these

13. Given a real-valued function f such that

$$f(x)=\begin{cases}\frac{\tan^2\{x\}}{(x^2-[x]^2)}, & \text{for } x>0\\ 1, & \text{for } x=0\\ \sqrt{\{x\}\cot\{x\}}, & \text{for } x<0\end{cases}$$

where $[x]$ is the integral part and $\{x\}$ is the fractional part of x, then

(1) $\lim_{x\to 0^+} f(x)=1$

(2) $\lim_{x\to 0^-} f(x)=\cot 1$

(3) $\cot^{-1}\left(\lim_{x\to 0^-} f(x)\right)^2=1$

(4) $\tan^{-1}\left(\lim_{x\to 0^+} f(x)\right)=\frac{\pi}{4}$

14. If $f(x)=\frac{3x^2+ax+a+1}{x^2+x-2}$, then which of the following can be correct?

(1) $\lim_{x\to 1} f(x)$ exists $\Rightarrow a=-2$

(2) $\lim_{x\to -2} f(x)$ exists $\Rightarrow a=13$

(3) $\lim_{x\to 1} f(x)=4/3$ if it exists

(4) $\lim_{x\to -2} f(x)=-1/3$ if it exists

15. The value of $\lim_{n\to\infty}\frac{1}{1+n\sin^2 nx}$ can be ($n \in N$)

(1) -1 (2) 0 (3) 1 (4) ∞

16. Let $f(x)=\frac{x^2-9x+20}{x-[x]}$ (where $[x]$ is the greatest integer not greater than x). Then

(1) $\lim_{x\to 5^-} f(x)=0$

(2) $\lim_{x\to 5^+} f(x)=1$

(3) $\lim_{x\to 5} f(x)$ does not exist

(4) none of these

17. Given $\lim_{x\to 0}\frac{f(x)}{x^2}=2$, where $[\cdot]$ denotes the greatest integer function, then

(1) $\lim_{x\to 0}[f(x)]=0$

(2) $\lim_{x\to 0}[f(x)]=1$

(3) $\lim_{x\to 0}\left[\frac{f(x)}{x}\right]$ does not exist

(4) $\lim_{x\to 0}\left[\frac{f(x)}{x}\right]$ exists

18. If $f(\alpha)=\lim_{x\to 2}\left(\sin^x\alpha+\cos^x\alpha\right)^{\frac{1}{(x-2)}}$ for $\alpha\in\left[0,\frac{\pi}{2}\right]$, then

(1) $f(0)=1$

(2) $f\left(\frac{\pi}{2}\right)=1$

(3) $f(\alpha)=(\cos\alpha)^{\cos^2\alpha}\cdot(\sin\alpha)^{\sin^2\alpha}$ if $\alpha\in\left(0,\frac{\pi}{2}\right)$

(4) $f(\alpha)=\frac{(\sin\alpha)^{\sin^2\alpha}}{(\cos\alpha)^{\cos^2\alpha}}$ if $\alpha\in\left(0,\frac{\pi}{2}\right)$

Linked Comprehension Type

For Problems 1–3

Let $f(x)=\frac{\sin^{-1}(1-\{x\})\times\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\times(1-\{x\})}$, where $\{x\}$ denotes the fractional part of x.

1. $R=\lim_{x\to 0+}f(x)$ is equal to

(1) $\frac{p}{2}$ (2) $\frac{\pi}{2\sqrt{2}}$ (3) $\frac{\pi}{\sqrt{2}}$ (4) $\sqrt{2}\pi$

2. $L=\lim_{x\to 0-}f(x)$ is equal to

(1) $\frac{p}{2}$ (2) $\frac{\pi}{2\sqrt{2}}$ (3) $\frac{\pi}{\sqrt{2}}$ (4) $\sqrt{2}\pi$

3. Which of the following is true?

(1) $\cos L<\cos R$ (2) $\tan(2L)<\tan 2R$

(3) $\sin L>\sin R$ (4) None of these

For Problems 4–6

$A_i=\frac{x-a_i}{|x-a_i|}$, $i=1,2,\ldots,n$, and $a_1<a_2<a_3<\ldots<a_n$.

4. If $1\le m\le n$, $m\in N$, then the value of

$L=\lim_{x\to a_m^-}(A_1A_2\cdots A_n)$ is

(1) always 1 (2) always -1

(3) $(-1)^{n-m+1}$ (4) $(-1)^{n-m}$

5. If $1\le m\le n$, $m\in N$, then the value of

$R=\lim_{x\to a_m^+}(A_1A_2\cdots A_n)$ is

(1) always 1 (2) always -1

(3) $(-1)^{m+1}$ (4) $(-1)^{n-m}$

6. If $1\le m\le n$, $m\in N$, then $\lim_{x\to a_m}(A_1A_2\cdots A_n)$

(1) is always equal to -1 (2) is always equal to $+1$

(3) does not exist (4) None of these

For Problems 7–9

If $L=\lim_{x\to 0}\frac{\sin x+ae^x+be^{-x}+c\log_e(1+x)}{x^3}$ exists finitely, then

7. The value of L is

(1) 1/2 (2) $-1/3$ (3) $-1/6$ (4) 3

8. Equation $ax^2+bx+c=0$ has

(1) real and equal roots

(2) complex roots

(3) unequal positive real roots

(4) unequal roots

9. The solutions set of $||x+c|-2a|<4b$ is

(1) $[-2, 2]$ (2) $[0, 2]$ (3) $[-1, 1]$ (4) $[-2, 1]$

For Problems 10–12

Let $a_1>a_2>a_3>\ldots>a_n>1$.

$p_1>p_2>p_3>\ldots>p_n>0$ such that $p_1+p_2+p_3+\cdots+p_n=1$.

Also, $F(x)=(p_1a_1^x+p_2a_2^x+\cdots+p_na_n^x)^{1/x}$.

10. $\lim_{x\to 0^+}F(x)$ equals

(1) $p_1\ln a_1+p_2\ln a_2+\cdots+p_n\ln a_n$

(2) $a_1^{p_1}+a_2^{p_2}+\cdots+a_n^{p_n}$

(3) $a_1^{p_1}\cdot a_2^{p_2}\cdots a_n^{p_n}$

(4) $\sum_{r=1}^{n}a_rp_r$

11. $\lim_{x\to\infty}F(x)$ equals

(1) $\ln a_1$ (2) e^{an} (3) a_1 (4) a_n

12. $\lim_{x\to-\infty}f(x)$ equals

(1) $\ln a_n$ (2) e^{a1} (3) a_1 (4) a_n

For Problems 13 and 14

If $L=\lim_{x\to\infty}\left(x+1-\sqrt{ax^2+x+3}\right)$ exists finitely then

13. The value of a is

(1) -1 (2) 1/2 (3) 1 (4) 3/2

14. The value of L is

(1) $\frac{1}{3}$ (2) $\frac{1}{4}$ (3) $\frac{1}{2}$ (4) $\frac{2}{3}$

For Problems 15 and 16

Let $f: R\to R$ be a real function. The function f is double differentiable. If there exists $n\in N$ and $p\in R$ such that $\lim_{x\to\infty}x^nf(x)=p$ and there exists $\lim_{x\to\infty}x^{n+1}f(x)$, then

15. $\lim_{x\to\infty}x^{n+1}f'(x)$ is equal to

(1) $-np$ (2) np (3) n^2p (4) np^2

16. $\lim_{x\to\infty}x^{n+2}f''(x)$ is equal to

(1) $np(1-n)$ (2) $-np(1+n)$

(3) n^2p (4) $np(1+n)$

For Problems 17 and 18

Let $f(x)$ be a polynomial satisfying $\lim_{x\to\infty}\frac{x^2 f(x)}{2x^5+3}=6$ and $f(1)=3, f(3)=7$ and $f(5)=11$. Then

17. The value of $f(0)$ is

(1) 120 (2) 119 (3) 180 (4) -179

18. $\lim_{x\to1}\frac{x-1}{\sin(f(x)-2x-1)}$ is equal to

(1) $\frac{1}{96}$ (2) $\frac{1}{48}$ (3) $\frac{1}{24}$ (4) 1

For Problems 19 and 20

If $\lim_{x\to0}\frac{f(x)}{\sin^2 x}=8$, $\lim_{x\to0}\frac{g(x)}{2\cos x - xe^x + x^3 + x - 2}=\lambda$ and $\lim_{x\to0}(1+2f(x))^{\frac{1}{g(x)}}=\frac{1}{e}$, then

19. The value of λ is

(1) 2 (2) 4 (3) 8 (4) 16

20. $\lim_{x\to0}(1+f(x))^{\frac{1}{2g(x)}}$ is equal to

(1) $e^{-\frac{1}{4}}$ (2) $e^{-\frac{1}{2}}$ (3) e^{-2} (4) e^{-4}

Matrix Match Type

1. Match the following lists:

List I	List II
a. If $L=\lim_{x\to-1}\frac{\sqrt[3]{(7-x)}-2}{(x+1)}$, then $12L=$	**p.** -2
b. If $L=\lim_{x\to\pi/4}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}$, then $-L/4=$	**q.** 2
c. If $L=\lim_{x\to1}\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$, then $20L=$	**r.** 1
d. If $L=\lim_{x\to\infty}\frac{\log x^n-[x]}{[x]}$, where $n\in N$, ($[x]$ denotes greatest integer less than or equal to x), then $-2L=$	**s.** -1

2. Match the following lists:

List I ($[\cdot]$ denotes the greatest integer function)	List II
a. $\lim_{x\to0}\left(\left[100\frac{\sin x}{x}\right]+\left[100\frac{\tan x}{x}\right]\right)$	**p.** 198
b. $\lim_{x\to0}\left(\left[100\frac{x}{\sin x}\right]+\left[100\frac{\tan x}{x}\right]\right)$	**q.** 199
c. $\lim_{x\to0}\left(\left[100\frac{\sin^{-1}x}{x}\right]+\left[100\frac{\tan^{-1}x}{x}\right]\right)$	**r.** 200
d. $\lim_{x\to0}\left(\left[100\frac{x}{\sin^{-1}x}\right]+\left[100\frac{\tan^{-1}x}{x}\right]\right)$	**s.** 201

3. Match the following lists:

List I	List II
a. If $\lim_{x\to\infty}(\sqrt{(x^2-x-1)}-ax-b)=0$, where $a>0$, then there exists at least one a and b for which point $(a, 2b)$ lies on the line.	**p.** $y=-3$
b. If $\lim_{x\to\infty}\frac{(1+a^3)+8e^{1/x}}{1+(1-b^3)e^{1/x}}=2$, then there exists at least one a and b for which point (a, b^3) lies on the line.	**q.** $3x-2y-5=0$
c. If $\lim_{x\to\infty}(\sqrt{(x^4-x^2+1)}-ax^2-b)=0$, then there exists at least one a and b for which point $(a, -4b)$ lies on the line.	**r.** $15x-2y-11=0$
d. If $\lim_{x\to-a}\frac{x^7+a^7}{x+a}=7$, where $a<0$, then there exists at least one a for which point $(-a, 2)$ lies on the line.	**s.** $y=2$

4. Consider $\lim_{x\to\infty}\left(\frac{x^3+x^2+x+\sin x}{x^2+2\cos x}-a\sin x-bx+c\right)=4$. Now, match the following lists and then choose the correct code.

List I	List II
a. The value of a is	**p.** 1
b. The value of b is	**q.** 3
c. The value of c is	**r.** 2
d. Number of real roots of equation $cx^2+bx+a=0$ is	**s.** 0

Codes:

	a	b	c	d
(1)	s	r	q	p
(2)	q	s	r	p
(3)	s	r	p	q
(4)	s	p	q	r

5. Match the following lists (where $[x]$ represents the greatest integer function) and then choose the correct code.

List I	List II
a. $\lim_{x\to0} x(-1)^{[1/x]}$	**p.** Does not exist
b. $\lim_{x\to2}(-1)^{[x]}$	**q.** is 0
c. $\lim_{x\to\frac{3}{2}}(x-[x])$	**r.** is 1
d. $\lim_{x\to0}[x]\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$	**s.** is 2

Codes:

	a	b	c	d
(1)	s	r	q	p
(2)	q	p	s	p
(3)	s	r	p	q
(4)	p	p	q	r

Numerical Value Type

1. The reciprocal of the value of

$\lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$ is______.

2. $\lim_{x\to\infty} f(x)$, where $\frac{2x-3}{x} < f(x) < \frac{2x^2+5x}{x^2}\ \forall x>0$, is ______.

3. If $f(x) = \begin{cases} x-1, & x\ge 1 \\ 2x^2-2, & x<1 \end{cases}$, $g(x) = \begin{cases} x+1, & x>0 \\ -x^2+1, & x\le 0 \end{cases}$, and $h(x)=|x|$, then $\lim_{x\to 0} f(g(h(x)))$ is ______.

4. If $\lim_{x\to\infty} f(x)$ exists and is finite and nonzero and if $\lim_{x\to\infty}\left\{f(x)+\frac{3f(x)-1}{f^2(x)}\right\} = 3$, then the value of $\lim_{x\to\infty} f(x)$ is ______.

5. If $L = \lim_{x\to 2}\frac{(10-x)^{1/3}-2}{x-2}$, then the value of $|1/(4L)|$ is______.

6. If $\lim_{x\to 0}\frac{p\sin 2x+(1-\cos 2x)}{x+\tan x} = 1$, then the value of p is ______.

7. The value of $\lim_{x\to 1}\left(\frac{100}{1-x^{100}}-\frac{50}{1-x^{50}}\right)$ is ______.

8. If $L = \lim_{x\to 2}\frac{\sqrt[3]{60+x^2}-4}{\sin(x-2)}$, then the value of $1/L$ is ______.

9. The value of $\lim_{x\to\infty}\left(\frac{20^x-1}{19(5^x)}\right)^{1/x}$ is ______.

10. The value of $\lim_{n\to\infty}\left[\sqrt[3]{(n+1)^2}-\sqrt[3]{(n-1)^2}\right]$ is ______.

11. If $L = \lim_{n\to\infty}(2\times 3^2\times 2^3\times 3^4\cdots\times 2^{n-1}\times 3^n)^{\frac{1}{(n^2+1)}}$, then the value of L^4 is ______.

12. The value of $\lim_{x\to\infty}\frac{\log_e(\log_e x)}{e^{\sqrt{x}}}$ is______.

13. If $\lim_{x\to 0}\frac{1-\sqrt{\cos 2x}\cdot\sqrt[3]{\cos 3x}\cdot\sqrt[4]{\cos 4x}\cdots\sqrt[n]{\cos nx}}{x^2}$ has the value equal to 10, then the value of n equals ______.

14. The value of $\lim_{x\to\infty}\left(x-x^2\log_e\left(1+\frac{1}{x}\right)\right)$ is ______.

15. Let $S_n = 1+2+3+\cdots+n$ and $P_n = \frac{S_2}{S_2-1}\cdot\frac{S_3}{S_3-1}\cdot\frac{S_4}{S_4-1}\cdots\frac{S_n}{S_n-1}$, where $n\in N, (n\ge 2)$.

Then $\lim_{n\to\infty} P_n =$ ______.

16. If $\lim_{x\to 1}\frac{a\sin(x-1)+b\cos(x-1)+4}{x^2-1} = -2$, then $|a+b|$ is______.

17. Let $\lim_{x\to 1}\frac{x^a-ax+a-1}{(x-1)^2} = f(a)$. Then the value of $f(4)$ is______.

18. Number of integral values of k for which $\lim_{x\to 1}\sin^{-1}\left(\frac{k}{\log_e x}-\frac{k}{x-1}\right)$ exists is ______.

19. If $\lim_{x\to 1}(1+ax+bx^2)^{\frac{c}{(x-1)}} = e^3$, then the value of bc is______.

20. Let $f''(x)$ be continuous at $x = 0$.

If $\lim_{x\to 0}\frac{2f(x)-3a\,f(2x)+bf(8x)}{\sin^2 x}$ exists and $f(0)\ne 0$, $f'(0)\ne 0$, then the value of $3a/b$ is______.

21. If $L = \lim_{x\to 0}\frac{e^{-x^2/2}-\cos x}{x^3\sin x}$, then the value of $1/(3L)$ is______.

22. The integer n for which $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite nonzero number is______.

23. If $\lim_{x\to 0}\left[1+x+\frac{f(x)}{x}\right]^{1/x} = e^3$, then the value of $\ln\left(\lim_{x\to 0}\left[1+\frac{f(x)}{x}\right]^{1/x}\right)$ is ______.

Archives

JEE Main

Single Correct Answer Type

1. Let $f: R \to R$ be a positive, increasing function with $\lim_{x\to\infty} \frac{f(3x)}{f(x)} = 1$. Then $\lim_{x\to\infty} \frac{f(2x)}{f(x)}$ is equal to

(1) 3 (2) 1 (3) $\frac{2}{3}$ (4) $\frac{3}{2}$

(AIEEE 2010)

2. $\lim_{x\to 2}\left(\frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$

(1) Equals $\frac{1}{\sqrt{2}}$ (2) Does not exist

(3) Equals $\sqrt{2}$ (4) Equals $-\sqrt{2}$

(AIEEE 2011)

3. $\lim_{x\to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$ is equal to

(1) $-\frac{1}{4}$ (2) $\frac{1}{2}$ (3) 1 (4) 2

(JEE Main 2013)

4. $\lim_{x\to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to

(1) $\frac{\pi}{2}$ (2) 1 (3) $-\pi$ (4) π

(JEE Main 2014)

5. $\lim_{x\to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$ is equal to

(1) 4 (2) 3 (3) 2 (4) 1/2

(JEE Main 2015)

6. Let $p = \lim_{x\to 0^+} (1+\tan^2\sqrt{x})^{\frac{1}{2x}}$. Then $\log_e p$ is equal to

(1) 1 (2) $\frac{1}{2}$ (3) $\frac{1}{4}$ (4) 2

(JEE Main 2016)

7. $\lim_{x\to \frac{\pi}{2}} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals

(1) $\frac{1}{4}$ (2) $\frac{1}{24}$ (3) $\frac{1}{16}$ (4) $\frac{1}{8}$

(JEE Main 2017)

8. For each $t \in R$, let $[t]$ be the greatest integer less than or equal to t. Then $\lim_{x\to 0^+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\ldots+\left[\frac{15}{x}\right]\right)$

(1) does not exist (in R) (2) is equal to 0

(3) is equal to 15 (4) is equal to 120

(JEE Main 2018)

JEE Advanced

Single Correct Answer Type

1. If $\lim_{x\to 0}[1+x\ln(1+b^2)]^{1/x} = 2b\sin^2\theta, b>0$, and $\theta \in (-\pi, \pi]$, then the value of θ is

(1) $\pm\frac{\pi}{4}$ (2) $\pm\frac{\pi}{3}$ (3) $\pm\frac{\pi}{6}$ (4) $\pm\frac{\pi}{2}$

(IIT-JEE, 2011)

2. If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$, then

(1) $a = 1, b = 4$ (2) $a = 1, b = -4$

(3) $a = 2, b = -3$ (4) $a = 2, b = 3$

(IIT-JEE, 2012)

Multiple Correct Answers Type

1. Let $L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$, $a > 0$. If L is finite, then

(1) $a = 2$ (2) $a = 1$ (3) $L = \frac{1}{64}$ (4) $L = \frac{1}{32}$

(IIT-JEE, 2009)

2. Let $f(x) = \frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$ for $x \neq 1$. then

(1) $\lim_{x\to 1^+} f(x) = 0$

(2) $\lim_{x\to 1^-} f(x)$ does not exist

(3) $\lim_{x\to 1^-} f(x) = 0$

(4) $\lim_{x\to 1^+} f(x)$ does not exist

(JEE Advanced 2017)

Numerical Value Type

1. The largest value of the non-negative integer a for which

$$\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4} \text{ is______.}$$

(JEE Advanced 2014)

2. Let m and n be two positive integers greater than 1. If $\lim_{\alpha\to 0}\frac{e^{\cos(\alpha^n)} - e}{\alpha^m} = -\frac{e}{2}$, then the value of $\frac{m}{n}$ is ______.

(JEE Advanced 2015)

3. Let $\alpha, \beta \in R$ be such that $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x - \sin x} = 1$. Then $6(\alpha+\beta)$ equals ______.

(JEE Advanced 2016)

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	2. (2)	3. (2)	4. (3)	5. (3)
6. (1)	7. (2)	8. (3)	9. (2)	10. (3)
11. (4)	12. (1)	13. (2)	14. (1)	15. (3)
16. (2)	17. (1)	18. (1)	19. (4)	20. (3)
21. (3)	22. (2)	23. (4)	24. (4)	25. (3)
26. (1)	27. (3)	28. (1)	29. (2)	30. (3)
31. (3)	32. (3)	33. (1)	34. (3)	35. (1)
36. (2)	37. (4)	38. (1)	39. (1)	40. (2)
41. (2)	42. (2)	43. (2)	44. (3)	45. (2)
46. (4)	47. (1)	48. (4)	49. (2)	50. (1)
51. (4)	52. (2)	53. (1)	54. (2)	55. (2)
56. (4)	57. (3)	58. (3)	59. (3)	60. (2)
61. (3)	62. (2)	63. (2)	64. (1)	65. (2)
66. (3)	67. (3)	68. (1)	69. (2)	70. (3)
71. (3)	72. (2)	73. (3)	74. (1)	75. (2)
76. (3)				

Multiple Correct Answers Type

1. (2), (3)	2. (1), (4)
3. (1), (3)	4. (1), (2), (3)
5. (1), (2)	6. (1), (2), (3)
7. (1), (2), (3)	8. (1), (2), (3)
9. (2), (3)	10. (1), (2), (3)
11. (2), (3), (4)	12. (1), (3)
13. (1), (2), (3), (4)	14. (1), (2), (3), (4)
15. (2), (3)	16. (1), (2), (3)
17. (1), (3)	18. (1), (2), (3)

Linked Comprehension Type

1. (1)	2. (2)	3. (4)	4. (3)	5. (4)
6. (3)	7. (2)	8. (4)	9. (3)	10. (3)
11. (3)	12. (4)	13. (3)	14. (2)	15. (1)
16. (4)	17. (4)	18. (1)	19. (3)	20. (1)

Matrix Match Type

1. a → s; b → r; c → p; d → q
2. a → q; b → r; c → q; d → p
3. a → q; b → p, q, r; c → r, s; d → r, s
4. (4) 5. (2)

Numerical Value Type

1. (2)	2. (2)	3. (0)	4. (1)	5. (3)
6. (1)	7. (25)	8. (12)	9. (4)	10. (0)
11. (6)	12. (0)	13. (6)	14. (0.5)	15. (3)
16. (8)	17. (6)	18. (5)	19. (3)	20. (7)
21. (4)	22. (3)	23. (2)		

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2)	2. (2)	3. (4)	4. (4)	5. (3)
6. (2)	7. (3)	8. (4)		

JEE Advanced

Single Correct Answer Type

1. (4) 2. (2)

Multiple Correct Answers Type

1. (1), (3) 2. (3), (4)

Numerical Value Type

1. (2) 2. (2) 3. (7)

3 Differentiation

DIFFERENTIATION: DEFINITION

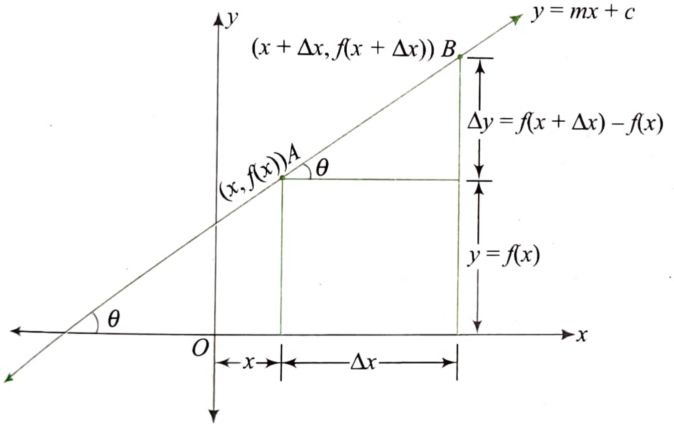
DIFFERENTIATION AS SLOPE OF A TANGENT TO A CURVE

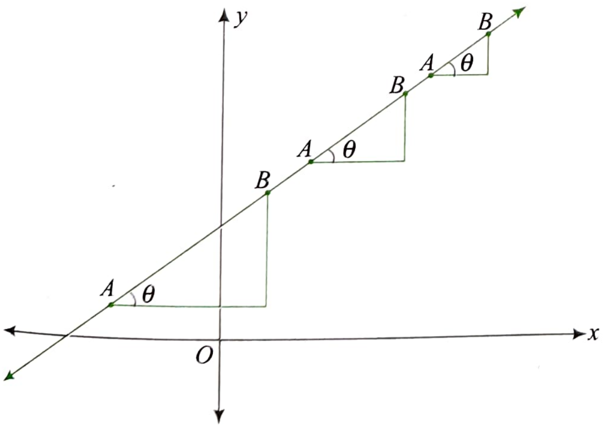
In calculus, the derivative is a measure of how a function changes as its input changes. In general, a derivative can be thought of as how much one quantity is changing in response to changes in some other quantity when these two quantities are related by some relation.

To understand the concept of differentiation, let us start with the simplest case, when y is linear function of x, i.e.,

$y = f(x) = mx + c$, where m is slope of the line.

Consider two points $A(x, f(x))$ and $B(x + \Delta x, f(x + \Delta x))$ on the line, where $f(x + \Delta x) = y + \Delta y$ and $f(x) = y$.





Now, $f(x + \Delta x) - f(x) = m(x + \Delta x) + c - (mx + c) = m(\Delta x)$

$\therefore \quad y + \Delta y - y = m(\Delta x)$ or $\Delta y = m\Delta x$ or $\dfrac{\Delta y}{\Delta x} = m = \tan\theta$.

Thus, change in y (Δy) w.r.t. change is x (Δx) is constant m, which is slope the line.

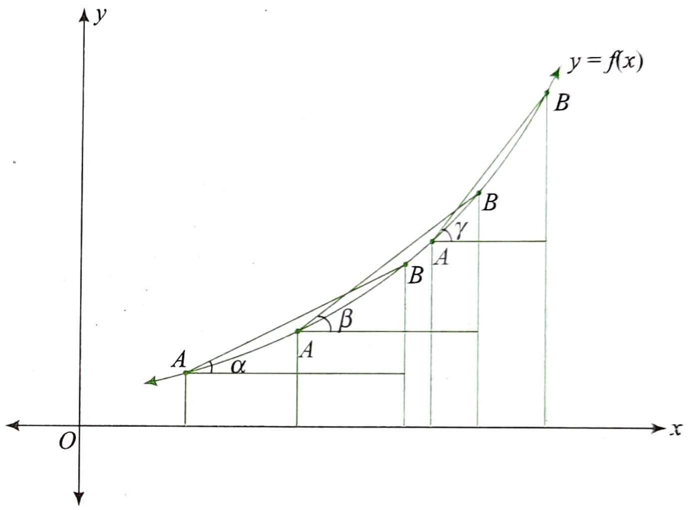
So, for any two points A and B on the line

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = m.$$

If $\Delta x \to 0$, points A and B are very close, but ratio $\dfrac{\Delta y}{\Delta x}$ is constant 'm'.

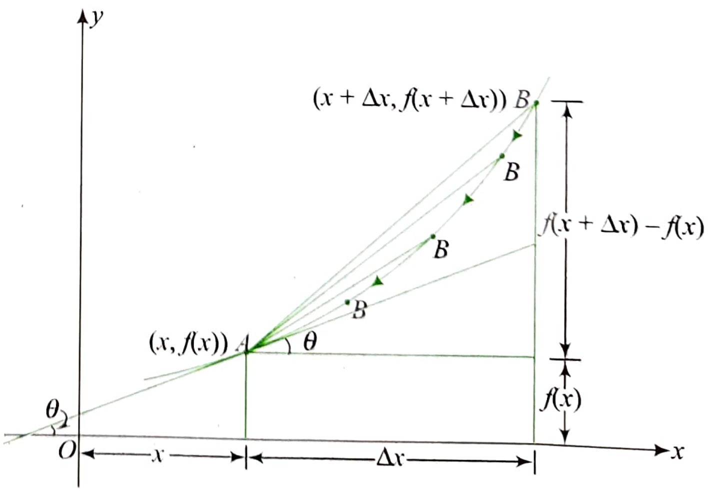
We can say that slope of the line is same for any segment of the line selected.

Now consider the function $y = f(x)$, the graph of the which is not a straight line.



Line segments joining different pair of points A and B on the graph has different slopes.

Consider one such line segment AB having end points $A(x, f(x))$ and $B(x + \Delta x, f(x + \Delta x))$ on the curve.



Suppose we want to find the slope of the curve at point A.

Slope of the curve at point A is same as the slope of the tangent to the curve at point A. Now we consider tangent point A as chord AB when point B is very close to A, so that length of chord AB is infinitely small.

For that we drag point B on the curve towards A.

That means we are lowering the value of Δx.

By making Δx infinitely small, i.e., when $\Delta x \to 0$, chord AB behaves as tangent to curve at point A.

Now for any position of point B on the curve, slope of chord AB is $\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\Delta f(x)}{\Delta x}$ for varying value of Δx. Here $\Delta f(x)$ is change in value of $f(x)$ when change is x is Δx.

For the situation when point B is very close to A, we have slope of chord AB equals $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{df(x)}{dx} = \tan\theta$.

Here, $df(x)$ means very small change is value of $f(x)$ produced by very small change dx in the value of x, and θ is the angle of tangent at point A to the curve with positive x-axis.

The ratio $\frac{df(x)}{dx}$ is called derivative of $f(x)$ w.r.t. x, which is same as slope of tangent to the curve at point A.

Ratio $\frac{df(x)}{dx}\left(=\frac{d}{dx}(f(x))\right)$ is also called differential coefficient.

Derivative of $f(x)$ w.r.t x is also written as $f'(x)$ and read as f-prime of x.

Conventionally, Δx is denoted as h and we write

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Also, if we let $x + h = z$, then

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{z\to x}\frac{f(z)-f(x)}{z-x}.$$

If above graph of the function is considered as the graph of the displacement(s) vs. time(t), then it represents function $s = f(t)$. Slope of the chord joining points A and B is the average velocity $\frac{\Delta s}{\Delta t}$ over the length of the time interval Δt. The velocity at point A is called the instantaneous velocity at point A, which is given by $v(t) = \frac{ds}{dt} = \lim\limits_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}$.

Similarly, if Δv is the change in the velocity of the particle as it moves from point A to B during the time interval Δt, then average acceleration in that duration is $\frac{\Delta v}{\Delta t}$ and instantaneous acceleration at point A is $\frac{dv}{dt} = \lim\limits_{\Delta t\to 0}\frac{\Delta v}{\Delta t}$.

While finding the derivative of the function $f(x)$ or slope of tangent to curve at $x = a$, we are assuming that function is differentiable at $x = a$, i.e., it is possible to draw at unique tangent at $x = a$.

Using this definition of derivative, we can find the derivative of some elementary functions.

(i) $\frac{d(c)}{dx} = 0$, where c is constant

As derivative is change in value of y with respect to x. But in constant function if input x is changed, the value of y does not change. Hence derivative is zero.

(ii) $\frac{d(\sin x)}{dx} = \cos x$

Proof:

$$\frac{d(\sin x)}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{2\sin\frac{h}{2}\cos\left(\frac{2x+h}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right) = \cos x$$

(iii) $\frac{d(\cos x)}{dx} = -\sin x$

(iv) $\frac{d(\tan x)}{dx} = \sec^2 x$

Proof:

$$\frac{d(\tan x)}{dx} = \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos x\cos(x+h)}$$

$$= \lim_{h\to 0}\frac{\sin h}{h}\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}$$

$$= 1\times\frac{1}{\cos x\cdot\cos x} = \sec^2 x$$

(v) $\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

(vi) $\frac{d(\sec x)}{dx} = \sec x\tan x$

(vii) $\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x\cot x$

(viii) $\frac{d(x^n)}{dx} = nx^{n-1}, n \in Q$

Proof:

$$\frac{d(x^n)}{dx} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= x^n\lim_{h\to 0}\frac{\left(1+\frac{h}{x}\right)^n - 1}{h}$$

$$= x^{n-1}\lim_{h\to 0}\frac{\left(1+\frac{h}{x}\right)^n - 1}{\frac{h}{x}} = nx^{n-1}$$

(ix) $\frac{d(a^x)}{dx} = a^x \log_e a,\ a > 0,\ a \neq 1$

Proof:

$$\frac{d(a^x)}{dx} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

$$= a^x \log_e a$$

(x) $\frac{d(e^x)}{dx} = e^x$

(xi) $\frac{d(\log_e |x|)}{dx} = \frac{1}{x},\ x \neq 0$

Proof:

If $x > 0$,

$$\frac{d(\log_e x)}{dx} = \lim_{h\to 0} \frac{\log_e (x+h) - \log_e x}{h}$$

$$= \lim_{h\to 0} \frac{\log_e\left(1+\frac{h}{x}\right)}{\frac{h}{x}\cdot x} = \frac{1}{x}$$

Also it can be prove that $\frac{d(\log_e(-x))}{dx} = \frac{1}{x},\ x < 0$

$$\therefore \quad \frac{d(\log_e |x|)}{dx} = \frac{1}{x},\ x \neq 0$$

(xii) $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

Proof:

Let $\sin^{-1} x = y$, where $y \in [-\pi/2, \pi/2]$

$\therefore \quad x = \sin y$

$\Rightarrow \quad \frac{dx}{dy} = \cos y = \sqrt{1-x^2}$ $\quad (\because \cos y \geq 0$ for $y \in [-\pi/2, \pi/2])$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

(xiii) $\frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$

(xiv) $\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$

Proof:

Let $\tan^{-1} x = y,\ y \in (0, \pi)$

$\therefore \quad x = \tan y$

$\Rightarrow \quad \frac{dx}{dy} = \sec^2 y$

$= 1 + \tan^2 y$

$= 1 + x^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+x^2}$

(xv) $\frac{d(\cot^{-1} x)}{dx} = -\frac{1}{1+x^2}$

(xvi) $\frac{d(\sec^{-1} x)}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

Proof:

Let $\sec^{-1} x = y$, where $y \in [0, \pi] - \{\pi/2\}$

$\therefore \quad x = \sec y$

$\Rightarrow \quad \frac{dx}{dy} = \sec y \tan y$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y}$ ...(1)

Now when $x > 1$ or $y \in (0, \pi/2)$, $\tan y > 0$

$\therefore \quad \tan y = \sqrt{x^2-1}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}},\ x > 1$ ...(2)

When $x < -1$ or $y \in (\pi/2, \pi)$, $\tan y < 0$

$\therefore \quad \tan y = -\sqrt{x^2-1}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{-x\sqrt{x^2-1}},\ x < -1$ ...(3)

From (2) and (3), $\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

(xvii) $\frac{d(\operatorname{cosec}^{-1} x)}{dx} = -\frac{1}{|x|\sqrt{x^2-1}}$

ILLUSTRATION 3.1

If $y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}},\ x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$, then find $\frac{dy}{dx}$.

Sol. We have

$$y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\frac{2\sin^2 x}{2\cos^2 x}} = \sqrt{\tan^2 x}$$

$$= |\tan x|, \text{ where } x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$= \begin{cases} \tan x, & x \in \left(0, \frac{\pi}{2}\right) \\ -\tan x, & x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

$$\therefore \quad \frac{dy}{dx} = \begin{cases} \sec^2 x, & x \in \left(0, \frac{\pi}{2}\right) \\ -\sec^2 x, & x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

ILLUSTRATION 3.2

Find the derivative of $e^{\sqrt{x}}$ w.r.t. x using the first principle.

Sol. Let $f(x) = e^{\sqrt{x}}$. Then $f(x+h) = e^{\sqrt{x+h}}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{e^{\sqrt{x+h}}-e^{\sqrt{x}}}{h}$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{h}\right)$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{\sqrt{x+h}-\sqrt{x}}\right)\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{\sqrt{x+h}-\sqrt{x}}\right) \times \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{y}-1}{y}\right)\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})},$$

where $y = \sqrt{x+h}-\sqrt{x}$ $\quad(\because$ when $h\to 0, y\to 0)$

$$\therefore \quad \frac{d}{dx}(f(x)) = e^{\sqrt{x}}\times 1\times\left(\frac{1}{\sqrt{x}+\sqrt{x}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

ILLUSTRATION 3.3

If $f(x) = x\tan^{-1}x$, find $f'(\sqrt{3})$ using the first principle.

Sol. We have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\therefore \quad f'(\sqrt{3}) = \lim_{h\to 0}\frac{f(\sqrt{3}+h)-f(\sqrt{3})}{h}$$

$$= \lim_{h\to 0}\frac{(\sqrt{3}+h)\tan^{-1}(\sqrt{3}+h)-\sqrt{3}\tan^{-1}\sqrt{3}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3}\left[\tan^{-1}(\sqrt{3}+h)-\tan^{-1}\sqrt{3}\right]+h\tan^{-1}(\sqrt{3}+h)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3}}{h}\tan^{-1}\left(\frac{\sqrt{3}+h-\sqrt{3}}{1+\sqrt{3}(\sqrt{3}+h)}\right)+\lim_{h\to 0}\tan^{-1}(\sqrt{3}+h)$$

$$= \sqrt{3}\lim_{h\to 0}\left\{\frac{\tan^{-1}\left(\dfrac{h}{4+\sqrt{3}h}\right)}{\dfrac{h}{4+\sqrt{3}h}}\right\}\frac{1}{4+\sqrt{3}h}+\lim_{h\to 0}\tan^{-1}(\sqrt{3}+h)$$

$$= \sqrt{3}\times 1\times\frac{1}{4}+\tan^{-1}\sqrt{3} = \frac{\sqrt{3}}{4}+\tan^{-1}\sqrt{3}$$

ILLUSTRATION 3.4

If $f(x) = [2x]\sin 3\pi x$ then prove that $f'(k^+) = 6k\pi(-1)^k$, (where [.] denotes the greatest integer function and $k \in N$).

Sol. $$f'(k^+) = \lim_{h\to 0}\frac{f(k+h)-f(k)}{h}$$

$$= \lim_{h\to 0}\frac{[2k+2h]\sin(3\pi(k+h))-[2k]\sin(3k\pi)}{h}$$

$$= \lim_{h\to 0}\frac{[2k]\sin(3k\pi+3\pi h)}{h} \qquad (\text{as } \sin(3k\pi)=0)$$

$$= 2k\times\lim_{h\to 0}\frac{(-1)^k\sin(3\pi h)}{h} \qquad (\text{as } 2k \text{ is integer})$$

$$= 2k\times(-1)^k\times(3\pi)\lim_{h\to 0}\frac{\sin(3\pi h)}{(3\pi h)}$$

$$= 6k\pi(-1)^k$$

ILLUSTRATION 3.5

Let $f: R\to R$ satisfying $|f(x)| \le x^2$, $\forall\, x\in R$, differentiable at $x=0$ then find $f'(0)$

Sol. Since, $|f(x)|\le x^2$, $\forall\, x\in R$ ...(1)

At $x=0$, $|f(0)|\le 0$

$\Rightarrow f(0)=0$ ...(2)

Now $f'(0) = \lim_{h\to 0}\dfrac{f(h)-f(0)}{h} = \lim_{h\to 0}\dfrac{f(h)}{h}$ ...(3)

From (1), $|f(h)|\le |h|^2$

$$\Rightarrow \frac{|f(h)|}{|h|}\le |h|$$

$$\Rightarrow \lim_{h\to 0}\left|\frac{f(h)}{h}\right| \le \lim_{h\to 0}|h|$$

$$\Rightarrow \left|\lim_{h\to 0}\frac{f(h)-0}{h}\right|\le 0$$

$$\Rightarrow |f'(0)|\le 0$$

$$\Rightarrow f'(0)=0$$

ILLUSTRATION 3.6

A function $f: R\to R$ satisfies the equation $f(x+y)=f(x)f(y)$ for all $x, y\in R$ and $f(x)\ne 0$ for all $x\in R$. If $f(x)$ is differentiable at $x=0$ and $f'(0)=2$, then prove that $f'(x)=2f(x)$.

Sol. We have $f(x+y)=f(x)f(y)$ for all $x, y\in R$

$\therefore \quad f(0)=f(0)f(0)$ or $f(0)\{f(0)-1\}=0$

or $\quad f(0)=1$ $\qquad [\because f(0)\ne 0]$

Now, $f'(0)=2$

or $$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=2$$

or $$\lim_{h\to 0}\frac{f(h)-1}{h}=2 \quad (\because f(0)=1) \qquad (1)$$

Now, $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)f(h)-f(x)}{h} \quad [\because f(x+y)=f(x)f(y)]$$

$$= f(x)\left(\lim_{h\to 0}\frac{f(h)-1}{h}\right) = 2f(x) \quad \text{[Using (1)]}$$

CONCEPT APPLICATION EXERCISE 3.1

1. Using the definition of derivative find the derivative of
   (a) $\sqrt{\sin x}$ (b) $\cos^2 x$ (c) $\sin^{-1}x$
2. Find the derivative of $\sqrt{4-x}$ w.r.t. x using the first principle.
3. Let $f(x) = \dfrac{(2^x+2^{-x})\sin x\sqrt{\tan^{-1}(x^2-x+1)}}{(7x^2+3x+1)^3}$. Then find the value of $f'(0)$.
4. Let $f: R\to R$ be a real valued function $\forall\, x, y \in R$ such that $|f(x)-f(y)| \le |x-y|^3$, then prove that $f(x)$ is constant function.

ANSWERS

1. (a) $\dfrac{\cos x}{2\sqrt{\sin x}}$ (b) $-\sin 2x$ (c) $\dfrac{1}{\sqrt{1-x^2}}$
2. $\dfrac{-1}{2\sqrt{4-x}}$ 3. $\sqrt{\pi}$

RULES OF DIFFERENTIATION

There are many rules of differentiation. Let us start with elementary rules.

(i) $\dfrac{d}{dx}(kf(x)) = k\dfrac{d}{dx}f(x)$, where k is any constant

(ii) $\dfrac{d}{dx}[f_1(x)\pm f_2(x)\pm f_3(x)....\pm f_n(x)]$

$$= \frac{d}{dx}f_1(x)\pm\frac{d}{dx}f_2(x)\pm\frac{d}{dx}f_3(x)...\pm\frac{d}{dx}f_n(x)$$

i.e., derivative is distributive over addition and subtraction.

ILLUSTRATION 3.7

If $y=(1+x^{1/4})(1+x^{1/2})(1-x^{1/4})$, then find $\dfrac{dy}{dx}$.

Sol. $y=(1+x^{1/4})(1+x^{1/2})(1-x^{1/4})$

$= (1+x^{1/4})(1-x^{1/4})(1+x^{1/2})$

$= (1-x^{1/2})(1+x^{1/2})$

$= 1-x$

$\therefore \dfrac{dy}{dx} = -1$

ILLUSTRATION 3.8

If $f(x) = x|x|, x\ne 0$ then prove that $f'(x) = 2|x|$

Sol. $f(x) = x|x|, x\ne 0$

$$\Rightarrow f(x) = \begin{cases} -x^2, & x<0 \\ x^2, & x>0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -2x, & x<0 \\ 2x, & x>0 \end{cases}$$

$\therefore f'(x) = 2|x|, x\ne 0.$

ILLUSTRATION 3.9

If $y = 1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots+\dfrac{x^n}{n!}$, then show that $\dfrac{dy}{dx}-y+\dfrac{x^n}{n!}=0$.

Sol. $\dfrac{dy}{dx} = 0+\dfrac{1}{1!}+\dfrac{1}{2!}(2x)+\dfrac{1}{3!}(3x^2)+\cdots+\dfrac{1}{n!}(nx^{n-1})$

$= 1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\cdots+\dfrac{x^{n-1}}{(n-1)!}$

$= \left\{1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\cdots+\dfrac{x^{n-1}}{(n-1)!}+\dfrac{x^n}{n!}\right\}-\dfrac{x^n}{n!}$

$= y-\dfrac{x^n}{n!}$

or $\dfrac{dy}{dx}-y+\dfrac{x^n}{n!}=0$

ILLUSTRATION 3.10

Find $\dfrac{dy}{dx}$ for $y=\sin^{-1}(\cos x), x\in(0,\pi)\cup(\pi, 2\pi)$.

Sol. We have,

$$\sin^{-1}(\cos x) = \frac{\pi}{2}-\cos^{-1}(\cos x)$$

$$= \begin{cases} \dfrac{\pi}{2}-x, & \text{if } 0<x<\pi \\ \dfrac{\pi}{2}-(2\pi-x), & \text{if } \pi<x<2\pi \end{cases}$$

$$= \begin{cases} \dfrac{\pi}{2}-x, & \text{if } 0<x<\pi \\ x-\dfrac{3\pi}{2}, & \text{if } \pi<x<2\pi \end{cases}$$

$$\therefore \frac{d}{dx}\{\sin^{-1}(\cos x)\} = \begin{cases} -1, & \text{if } 0<x<\pi \\ 1, & \text{if } \pi<x<2\pi \end{cases}$$

ILLUSTRATION 3.11

Differentiate $\sin^{-1}(2x\sqrt{1-x^2})$ with respect to x, if

(i) $-\dfrac{1}{\sqrt{2}}<x<\dfrac{1}{\sqrt{2}}$ (ii) $\dfrac{1}{\sqrt{2}}<x<1$ (iii) $-1<x<-\dfrac{1}{\sqrt{2}}$.

Sol. We have $y=\sin^{-1}(2x\sqrt{1-x^2})$

Let $x=\sin\theta$, where $\theta=\sin^{-1}x$, and $\theta\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$

$\therefore y=\sin^{-1}(2\sin\theta\cos\theta)=\sin^{-1}(\sin 2\theta)$

(i) $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ or $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ or $-\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$

$\therefore \quad y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1} x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

(ii) $\frac{1}{\sqrt{2}} < x < 1$ or $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < 2\theta < \pi$

$\therefore \quad y = \sin^{-1}(\sin 2\theta) = \sin^{-1}(\sin(\pi - 2\theta)) = \pi - 2\theta$

$\therefore \quad y = \pi - 2\sin^{-1} x$

$\Rightarrow \quad \frac{dy}{dx} = 0 - \frac{2}{\sqrt{1-x^2}} = -\frac{2}{\sqrt{1-x^2}}$

(iii) $-1 < x < -\frac{1}{\sqrt{2}}$ or $-\frac{\pi}{2} < \theta < -\frac{\pi}{4}$ or $-\pi < 2\theta < -\frac{\pi}{2}$

$\therefore \quad y = \sin^{-1}(\sin 2\theta) = \sin^{-1}(-\sin(\pi + 2\theta))$

$= \sin^{-1}(\sin(-\pi - 2\theta)) = -\pi - 2\theta$

$\therefore \quad y = -\pi - 2\sin^{-1} x$

$\Rightarrow \quad \frac{dy}{dx} = 0 - \frac{2}{\sqrt{1-x^2}} = -\frac{2}{\sqrt{1-x^2}}$

ILLUSTRATION 3.12

If $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1$, then find $\frac{dy}{dx}$.

Sol. Let $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

$= \tan^{-1}\left(\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right)$

$= \tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}(\tan x)$

$= \tan^{-1}\left(\frac{a}{b}\right) - x \qquad \left[\because -\frac{\pi}{2} < x < \frac{\pi}{2}\right]$

$\therefore \quad \frac{dy}{dx} = 0 - 1 = -1$

ILLUSTRATION 3.13

$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, where $0 < x < \infty$

Sol. From triangular conversions,

$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \tan^{-1} x + \tan^{-1} x = 2\tan^{-1} x$

$\therefore \quad \frac{dy}{dx} = \frac{2}{1+x^2}$

DERIVATIVE OF COMPOSITE FUNCTION (CHAIN RULE)

The Chain rule is used for differentiation of composite function of two or more functions.

The rule is as follows:

(i) If f and g are two real valued functions, then

$(f(g(x)))' = f'(g(x)) \times g'(x)$

If a variable z depends on the variable y, which itself depends on the variable x, so that y and z are therefore dependent variables, then z, via the intermediate variable of y, depends on x as well. The chain rule then states,

$$\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx}$$

Let $f(g(x)) = z$ and $g(x) = y$.

So, we have $z = f(y)$

$\therefore \quad (f(g(x)))' = \frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx}$

$= f'(y)g'(x) = f'(g(x)) \times g'(x)$

(ii) If f, g and h are real valued functions, then

$(f(g(h(x))))' = f'(g(h(x))) \times g'(h(x)) \times h'(x)$

Here we let $f(g(h(x))) = z$, $g(h(x)) = y$ and $h(x) = t$.

So, we have $z = f(y)$ and $y = g(t)$

$\therefore \quad \frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dt}\cdot\frac{dt}{dx} = f'(y)g'(t)h'(x)$

$= f'(g(h(x))) \times g'(h(x)) \times h'(x)$

Similarly, we extend this rule for derivative of composite functions of more functions.

We can find the derivative using direct differentiation method also.

e.g. $\frac{d(\sin e^x)}{dx} = \cos(e^x)\cdot\frac{d(e^x)}{dx} = (\cos(e^x))e^x$

$\frac{d(\sqrt{\sin x})}{dx} = \frac{d\left((\sin x)^{\frac{1}{2}}\right)}{dx}$

$= \frac{1}{2}(\sin x)^{\frac{1}{2}-1}\cdot\frac{d(\sin x)}{dx}$

$= \frac{1}{2\sqrt{\sin x}}\cdot\cos x$ etc.

ILLUSTRATION 3.14

Find $\frac{dy}{dx}$ for $y = \sin(x^2 + 1)$

Sol. Let $y = \sin(x^2 + 1)$.

Putting $u = x^2 + 1$, we get $y = \sin u$

$\therefore \quad \frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$

Now, $\frac{du}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$= (\cos u)(2x) = 2x\cos(x^2 + 1)$

ILLUSTRATION 3.15

If $y = \sqrt{\log\left\{\sin\left(\frac{x^2}{3} - 1\right)\right\}}$, then find $\frac{dy}{dx}$.

Sol. $y = \sqrt{\log\left\{\sin\left(\frac{x^2}{3} - 1\right)\right\}}$

Putting $\frac{x^2}{3} - 1 = v$, we get $\sin\left(\frac{x^2}{3} - 1\right) = \sin v = u$.

Putting $\log\left\{\sin\left(\frac{x^2}{3} - 1\right)\right\} = \log u = z$, we get $y = \sqrt{z}$, $z = \log u$,

$u = \sin v$, and $v = \frac{x^2}{3} - 1$. Therefore,

$$\frac{dy}{dz} = \frac{1}{2\sqrt{z}}, \frac{dz}{du} = \frac{1}{u}, \frac{du}{dv} = \cos v, \text{ and } \frac{dv}{dx} = \frac{2x}{3}$$

Now, $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$

$$= \left(\frac{1}{2\sqrt{z}}\right)\left(\frac{1}{u}\right)(\cos v)\left(\frac{2x}{3}\right)$$

$$= \frac{x}{3} \cdot \frac{\cos v}{u\sqrt{\log u}}$$

$$= \frac{x \cot\left(\frac{x^2}{3} - 1\right)}{3\sqrt{\log\left\{\sin\left(\frac{x^2}{3} - 1\right)\right\}}}$$

ILLUSTRATION 3.16

Differentiate the function $f(x) = \sec(\tan(\sqrt{x}))$ with respect to x.

Sol. $f(x) = \sec(\tan(\sqrt{x}))$

$\therefore \quad f'(x) = \frac{d}{dx}[\sec(\tan\sqrt{x})]$

$$= \sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x}) \cdot \frac{d}{dx}(\tan\sqrt{x})$$

$$= \sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x})\sec^2(\sqrt{x})}{2\sqrt{x}}$$

ILLUSTRATION 3.17

Find $\frac{dy}{dx}$ for $y = \log(x + \sqrt{a^2 + x^2})$.

Sol. $y = \log(x + \sqrt{a^2 + x^2})$

Then $\frac{dy}{dx} = \frac{d}{dx}\{\log(x + \sqrt{a^2 + x^2})\}$

$$= \frac{1}{x + \sqrt{a^2 + x^2}} \frac{d}{dx}(x + \sqrt{a^2 + x^2})$$

$$= \frac{1}{x + \sqrt{a^2 + x^2}} \times \left\{1 + \frac{1}{2}(a^2 + x^2)^{-1/2} \frac{d}{dx}(a^2 + x^2)\right\}$$

$$= \frac{1}{x + \sqrt{a^2 + x^2}} \left\{1 + \frac{1}{2\sqrt{a^2 + x^2}} \times 2x\right\}$$

$$= \frac{1}{x + \sqrt{a^2 + x^2}} \times \frac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2}}$$

$$= \frac{1}{\sqrt{a^2 + x^2}}$$

ILLUSTRATION 3.18

$y = \tan^{-1}\left(\frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}\right)$, where $-1 < x < 1, x \neq 0$

Sol. Putting $x^2 = \cos 2\theta$, we get

$$y = \tan^{-1}\left(\frac{\sqrt{1 + \cos 2\theta} + \sqrt{1 - \cos 2\theta}}{\sqrt{1 + \cos 2\theta} - \sqrt{1 - \cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1 + \tan\theta}{1 - \tan\theta}\right)$$

$$= \tan^{-1}(\tan(\pi/4 + \theta))$$

$$= \frac{\pi}{4} + \theta \qquad \left[\begin{array}{l} \because\ 0 < x^2 < 1 \Rightarrow 0 < \cos 2\theta < 1 \\ \text{or } 0 < 2\theta < \pi/2 \\ \text{or } 0 < \theta < \pi/4 \\ \text{or } \pi/4 < \pi/4 + \theta < \pi/2 \end{array}\right]$$

$$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

$$\therefore \quad \frac{dy}{dx} = 0 - \frac{1}{2} \times \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx}(x^2)$$

$$= -\frac{1}{2} \times \frac{2x}{\sqrt{1 - x^4}} = \frac{-x}{\sqrt{1 - x^4}}$$

ILLUSTRATION 3.19

Find $\frac{dy}{dx}$ for $y = \tan^{-1}\sqrt{\frac{a - x}{a + x}}$, $-a < x < a$.

Sol. $y = \tan^{-1}\left\{\sqrt{\frac{a - x}{a + x}}\right\}$, where $-a < x < a$

Substituting $x = a\cos\theta$, we get

$$y = \tan^{-1}\left\{\sqrt{\frac{a - a\cos\theta}{a + a\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\sqrt{\tan^2\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left|\tan\frac{\theta}{2}\right|$$

Also, for $-a < x < a$, $-1 < \cos\theta < 1$

or $\quad \theta \in (0, \pi)$ or $\frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad y = \tan^{-1}\left|\tan\frac{\theta}{2}\right| = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$$= \frac{\theta}{2} = \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)$$

or $\quad \frac{dy}{dx} = -\frac{1}{2} \times \frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\frac{d}{dx}\left(\frac{x}{a}\right) = -\frac{1}{2\sqrt{a^2-x^2}}$

ILLUSTRATION 3.20

If $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$ and $0 < x < 1$, then find $\frac{dy}{dx}$.

Sol. $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$, where $0 < x < 1$

$= \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}]$

$= \sin^{-1}x - \sin^{-1}\sqrt{x}$

$[\because \sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$

ILLUSTRATION 3.21

If $y = \tan^{-1}\frac{1}{1+x+x^2} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \cdots$ upto n terms, then find the value of $y'(0)$.

Sol. $y = \tan^{-1}\frac{1}{1+x+x^2} + \tan^{-1}\frac{1}{x^2+3x+3} + \cdots + n$ terms

$$= \tan^{-1}\frac{(x+1)-x}{1+x(1+x)} + \tan^{-1}\frac{(x+2)-(x+1)}{1+(x+1)(x+2)} + \cdots + n \text{ terms}$$

$$= \tan^{-1}(x+1) - \tan^{-1}x + \tan^{-1}(x+2) - \tan^{-1}(x+1) + \cdots + \tan^{-1}(x+n) - \tan^{-1}(x+(n-1))$$

$$= \tan^{-1}(x+n) - \tan^{-1}x$$

$\therefore \quad y'(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$

or $\quad y'(0) = \frac{1}{1+n^2} - 1 = \frac{-n^2}{1+n^2}$

ILLUSTRATION 3.22

Let $f: R \to R$ be a one-one onto differentiable function, such that $f(2) = 1$ and $f'(2) = 3$. Then find the value of $\left(\frac{d}{dx}(f^{-1}(x))\right)_{x=1}$.

Sol. Let $f^{-1}(x) = g(x)$

$\therefore \quad f(g(x)) = x$

$\therefore \quad f'(g(x))g'(x) = 1$

$\therefore \quad g'(x) = \frac{1}{f'(g(x))}$

$\therefore \quad g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(2)} = \frac{1}{3}$

ILLUSTRATION 3.23

If $f(x) = \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x$, then find $f'\left(\frac{\pi}{4}\right)$.

Sol. $f(x)$

$$= \frac{2\sin x \cdot \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x}{2\sin x}$$

$$= \frac{\sin 32x}{2^5 \sin x}$$

$\therefore \quad f'(x) = \frac{1}{32} \times \frac{32\cos 32x \times \sin x - \cos x \times \sin 32x}{\sin^2 x}$

$\therefore \quad f'\left(\frac{\pi}{4}\right) = \frac{32 \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \times 0}{32 \times \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{2}$

ILLUSTRATION 3.24

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

Sol. Given relation is $\cos y = x\cos(a+y)$. Therefore,

$$x = \frac{\cos y}{\cos(a+y)}$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy}=\frac{d}{dy}\left(\frac{\cos y}{\cos(a+y)}\right)$$

$$=\left(\frac{\cos(a+y)(-\sin y)-\cos y(-\sin(a+y))}{\cos^2(a+y)}\right)$$

$$=\left(\frac{-\cos(a+y)\sin y+\cos y\sin(a+y)}{\cos^2(a+y)}\right)$$

$$=\left(\frac{\sin(a+y-y)}{\cos^2(a+y)}\right)$$

$$=\frac{\sin a}{\cos^2(a+y)}$$

$$\therefore\ \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$

PRODUCT RULE FOR DIFFERENTIATION

This rule is used to find the derivative of product of two or more functions.

The rule for two functions $f(x)$ and $g(x)$ is

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x))+g(x)\frac{d}{dx}(f(x))$$

Proof:

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\right]$$

$$=f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=f(x)\frac{d}{dx}(g(x))+g(x)\frac{d}{dx}(f(x))$$

For three functions $f(x)$, $g(x)$ and $h(x)$ is

$$\frac{d}{dx}(f(x)g(x)h(x))$$

$$=g(x)h(x)\frac{d}{dx}(f(x))+f(x)h(x)\frac{d}{dx}(g(x))+f(x)g(x)\frac{d}{dx}(h(x))$$

In general,

$$\frac{d}{dx}(f_1(x).f_2(x).f_3(x)...f_n(x))$$

$$=\left(\frac{d}{dx}f_1(x)\right)(f_2(x)\cdot f_3(x)\cdot\ldots\cdot f_n(x))$$

$$+\left(\frac{d}{dx}f_2(x)\right)(f_1(x)\cdot f_3(x)\cdot\ldots\cdot f_n(x))$$

$$+\left(\frac{d}{dx}f_3(x)\right)(f_1(x))\cdot f_2(x)\cdot f_4(x)\cdot\ldots\cdot f_n(x))$$

$$\ldots$$

$$\ldots$$

$$+\left(\frac{d}{dx}f_n(x)\right)(f_1(x))\cdot f_2(x)\cdot\ldots\cdot f_{n-1}(x))$$

ILLUSTRATION 3.25

Find $\frac{dy}{dx}$ for $y = x\sin x\log x$.

Sol. We have

$$\frac{d}{dx}(x\sin x\log x)$$

$$=\left\{\frac{d}{dx}(x)\right\}\sin x\log x+x\left\{\frac{d}{dx}(\sin x)\right\}\log x+x\sin x\left\{\frac{d}{dx}(\log x)\right\}$$

$$=1\times\sin x\times\log x+x\times\cos x\times\log x+x\times\sin x\times\frac{1}{x}$$

$$=\sin x\log x+x\cos x\log x+\sin x$$

ILLUSTRATION 3.26

Evaluate $\lim_{h\to 0}\frac{(a+h)^2.\sin^{-1}(a+h)-a^2\sin^{-1}a}{h}$.

Sol. $\lim_{h\to 0}\frac{(a+h)^2.\sin^{-1}(a+h)-a^2\sin^{-1}a}{h}$

$$=\frac{d}{da}(a^2\sin^{-1}a)$$

$$=(\sin^{-1}a)\frac{d}{da}(a^2)+(a^2)\frac{d}{da}(\sin^{-1}a)$$

$$=\frac{a^2}{\sqrt{1-a^2}}+2a\sin^{-1}a$$

QUOTIENT RULE FOR DIFFERENTIATION

This rule is used to find the derivative of division of two functions.

The rule is

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{d(f(x))}{dx}-f(x)\frac{d(g(x))}{dx}}{(g(x))^2}$$

$$=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

In particular case if $f(x)=1$ then $\frac{d}{dx}\left(\frac{1}{g(x)}\right)=\frac{-g'(x)}{(g(x))^2}$

Proof:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}$$

$$=\lim_{h\to 0}\frac{g(x)f(x+h)-f(x)g(x+h)}{h\cdot g(x)g(x+h)}$$

$$= \lim_{h\to 0} \frac{g(x)f(x+h)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{h\cdot g(x)g(x+h)}$$

$$= \frac{1}{(g(x))^2}\left[g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right]$$

$$= \frac{g(x)\dfrac{d(f(x))}{dx} - f(x)\dfrac{d(g(x))}{dx}}{(g(x))^2}$$

ILLUSTRATION 3.27

Differentiate $y = \dfrac{e^x}{1+\sin x}$.

Sol. Using quotient rule, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{1+\sin x}\right)$$

$$= \frac{(1+\sin x)\cdot\dfrac{d}{dx}(e^x) - e^x\cdot\dfrac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)\cdot e^x - e^x\cdot(0+\cos x)}{(1+\sin x)^2} = \frac{e^x(1+\sin x-\cos x)}{(1+\sin x)^2}$$

ILLUSTRATION 3.28

If $f(x) = \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x$, then find $f'\left(\dfrac{\pi}{4}\right)$.

Sol. $$f(x) = \frac{2\sin x\cdot\cos x\cdot\cos 2x\cdot\cos 4x\cdot\cos 8x\cdot\cos 16x}{2\sin x}$$

$$= \frac{\sin 32x}{2^5 \sin x}$$

$$\therefore \quad f'(x) = \frac{1}{32}\times\frac{32\cos 32x\times\sin x - \cos x\times\sin 32x}{\sin^2 x}$$

$$\therefore \quad f'\left(\frac{\pi}{4}\right) = \frac{32\times\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\times 0}{32\times\left(\dfrac{1}{\sqrt{2}}\right)^2} = \sqrt{2}$$

ILLUSTRATION 3.29

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$.

Sol. Given relation is $\cos y = x\cos(a+y)$. Therefore,

$$x = \frac{\cos y}{\cos(a+y)}$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{\cos y}{\cos(a+y)}\right)$$

$$= \left(\frac{\cos(a+y)(-\sin y) - \cos y(-\sin(a+y))}{\cos^2(a+y)}\right)$$

$$= \left(\frac{-\cos(a+y)\sin y + \cos y\sin(a+y)}{\cos^2(a+y)}\right)$$

$$= \left(\frac{\sin(a+y-y)}{\cos^2(a+y)}\right) = \frac{\sin a}{\cos^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

ILLUSTRATION 3.30

If $y = \sqrt{\dfrac{1-x}{1+x}}$, prove that $(1-x^2)\dfrac{dy}{dx} + y = 0$.

Sol. We have

$$y = \sqrt{\frac{1-x}{1+x}}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{(1/2)-1}\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\,\frac{(1+x)\dfrac{d}{dx}(1-x)-(1-x)\dfrac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\,\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2} = -\sqrt{\frac{1+x}{1-x}}\,\frac{1}{(1+x)^2}$$

or $$(1-x^2)\frac{dy}{dx} = -\sqrt{\frac{1+x}{1-x}}\,\frac{1}{(1+x)^2}(1-x^2)$$

or $$(1-x^2)\frac{dy}{dx} = -\sqrt{\frac{1-x}{1+x}}$$

or $$(1-x^2)\frac{dy}{dx} = -y$$

or $$(1-x^2)\frac{dy}{dx} + y = 0$$

ILLUSTRATION 3.31

Find the sum of the series $1 + 2x + 3x^2 + (n-1)x^{n-2}$ using differentiation.

Sol. We know that $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1-x^n}{1-x}$.

Differentiating both sides w.r.t x, we get

$$0 + 1 + 2x + 3x^2 + \ldots + (n-1)x^{n-2}$$

$$= \frac{(1-x)\frac{d}{dx}(1-x^n)-(1-x^n)\frac{d}{dx}(1-x)}{(1-x)^2}$$

or $$1+2x+3x^2+\cdots+(n-1)x^{n-2} = \frac{-(1-x)nx^{n-1}+(1-x^n)}{(1-x)^2}$$

or $$1+2x+3x^2+\cdots+(n-1)x^{n-2} = \frac{-nx^{n-1}+(n-1)x^n+1}{(1-x)^2}$$

CONCEPT APPLICATION EXERCISE 3.2

Find the derivative of y w.r.t. x.

1. $y=\sin^{-1}\frac{2x}{1+x^2}, -1\le x\le 1$
2. $y=\tan^{-1}\frac{3x-x^3}{1-3x^2}, -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$
3. $y=\sec^{-1}\frac{1}{2x^2-1}, 0<x<\frac{1}{\sqrt{2}}$
4. $y=\tan^{-1}\frac{4x}{1+5x^2}+\tan^{-1}\frac{2+3x}{3-2x}$
5. $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), x\ne 0$
6. $y=\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$
7. $y=\sin^{-1}\sqrt{(1-x)}+\cos^{-1}\sqrt{x}$
8. $y=\sqrt{\sin\sqrt{x}}$
9. $y=e^{\sin x^2}$
10. $y=\log\sqrt{\sin\sqrt{e^x}}$
11. $y=a^{(\sin^{-1}x)^2}$
12. $y=\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}$
13. $y=\sin^{-1}[\sqrt{x-ax}-\sqrt{a-ax}]$
14. $y=x^3e^x\sin x$
15. $y=\log_e\sqrt{\frac{1+\sin x}{1-\sin x}}$, where $x=\pi/3$
16. $y=\frac{x+\sin x}{x+\cos x}$
17. If $y=(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n})$, then find $\frac{dy}{dx}$ at $x=0$.
18. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, prove that $\frac{dy}{dx}=-\frac{1}{(x+1)^2}$.
19. If g is the inverse function of f and $f'(x)=\sin x$ then prove that $g'(x)=\operatorname{cosec}(g(x))$

ANSWERS

1. $\frac{2}{1+x^2}$ 2. $\frac{3}{1+x^2}$ 3. $\frac{-2}{\sqrt{1-x^2}}$ 4. $\frac{5}{1+25x^2}$

5. $\frac{1}{2}\left(\frac{1}{1+x^2}\right)$ 6. $\frac{1}{2\sqrt{1-x^2}}$ 7. $\frac{-1}{\sqrt{x-x^2}}$

8. $\frac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$ 9. $e^{\sin x^2}\cos x^2\, 2x$

10. $\frac{1}{4}e^{x/2}\cot\left(e^{x/2}\right)$ 11. $\frac{2\log a\sin^{-1}x}{\sqrt{1-x^2}}a^{(\sin^{-1}x)^2}$

12. $\frac{x^2-1}{x^2-4}$ 13. $\frac{1}{2\sqrt{x(1-x)}}$

14. $x^2e^x(3\sin x+x\sin x+x\cos x)$ 15. 2

16. $\frac{\cos x-\sin x+x(\cos x+\sin x)+1}{(x+\cos x)^2}$ 17. 1

DIFFERENTIATION OF IMPLICIT FUNCTIONS

Until now we have been differentiating various functions given in the form $y=f(x)$, which are explicit functions.

An explicit function is one which can be expressed in terms of the independent variable. i.e., $y=x^2+3x+2$, $y=e^x\sin x$ etc.

Sometimes we have instead an equation $f(x, y)=0$ in x and y, for example, $x+y=\sin(x+y)$, which may not be solvable for y. The solutions to this equation are a set of points $\{(x, y)\}$ which implicitly define a relation between x and y which we call an implicit function.

Implicit functions are often not actually function in the strict definition of the word, because they often have multiple y values for a single x value. However, these functions have graphs and derivatives.

It is often easier to differentiate an implicit function without rearranging it and differentiating each term in turn using chain rule, product rule and quotient rule, i.e.,

$$\frac{d}{dx}(y^2)=\left(\frac{d}{dy}(y^2)\right)\frac{dy}{dx}=2y\frac{dy}{dx}$$

$$\frac{d}{dx}(x^2\sin y)=\sin y\cdot\frac{d}{dx}(x^2)+x^2\cdot\frac{d}{dx}(\sin y)$$

$$=\sin y\cdot(2x)+x^2\cdot\left(\frac{d}{dy}(\sin y)\right)\frac{dy}{dx}$$

$$=\sin y\cdot(2x)+x^2\cdot\cos y\cdot\frac{dy}{dx}$$

Note:

To find the derivative of $f(x, y)=0$ we may use following formula.

$$\frac{dy}{dx}=-\frac{\text{differentitaion of } f(x, y) \text{ w.r.t. } x \text{ keeping } y \text{ as constant}}{\text{differenetiation of } f(x, y) \text{ w.r.t. } y \text{ keeping } x \text{ as constant}}$$

ILLUSTRATION 3.32

If $\sqrt{x}+\sqrt{y}=4$, then find $\frac{dy}{dx}$.

Sol. We have $\sqrt{x}+\sqrt{y}=4$ (1)

Differentiating the given equation both sides w.r.t. x, we get

$$\frac{1}{2\sqrt{x}}+\frac{d}{dx}(\sqrt{y})=0$$

$$\Rightarrow \quad \frac{1}{2\sqrt{x}}+\left(\frac{d}{dy}(\sqrt{y})\right)\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

ILLUSTRATION 3.33

If $xy+y^2=\tan x+y$, then find $\frac{dy}{dx}$.

Sol. The given relation is $xy+y^2=\tan x+y$.

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(xy)+\frac{d}{dx}(y^2)=\frac{d}{dx}(\tan x)+\frac{dy}{dx}$$

or $$\left[y\cdot 1+x\cdot\frac{dy}{dx}\right]+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$

or $$(x+2y-1)\frac{dy}{dx}=\sec^2 x-y$$

$$\therefore \quad \frac{dy}{dx}=\frac{\sec^2 x-y}{(x+2y-1)}$$

ILLUSTRATION 3.34

If $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cdots}}}$ prove that $\frac{dy}{dx}=\frac{y}{2y-x}$.

Sol. We have

$$y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cdots}}}=x+\frac{1}{y}$$

or $$y^2=xy+1$$

or $$2y\frac{dy}{dx}=y+x\frac{dy}{dx}+0 \quad \text{[Differentiating both sides w.r.t. } x\text{]}$$

or $$\frac{dy}{dx}(2y-x)=y$$

or $$\frac{dy}{dx}=\frac{y}{2y-x}$$

ILLUSTRATION 3.35

If $\sec(x+y)=xy$, then find $\frac{dy}{dx}$

Sol. We have, $\sec(x+y)=xy$

On differentiating both sides w.r.t x, we get

$$\sec(x+y)\cdot\tan(x+y)\cdot\frac{d}{dx}(x+y)=x\frac{dy}{dx}+y$$

$$\Rightarrow \quad \sec(x+y)\cdot\tan(x+y)\cdot\left(1+\frac{dy}{dx}\right)=x\frac{dy}{dx}+y$$

$$\Rightarrow \quad \frac{dy}{dx}[\sec(x+y)\cdot\tan(x+y)-x]=y-\sec(x+y)\cdot\tan(x+y)$$

$$\therefore \quad \frac{dy}{dx}=\frac{y-\sec(x+y).\tan(x+y)}{\sec(x+y).\tan(x+y)-x}$$

ILLUSTRATION 3.36

If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots}}}$, prove that $\frac{dy}{dx}=\frac{\cos x}{2y-1}$.

Sol. We have $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots}}}$

$$\Rightarrow \quad y=\sqrt{\sin x+y}$$

$$\Rightarrow \quad y^2=\sin x+y \quad (1)$$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx}=\cos x+\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{\cos x}{2y-1}$$

Alternative method:

From (1), $y^2-\sin x-y=0$

$$\therefore \quad \frac{dy}{dx}=-\frac{\text{differentitaion of } f(x,y) \text{ w.r.t. } x \text{ keeping } y \text{ as constant}}{\text{differenetiation of } f(x,y) \text{ w.r.t. } y \text{ keeping } x \text{ as constant}}$$

$$=-\frac{-\cos x}{2y-1}=\frac{\cos x}{2y-1}$$

ILLUSTRATION 3.37

If $\log(x^2+y^2)=2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx}=\frac{x+y}{x-y}$.

Sol. Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left\{\log\left(x^2+y^2\right)\right\}=2\frac{d}{dx}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

or $$\frac{1}{x^2+y^2}\times\frac{d}{dx}\left(x^2+y^2\right)=2\frac{1}{1+(y/x)^2}\times\frac{d}{dx}\left(\frac{y}{x}\right)$$

or $$\frac{1}{x^2+y^2}\left\{\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(y^2\right)\right\}=2\times\frac{x^2}{x^2+y^2}\left\{\frac{x\frac{dy}{dx}-y\times 1}{x^2}\right\}$$

or $\frac{1}{x^2+y^2}\left\{2x+2y\frac{dy}{dx}\right\} = \frac{2}{x^2+y^2}\left\{x\frac{dy}{dx}-y\right\}$

or $2\left\{x+y\frac{dy}{dx}\right\}=2\left\{x\frac{dy}{dx}-y\right\}$

or $x+y\frac{dy}{dx}=x\frac{dy}{dx}-y$

or $\frac{dy}{dx}(y-x)=-(x+y)$

or $\frac{dy}{dx}=\frac{x+y}{x-y}$

ILLUSTRATION 3.38

If $4xe^{xy} = y + 5\sin^2 x$, then find the value of $y'(0)$.

Sol. We have $4xe^{xy} = y + 5\sin^2 x$ ...(1)

Putting, $x = 0$ in above equation we get $y = 0$.

$\therefore$ (0, 0) lies on the curve

Now on differentiating equation (1) w.r.t. x, we get

$$4e^{xy} + 4xe^{xy}\left(x\frac{dy}{dx}+y\right) = \frac{dy}{dx}+10\sin x\cos x$$

Putting $x = 0$ and $y = 0$, we get

$$4+0=\left(\frac{dy}{dx}\right)_{(0,0)}+0$$

$$\therefore \left(\frac{dy}{dx}\right)_{(0,0)}=4$$

CONCEPT APPLICATION EXERCISE 3.3

1. If $x^3+y^3=3axy$, find $\frac{dy}{dx}$.
2. If $y=b\tan^{-1}\left(\frac{x}{a}+\tan^{-1}\frac{y}{x}\right)$, find $\frac{dy}{dx}$.
3. If $\log_e(\log_e x - \log_e y) = e^{x^2 y}(1-\log_e x)$, then find the value of $y'(e)$.
4. If $y=\sqrt{x+\sqrt{y+\sqrt{x+\sqrt{y+\ldots\infty}}}}$, then prove that $\frac{dy}{dx}=\frac{y^2-x}{2y^3-2xy-1}$.

ANSWERS

1. $\frac{ay-x^2}{y^2-ax}$ 2. $\frac{\frac{1}{a}-\frac{y}{x^2+y^2}}{\frac{1}{b}\sec^2\left(\frac{y}{b}\right)-\frac{x}{x^2+y^2}}$ 3. $\frac{1+e^{e^2}}{e}$

DIFFERENTIATION OF FUNCTIONS IN PARAMETRIC FORM

Parametric forms are used to express functions that can't be written in single-equation form. In physics, they are used to define the change in position of an object with respect to time. In this form, the relation between the variables x and y is defined by representing both the variables in the terms of another variable t (known as parameter) as $x = f(t)$ and $y = g(t)$. The functions are then related to one another through the parameter. Also parametric from is assumed to be an alternative way to express the curve or function which makes study of properties of curve easier. In coordinate geometry we use parametric forms in conic sections in equations of circle, parabola, ellipse and hyperbola. Parametric equations can be differentiated using the chain rule.

i.e., $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$.

So, $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$

ILLUSTRATION 3.39

Find $\frac{dy}{dx}$ if $x=a(\theta-\sin\theta)$ and $y=a(1-\cos\theta)$.

Sol. We have $x=a(\theta-\sin\theta)$ and $y=a(1-\cos\theta)$

$\therefore$ $\frac{dx}{d\theta}=a(1-\cos\theta)$ and $\frac{dy}{d\theta}=a\sin\theta$

or
$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sin\theta}{a(1-\cos\theta)}=\frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}=\cot\frac{\theta}{2}$$

ILLUSTRATION 3.40

If $x=a\sec^3\theta$ and $y=a\tan^3\theta$, find $\frac{dy}{dx}$ at $\theta=\frac{\pi}{3}$.

Sol. We have $x=a\sec^3\theta$ and $y=a\tan^3\theta$

$\therefore$ $\frac{dx}{d\theta}=3a\sec^2\theta\frac{d}{d\theta}(\sec\theta)=3a\sec^3\theta\tan\theta$

and $\frac{dy}{d\theta}=3a\tan^2\theta\frac{d}{d\theta}(\tan\theta)=3a\tan^2\theta\sec^2\theta$

or $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta}=\frac{\tan\theta}{\sec\theta}=\sin\theta$

or $\left(\frac{dy}{dx}\right)_{\theta=\pi/3}=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

ILLUSTRATION 3.41

If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = \frac{-y\log x}{x\log y}$

Sol. $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

$\cos 2t = \log x$ and $\sin 2t = \log y$

$\therefore \quad \cos^2 2t + \sin^2 2t = (\log x)^2 + (\log y)^2$

$\Rightarrow \quad (\log x)^2 + (\log y)^2 = 1$

Differentiating both sides w.r.t. x, we get

$$2\log x\frac{1}{x} + 2\log y\frac{1}{y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y\log x}{x\log y}$$

ILLUSTRATION 3.42

Let $y = x^3 - 8x + 7$ and $x = f(t)$. If $\frac{dy}{dt} = 2$ and $x = 3$ at $t = 0$, then find the value of $\frac{dx}{dt}$ at $t = 0$.

Sol. We have $y = x^3 - 8x + 7$

$\therefore \quad \frac{dy}{dx} = 3x^2 - 8$

It is given that when $t = 0$, $x = 3$. Therefore, when $t = 0$,

$$\frac{dy}{dx} = 3 \times 3^2 - 8 = 19$$

Also, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ (1)

Since when $t = 0$, $\frac{dy}{dx} = 19$ and $\frac{dy}{dt} = 2$, from (1)

$$19 = \frac{2}{dx/dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{2}{19}$$

CONCEPT APPLICATION EXERCISE 3.4

1. If $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$, then find $\frac{dy}{dx}$ at $t = 2$.
2. If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, $a > 0$ and $-1 < t < 1$, show that $\frac{dy}{dx} = -\frac{y}{x}$.
3. Find $\frac{dy}{dx}$, if $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$.
4. Find $\frac{dy}{dx}$ if $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$.
5. Find $\frac{dy}{dx}$ at $t = \pi/4$ for $x = a\left[\cos t + \frac{1}{2}\log\tan^2\frac{t}{2}\right]$ and $y = a\sin t$.

ANSWERS

1. $\frac{4}{3}$ 3. $\frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$ 4. $\cot\theta$
5. 1

DIFFERENTIATION USING LOGARITHM

Logarithm and its properties are very much useful in differentiation. Consider the following expressions and see how it is convenient to differentiate after taking logarithm on both sides.

(i) $y = [f_1(x)]^{f_2(x)}$

$\therefore \quad \log_e y = f_2(x)\log_e(f_1(x))$

Differentiating, w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = (f_2(x))'\log_e(f_1(x)) + f_2(x)\frac{(f_1(x))'}{f_1(x)}$$

(ii) $y = \frac{f_1(x)\cdot f_2(x)\cdot f_3(x)\ldots}{g_1(x)\cdot g_2(x)\cdot g_3(x)\ldots}$

$\therefore \quad \log_e y = [\log_e(f_1(x)) + \log_e(f_2(x)) + \log_e(f_3(x)) + \ldots] - [\log_e(g_1(x)) + \log_e(g_2(x)) + \log_e(g_3(x)) + \ldots]$

Differentiating, w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \left[\frac{(f_1(x))'}{f_1(x)} + \frac{(f_2(x))'}{f_2(x)} + \frac{(f_3(x))'}{f_3(x)} + \ldots\right] - \left[\frac{(g_1(x))'}{g_1(x)} + \frac{(g_2(x))'}{g_2(x)} + \frac{(g_3(x))'}{g_3(x)} + \ldots\right]$$

Note:

We have some more methods to differentiate $y = [f(x)]^{g(x)}$ as given below.

(i) $y = [f(x)]^{g(x)} = e^{g(x)\log_e f(x)}$

$$\therefore \quad y' = e^{g(x)\log_e f(x)} \times \left[g'(x)\log_e f(x) + g(x)\frac{f'(x)}{f(x)}\right]$$

(ii) $y = [f(x)]^{g(x)}$

$\therefore \quad \frac{dy}{dx}$ = Differential of y treating $f(x)$ as constant + Differential of y treating $g(x)$ as constant.

e.g. $y = (\sin x)^{\log_e \cos x}$

$\frac{dy}{dx}$ = (differential of y keeping '$\sin x$' as constant) + (differential of y keeping '$\log\cos x$' as constant.)

$$= (\sin x)^{\log_e \cos x}\cdot(\log_e \sin x)(\log_e \cos x)' + (\log_e \cos x)(\sin x)^{(\log_e \cos x)-1}(\sin x)'$$

$$= (\sin x)^{\log_e \cos x}\cdot(\log_e \sin x)\left(\frac{-\sin x}{\cos x}\right) + (\log_e \cos x)(\sin x)^{(\log_e \cos x)-1}(\cos x)$$

ILLUSTRATION 3.43

Find the derivative of $\frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$ w.r.t. x.

Sol. Let $y = \frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$

Taking log of both sides, we get

$$\log y = \frac{1}{2}\log x + \frac{3}{2}\log(x+4) - \frac{4}{3}\log(4x-3)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} + \frac{3}{2}\frac{1}{x+4} - \frac{4}{3} \times \frac{1}{4x-3} \times 4$$

or $$\frac{dy}{dx} = y\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}$$

or $$\frac{dy}{dx} = \frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}$$

ILLUSTRATION 3.44

If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

Sol. We have $x^m y^n = (x+y)^{m+n}$.

Taking log on both sides, we get

$$m \log x + n \log y = (m+n)\log(x+y)$$

Differentiating both sides w.r.t. x, we get

$$m\frac{1}{x} + n\frac{1}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\frac{d}{dx}(x+y)$$

or $$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1 + \frac{dy}{dx}\right)$$

or $$\left\{\frac{n}{y} - \frac{m+n}{x+y}\right\}\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

or $$\left\{\frac{nx+ny-my-ny}{y(x+y)}\right\}\frac{dy}{dx} = \left\{\frac{mx+nx-mx-my}{(x+y)x}\right\}$$

or $$\frac{nx-my}{y(x+y)}\frac{dy}{dx} = \frac{nx-my}{(x+y)x}$$

or $$\frac{dy}{dx} = \frac{y}{x}$$

ILLUSTRATION 3.45

Differentiate $(\log x)^{\cos x}$ with respect to x.

Sol. Let $y = (\log x)^{\cos x}$.

Taking logarithm on both the sides, we obtain

$$\log y = \cos x \cdot \log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log(\log x)\frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}[\log(\log x)]$$

or $$\frac{1}{y}\cdot\frac{dy}{dx} = -\sin x \log(\log x) + \cos x \cdot \frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

or $$\frac{dy}{dx} = y\left[-\sin x \log(\log x) + \frac{\cos x}{\log x}\cdot\frac{1}{x}\right]$$

$$= (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right]$$

ILLUSTRATION 3.46

If $f(x) = |x|^{|\sin x|}$, then find $f'\left(-\frac{\pi}{4}\right)$.

Sol. In the neighbourhood of $-\pi/4$, we have

$$f(x) = (-x)^{-\sin x} = e^{-\sin x \log(-x)}$$

or $$f'(x) = e^{-\sin x \log(-x)}\left(-\cos x \cdot \log(-x) - \frac{\sin x}{x}\right)$$

$$= (-x)^{-\sin x}\left(-\cos x \cdot \log(-x) - \frac{\sin x}{x}\right)$$

or $$f'(-\pi/4) = \left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{-1}{\sqrt{2}}\log\frac{\pi}{4} + \frac{4}{\pi}\times\left(\frac{-1}{\sqrt{2}}\right)\right)$$

$$= \left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\log\frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}\right)$$

ILLUSTRATION 3.47

If $y = x^{x^{x^{\cdots\infty}}}$, find $\frac{dy}{dx}$.

Sol. $y = x^y$

or $\log y = y \log x$

or $$\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx} \times \log x + y\frac{1}{x}$$ [Diff. both sides w.r.t. x]

or $$\frac{dy}{dx}\frac{\{1 - y\log x\}}{y} = \frac{y}{x}$$

or $$\frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$$

ILLUSTRATION 3.48

If $f(x) = \lim_{h\to 0}\frac{(\sin(x+h))^{\log_e(x+h)} - (\sin x)^{\log_e x}}{h}$, then find $f(\pi/2)$.

Sol. Let $g(x) = (\sin x)^{\log_e x}$

$$\therefore \quad f(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = g'(x)$$

$$\therefore \quad f(x) = g'(x) = \frac{d}{dx}\left(e^{\log_e x . \log_e \sin x}\right)$$

$$= (\sin x)^{\log_e x}\left(\frac{1}{x}.\log_e \sin x + \log_e x . \frac{1}{\sin x}.\cos x\right)$$

$$\therefore \quad f(\pi/2) = 0$$

ILLUSTRATION 3.49

If $0 < x < 1$, prove that $\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots\infty = \frac{1}{1-x}$.

Sol. The given series is in the form

$$\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \frac{f_3'(x)}{f_3(x)} + \cdots\infty$$

Then consider the product $f_1(x)\cdot f_2(x)\cdot f_3(x)\cdots f_n(x)$. Now,

$$(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}}) \qquad (1)$$

$$=(1-x^2)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}})$$

$$=(1-x^4)(1+x^4)\cdots(1+x^{2^{n-1}})$$

$$\vdots$$

$$=(1-x^{2^{n-1}})(1+x^{2^{n-1}})$$

$$=1-x^{2^n}$$

Now, when $n\to\infty$, $x^{2^{n-1}}\to 0$ ($\because 0<x<1$).

Therefore, taking $n\to\infty$, in (1), we get

$$(1-x)(1+x)(1+x^2)(1+x^4)\cdots=1$$

Taking logarithm, we get

$$\log(1-x)+\log(1+x)+\log(1+x^2)+\log(1+x^4)+\cdots=0$$

Differentiating w.r.t. x, we get

$$-\frac{1}{1-x}+\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\cdots=0$$

or $$\frac{1}{x+1}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\cdots\infty=\frac{1}{1-x}$$

Concept Application Exercise 3.5

1. Differentiate $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ with respect to x.
2. If $x^y=e^{x-y}$, prove that $\dfrac{dy}{dx}=\dfrac{\log x}{(1+\log x)^2}$.
3. If $xy=e^{(x-y)}$, then find $\dfrac{dy}{dx}$.
4. If $y^x=x^y$, then find $\dfrac{dy}{dx}$.
5. If $x=e^{y+e^{y+\ldots \text{to}\,\infty}}$, where $x>0$, then find $\dfrac{dy}{dx}$.
6. Find $\dfrac{dy}{dx}$ for $y=x^x$.
7. Differentiate $(x\cos x)^x$ with respect to x.
8. If $y=(\tan x)^{(\tan x)^{\tan x}}$, then find $\dfrac{dy}{dx}$.

Answers

1. $\dfrac{1}{2}\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\dfrac{1}{x-1}+\dfrac{1}{x-2}-\dfrac{1}{x-3}-\dfrac{1}{x-4}-\dfrac{1}{x-5}\right]$

3. $\dfrac{y(x-1)}{x(y+1)}$ 4. $\dfrac{y(x\log y-y)}{x(y\log x-x)}$ 5. $\dfrac{1-x}{x}$

6. $x^x(1+\log x)$

7. $(x\cos x)^x\left[(\log x+1)+\{\log\cos x-x\cot x\}\right]$

8. $y\log y\sec^2 x\left[\log\tan x+1+\dfrac{1}{\tan x\log\tan x}\right]$

DIFFERENTIATION OF ONE FUNCTION W.R.T. ANOTHER FUNCTION

Let $u=f(x)$ and $v=g(x)$ be two functions of x.

Then to find the derivative of $f(x)$ w.r.t. $g(x)$ i.e., to find $\dfrac{du}{dv}$, we use the formula $\dfrac{du}{dv}=\dfrac{du/dx}{dv/dx}$

Thus, the derivative of $f(x)$ w.r.t. $g(x)$ is the ratio of the derivative of $f(x)$ and $g(x)$ w.r.t. x.

Illustration 3.50

Differentiate $\log\sin x$ w.r.t. $\sqrt{\cos x}$.

Sol. Let $u=\log\sin x$ and $v=\sqrt{\cos x}$. Then,

$$\frac{du}{dx}=\cot x \text{ and } \frac{dv}{dx}=-\frac{\sin x}{2\sqrt{\cos x}}$$

or $$\frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{\cot x}{-\dfrac{\sin x}{2\sqrt{\cos x}}}=-2\sqrt{\cos x}\cot x\,\operatorname{cosec} x$$

Illustration 3.51

Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}x$, where $x\neq 0$.

Sol. Let $u=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ and $v=\tan^{-1}x$.

Putting $x=\tan\theta$, we get

$$u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$=\frac{1}{2}\theta$$

$$=\frac{1}{2}\tan^{-1}x$$

Thus, we have $u=\dfrac{1}{2}\tan^{-1}x$ and $v=\tan^{-1}x$. Therefore,

$$\frac{du}{dx}=\frac{1}{2}\times\frac{1}{1+x^2} \text{ and } \frac{dv}{dx}=\frac{1}{1+x^2}$$

$$\therefore \frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{1}{2(1+x^2)}(1+x^2)=\frac{1}{2}$$

ILLUSTRATION 3.52

Find the derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'(\sqrt{2}) = 4$.

Sol. Let $u = f(\tan x)$ and $v = g(\sec x)$. Therefore,

$$\frac{du}{dx} = f'(\tan x)\sec^2 x$$

and $$\frac{dv}{dx} = g'(\sec x)\sec x \tan x$$

$$\therefore \quad \frac{du}{dv} = \frac{du}{dx} \Big/ \frac{dv}{dx} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x \tan x}$$

or $$\left[\frac{du}{dv}\right]_{x=\frac{\pi}{4}} = \frac{f'\left(\tan\frac{\pi}{4}\right)}{g'\left(\sec\frac{\pi}{4}\right)\sin\frac{\pi}{4}} = \frac{f'(1)\sqrt{2}}{g'(\sqrt{2})}$$

$$= \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

CONCEPT APPLICATION EXERCISE 3.6

1. Find the derivative of $\tan^{-1}\frac{2x}{1-x^2}$ w.r.t. $\sin^{-1}\frac{2x}{1+x^2}$, $|x|<1$.
2. Find the derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t. $\sqrt{1-x^2}$ at $x = \frac{1}{2}$.
3. Differentiate $\frac{x}{\sin x}$ w.r.t. $\sin x$.
4. If $y = f(x^3)$, $z = g(x^5)$, $f'(x) = \tan x$, and $g'(x) = \sec x$, then find the value of $\lim_{x\to 0}\frac{(dy/dz)}{x}$.

ANSWERS

1. 1 2. 4 3. $\frac{\tan x - x}{\sin^2 x}$ 4. $\frac{3}{5}$

DIFFERENTIATION OF DETERMINANT

To differentiate the determinant, often it is not required to expand it.

Consider two-order determinant, $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}$, where a_{ij} is function of x for all i, j.

$$\Delta = a_{11}a_{22} - a_{12}a_{21}$$

Differentiating w.r.t. x we get,

$$\Delta' = (a_{11})'a_{22} + a_{11}(a_{22})' - (a_{12})'a_{21} - a_{12}(a_{21})'$$

$$= [(a_{11})'a_{22} - (a_{12})'a_{21}] + [a_{11}(a_{22})' - a_{12}(a_{21})']$$

$$= \begin{vmatrix} (a_{11})' & (a_{12})' \\ a_{21} & a_{22}\end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ (a_{21})' & (a_{22})'\end{vmatrix}$$

Thus for $\Delta = \begin{vmatrix} R_1 \\ R_2 \end{vmatrix}$, $\Delta' = \begin{vmatrix} (R_1)' \\ R_2 \end{vmatrix} + \begin{vmatrix} R_1 \\ (R_2)' \end{vmatrix}$

Similarly, for three-order determinant,

$$\Delta = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}, \Delta' = \begin{vmatrix} (R_1)' \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ (R_2)' \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ (R_3)' \end{vmatrix}$$

Here we are differentiating row wise.

We can also differentiate column wise.

i.e. if $\Delta = |C_1\ C_2\ C_3|$, then

$$\Delta' = |(C_1)'\ C_2\ C_3| + |C_1\ (C_2)'\ C_3| + |C_1\ C_2\ (C_3)'|$$

Also, if $\Delta = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ a & b & c \\ p & q & r \end{vmatrix}$, where a, b, c, p, q, r are constants, then

$$\Delta' = \begin{vmatrix} (f_1(x))' & (f_2(x))' & (f_3(x))' \\ a & b & c \\ p & q & r \end{vmatrix}$$

and $$\frac{d^n\Delta}{dx^n} = \begin{vmatrix} \frac{d^n(f_1(x))}{dx^n} & \frac{d^n(f_2(x))}{dx^n} & \frac{d^n(f_3(x))}{dx^n} \\ a & b & c \\ p & q & r \end{vmatrix}$$

ILLUSTRATION 3.53

Let $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$ Then find the value of $f'(0)$ and $f'(\pi/2)$.

Sol. Differentiating with respect to x column wise, we get

$$f'(x) = \begin{vmatrix} -\sin x & \sin x & \cos x \\ -2\sin 2x & \sin 2x & 2\cos 2x \\ -3\sin 3x & \sin 3x & 3\cos 3x \end{vmatrix} + \begin{vmatrix} \cos x & \cos x & \cos x \\ \cos 2x & 2\cos 2x & 2\cos 2x \\ \cos 3x & 3\cos 3x & 3\cos 3x \end{vmatrix} + \begin{vmatrix} \cos x & \sin x & -\sin x \\ \cos 2x & \sin 2x & -4\sin 2x \\ \cos 3x & \sin 3x & -9\sin 3x \end{vmatrix}$$

$$= \begin{vmatrix} -\sin x & \sin x & \cos x \\ -2\sin 2x & \sin 2x & 2\cos 2x \\ -3\sin 3x & \sin 3x & 3\cos 3x \end{vmatrix} + \begin{vmatrix} \cos x & \sin x & -\sin x \\ \cos 2x & \sin 2x & -4\sin 2x \\ \cos 3x & \sin 3x & -9\sin 3x \end{vmatrix}$$

$$\therefore \quad f'(0) = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 + 0 = 0$$

$$f'(\pi/2) = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & -2 \\ 3 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 9 \end{vmatrix}$$

$$= (-6 + 2) + (-1 + 9) = 4$$

ILLUSTRATION 3.54

$f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$. Then find the value of $\lim\limits_{x\to 0} \dfrac{f(x)}{x}$.

Sol. $f'(x) = \begin{vmatrix} -\sin x & 1 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\cos x & 2x & 2 \\ \tan x & x & 1 \end{vmatrix}$

$$+ \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \sec^2 x & 1 & 0 \end{vmatrix}$$

$$\Rightarrow \quad f'(0) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

Now, $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = \lim\limits_{x\to 0} f'(x)$ (as $f(0) = 0$)

$$= f'(0) = 0$$

CONCEPT APPLICATION EXERCISE 3.7

1. If $f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$, then prove that

$f'(x) = 3x^2 + 2x(a^2 + b^2 + c^2)$.

2. Let $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin(2x^2) \end{vmatrix}$.

Find the value of $f'(0)$.

3. Let $g(x) = \begin{vmatrix} f(x+c) & f(x+2c) & f(x+3c) \\ f(c) & f(2c) & f(3c) \\ f'(c) & f'(2c) & f'(3c) \end{vmatrix}$,

where c is constant, then find $\lim\limits_{x\to 0} \dfrac{g(x)}{x}$.

ANSWERS

2. 2 **3.** 0

HIGHER ORDER DERIVATIVES

If $y = y(x)$, then $\dfrac{dy}{dx}$, the derivative of y with respect to x, is itself, in general, a function of x and can be differentiated again. We call $\dfrac{dy}{dx}$ as the first order derivative of y with respect to x and the derivates of $\dfrac{dy}{dx}$ w.r.t. x, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ as the second order derivative of y w.r.t. x and it is denoted by $\dfrac{d^2y}{dx^2}$. Similarly, the derivative of $\dfrac{d^2y}{dx^2}$ w.r.t. x, $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$ is termed as the third order derivative of y w.r.t. x and is denoted by $\dfrac{d^3y}{dx^3}$ and so on. The nth order derivative of y w.r.t. x is denoted by $\dfrac{d^ny}{dx^n}$.

If $y = f(x)$, then the other alternative notations for $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^2y}{dx^3}, \ldots, \dfrac{d^ny}{dx^n}$ are $y_1, y_2, y_3, \ldots, y_n$ or $y', y'', y''', \ldots,$ $y^{(n)}$ or $f'(x), f''(x), f'''(x), \ldots, f^n(x)$ etc.

ILLUSTRATION 3.55

If $y = \cos^{-1} x$, find $\dfrac{d^2y}{dx^2}$ in terms of y alone.

Sol. $y = \cos^{-1} x$,

or $\quad x = \cos y$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = -\sin y$$

or $$\frac{dy}{dx} = -\operatorname{cosec} y$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\operatorname{cosec} y)$$

$$= \frac{d}{dy}(-\operatorname{cosec} y)\frac{dy}{dx}$$

$$= \operatorname{cosec} y \cot y\,(-\operatorname{cosec} y)$$

$$= -\cot y \cdot \operatorname{cosec}^2 y$$

ILLUSTRATION 3.56

If $y = (x^2 - 1)^m$, then find the $(2m)$th differential coefficient of y, $m \in N$.

Sol. $y = (x^2 - 1)^m$

$$= {}^mC_0(x^2)^m - {}^mC_1(x^2)^{m-1} + {}^mC_2(x^2)^{m-2} + \ldots$$

Since the degree of expression is '$2m$', on differentiating '$2m$' times, all the terms except first term vanish.

$$\therefore \quad \frac{d^{2m}}{dx^{2m}}(x^2-1)^m = {}^mC_0\, 2m(2m-1)(2m-2)\ldots 1 = (2m)!$$

ILLUSTRATION 3.57

If $y = x \log\{x/(a+bx)\}$, then show that $x^3 \dfrac{d^2y}{dx^2} = \left(x\dfrac{dy}{dx} - y\right)^2$.

Sol. Given $y/x = [\log x - \log(a+bx)]$. Therefore,

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x} - \frac{b}{a+bx} \qquad \text{[Diff. both sides w.r.t. } x]$$

or $$x\frac{dy}{dx} - y = \frac{ax}{a+bx} \qquad (1)$$

Differentiating again w.r.t. x, we get

$$\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) - \frac{dy}{dx} = \frac{a^2}{(a+bx)^2}$$

$$\therefore \quad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(x\frac{dy}{dx} - y\right)^2 \qquad \text{[By (1)]}$$

ILLUSTRATION 3.58

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$ is a constant or independent of a and b.

Sol. We have $(x-a)^2 + (y-b)^2 = c^2,\ c > 0$

Differentiating w.r.t. x, we get

$$2(x-a) + 2(y-b)y' = 0$$

or $$(x-a) + (y-b)y' = 0 \qquad \ldots(1)$$

or $$y' = -\frac{x-a}{y-b} \qquad \ldots(2)$$

Differentiating (1), w.r.t. x again, we get

$$1 + (y')^2 + (y-b)y'' = 0 \qquad \ldots(3)$$

$$\therefore \quad \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+(y')^2\right]^{\frac{3}{2}}}{\frac{-[1+(y')^2]}{y-b}}$$

$$= -(y-b)\left[1+(y')^2\right]^{\frac{1}{2}}$$

$$= -(y-b)\left[1+\left(\frac{x-a}{y-b}\right)^2\right]^{\frac{1}{2}}$$

$$= -\left[(y-b)^2 + (x-a)^2\right]^{\frac{1}{2}}$$

$$= -c$$

ILLUSTRATION 3.59

If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2y = 0$.

Sol. $y = e^{a\cos^{-1}x}$

$$\therefore \quad \frac{dy}{dx} = e^{a\cos^{-1}x}\,\frac{-a}{\sqrt{1-x^2}} = \frac{-ay}{\sqrt{1-x^2}}$$

or $$(1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Differentiating both sides with respect to x, we get

$$\left(\frac{dy}{dx}\right)^2(-2x) + (1-x^2)\times 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = a^2\cdot 2y\cdot\frac{dy}{dx}$$

or $$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = a^2\cdot y \quad \left[\frac{dy}{dx} \ne 0\right]$$

or $$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

ILLUSTRATION 3.60

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$.

Sol. It is given that $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$. Therefore,

$$\frac{dx}{dt} = a[-\sin t + \sin t + t\cos t] = at\cos t$$

$$\frac{dy}{dt} = a[\cos t - \{\cos t - t\sin t\}] = at\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Then, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t)$$

$$= \frac{d}{dt}(\tan t)\frac{dt}{dx}$$

$$= \sec^2 t\cdot\frac{dt}{dx}$$

$$= \sec^2 t\cdot\frac{1}{at\cos t}$$

$$= \frac{\sec^3 t}{at}$$

ILLUSTRATION 3.61

If g is inverse of f then prove that $f''(g(x)) = -g''(x)(f'(g(x)))^3$.

Sol. Since g is inverse of f, $f(g(x)) = x$

$$\therefore \quad f'(g(x))g'(x) = 1 \qquad \ldots(1$$

$$\Rightarrow \quad f'(g(x)) = \frac{1}{g'(x)}$$

$$\Rightarrow \quad f''(g(x))g'(x) = -\frac{g''(x)}{(g'(x))^2}$$

$$\Rightarrow \quad f''(g(x)) = -\frac{g''(x)}{(g'(x))^3}$$

$$\Rightarrow \quad f''(g(x)) = -g''(x)(f'(g(x)))^3$$

ILLUSTRATION 3.62

Let $f(x)$ and $g(x)$ be real valued functions such that $f(x)g(x) = 1$, $\forall\, x \in R$. If $f''(x)$ and $g''(x)$ exists $\forall\, x \in R$ and $f'(x)$ and $g'(x)$ are never zero, then prove that $\dfrac{f''(x)}{f'(x)} - \dfrac{g''(x)}{g'(x)} = \dfrac{2f'(x)}{f(x)}$

Sol. $f(x) \times g(x) = 1$

Differentiating w.r.t x on both sides, we get

$$f'g + fg' = 0 \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{g}{g'} = -\frac{f}{f'} \qquad \ldots(2)$$

Differentiating (1), w.r.t. x we get

$$f''g + 2f'g' + fg'' = 0$$

$$\Rightarrow \quad \frac{f''g}{f'g'} + 2 + \frac{fg''}{f'g'} = 0$$

$$\Rightarrow \quad \frac{f''}{f'}\left(-\frac{f}{f'}\right) + 2 + \left(\frac{f}{f'}\right)\frac{g''}{g'} = 0 \qquad \text{(Using (2))}$$

$$\Rightarrow \quad \frac{f''(x)}{f'(x)} - \frac{g''(x)}{g'(x)} = \frac{2f'(x)}{f(x)}$$

ILLUSTRATION 3.63

If $f(x)$, $g(x)$, and $h(x)$ are three polynomials of degree 2, then prove that $\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$ is a constant polynomial.

Sol. Let $f(x) = a_1x^2 + a_2x + a_3$, $g(x) = b_1x^2 + b_2x + b_3$, and $h(x) = c_1x^2 + c_2x + c_3$. Then,

$$f'(x) = 2a_1x + a_2,\ g'(x) = 2b_1x + b_2,\ h'(x) = 2c_1x + c_2$$

$$f''(x) = 2a_1,\ g''(x) = 2b_1,\ h''(x) = 2c_1,$$

and $\quad f'''(x) = g'''(x) = h'''(x) = 0$

In order to prove that $\phi(x)$ is a constant polynomial, it is sufficient to show that $\phi'(x) = 0$ for all values of x, where

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$$\therefore \quad \phi'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ 0 & 0 & 0 \end{vmatrix}$$

$= 0 + 0 + 0 = 0$ for all values of x

$\therefore \quad \phi(x) =$ Constant for all

Hence, $\phi(x)$ is a constant polynomial.

CONCEPT APPLICATION EXERCISE 3.8

1. If $f(x) = (1 + x)^n$, then find the value of
$$f(0) + f'(0) + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \cdots + \frac{f^n(0)}{n!}.$$

2. If $e^y(x + 1) = 1$, show that $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.

3. Prove that $\dfrac{d^n}{dx^n}(e^{2x} + e^{-2x}) = 2^n[e^{2x} + (-1)^n e^{-2x}]$.

4. If $y = \sin(\sin x)$ and $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\tan x + f(x) = 0$, then find $f(x)$.

5. If $y = \log(1 + \sin x)$, prove that $y_4 + y_3y_1 + y_2^2 = 0$.

6. If $f(x) = \begin{vmatrix} x^n & n! & 2 \\ \cos x & \cos\left(\frac{n\pi}{2}\right) & 4 \\ \sin x & \sin\left(\frac{n\pi}{2}\right) & 8 \end{vmatrix}$, then find the value of $\dfrac{d^n}{dx^n}[f(x)]_{x=0}$

7. If $x = a\cos\theta$, $y = b\sin\theta$, then prove that $\dfrac{d^3y}{dx^3} = -\dfrac{3b}{a^3}\operatorname{cosec}^4\theta\cot\theta$.

8. If $x = a\cos^3\theta$, $y = b\sin^3\theta$, find $\dfrac{d^3y}{dx^3}$ at $\theta = 0$.

ANSWERS

1. 2^n **4.** $\cos^2 x \sin(\sin x)$ **6.** 0

8. Does not exist

DIFFERENTIATION OF FUNCTIONAL RELATIONS

In many problems we are given functional relation (of differentiable function) in two independent variables with some conditions and are asked to find the derivative of function at some value of x or to find the function. Since there are two independent variables in the functional relations we can differentiate the relation w.r.t. any one variable considering other variable constant. In turn we find the expression for derivative of the function. Using some elementary integration (reverse process of differentiation) we can find function. In alternative method we can use the first definition of derivative to find the derivative in which we use the given functional relation.

ILLUSTRATION 3.64

Let $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ for all real x and y. If $f'(0)$ exists and equals -1 and $f(0) = 1$, then find $f(2)$.

Sol. Since $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$

replacing x by $2x$ and y by 0, we get

$$f(x)=\frac{f(2x)+f(0)}{2}$$

or $\quad f(2x)+f(0)=2f(x)$ or $f(2x)-2f(x)=-f(0) \qquad (1)$

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\left\{\frac{\frac{f(2x)+f(2h)}{2}-f(x)}{h}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{f(2x)+f(2h)-2f(x)}{2h}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{f(2h)-f(0)}{2h}\right\} \qquad \text{[From (1)]}$$

$$=f'(0)$$

$$=-1\ \forall x\in R \qquad \text{(Given)}$$

Integrating, we get $f(x) = -x + c$.

Putting $x = 0$, we get

$f(0) = 0 + c = 1$ (Given)

$\therefore \quad c = 1$

Then $\quad f(x) = 1 - x$

$\therefore \quad f(2) = 1 - 2 = -1$

Alternative method 1

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$$

Differentiating both sides w.r.t. x treating y as constant, we get

$$f'\left(\frac{x+y}{2}\right)\cdot\frac{1}{2}=\frac{f'(x)+0}{2} \text{ or } f'\left(\frac{x+y}{2}\right)=f'(x)$$

Replacing x by 0 and y by $2x$, we get

$f'(x) = f'(0) = -1$ (Given)

Integrating, we have $f(x) = -x + c$.

Putting $x = 0$, we get

$f(0) = 0 + c = 1$ (Given)

$\therefore \quad c = 1$

Hence, $f(x) = -x + 1$.

Then $f(2) = -2 + 1 = -1$.

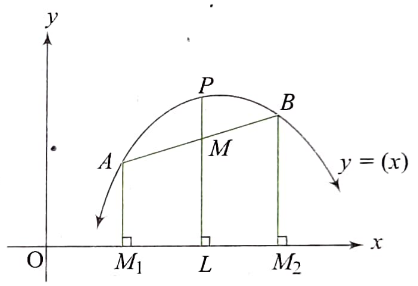
Alternative method 2 (Graphical method)

Suppose $A(x, f(x))$ and $B(y, f(y))$ are any two points on the curve $y = f(x)$.

If M is the midpoint of AB, then the coordinates of M are $\left(\frac{x+y}{2},\frac{f(x)+f(y)}{2}\right)$

From point M, draw a vertical line meeting curve at P and x-axis at L. So, the coordinates of P are

$$\left(\frac{x+y}{2},f\left(\frac{x+y}{2}\right)\right) \text{ and } PL>ML \text{ or } f\left(\frac{x+y}{2}\right)>\frac{f(x)+f(y)}{2}$$



But given

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$$

which is possible when $P \to M$, i.e., P lies on AB. Hence, $y = f(x)$ must be a linear function.

Let $\quad f(x) = ax + b$, $f(0) = 0 + b = 1$ (Given)

and $\quad f'(x) = a$ or $f'(0) = a = -1$ (Given)

$\therefore \quad f(x) = -x + 1$

$\therefore \quad f(2) = -2 + 1 = -1$

Also, the given relation $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ satisfies the section formula for abscissa and ordinate on LHS and RHS, respectively, which occurs only in the case of straight line. Hence, $f(x) = ax + b$. From $f'(0) = -1$, $a = -1$, and from $f(0) = 1$, $b = 1$. Therefore, $f(x) = -x + 1$.

ILLUSTRATION 3.65

If $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ for all $x, y \in R$, $(xy \neq 1)$, and $\lim_{x\to 0}\frac{f(x)}{x}=2$ then find $f\left(\frac{1}{\sqrt{3}}\right)$ and $f'(1)$.

Sol. $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ (1)

Putting $x=y=0$, we get $f(0)=0$.

Putting $y=-x$, we get

$f(+x)+f(-x)=f(0)$

or $f(-x)=-f(x)$ (2)

Also, $\lim_{x\to 0}\frac{f(x)}{x}=2$ (3)

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f(x+h)+f(-x)}{h}$ [Using (2)]

$=\lim_{h\to 0}\frac{f\left(\frac{x+h-x}{1-(x+h)(-x)}\right)}{h}$ [Using (1)]

$=\lim_{h\to 0}\left[\frac{f\left(\frac{h}{1+x(x+h)}\right)}{h}\right]$

$=\lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)}\times\left(\frac{1}{1+xh+x^2}\right)$

$=\lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)}\times\lim_{h\to 0}\frac{1}{1+xh+x^2}$

$\left(\text{Using }\lim_{x\to 0}\frac{f(x)}{x}=2\right)$

$=2\times\frac{1}{1+x^2}=\frac{2}{1+x^2}$

Integrating both sides, we get

$f(x)=2\tan^{-1}(x)+c$, where $f(0)=0\Rightarrow c=0$

Thus, $f(x)=2\tan^{-1}x$. Hence,

$f\left(\frac{1}{\sqrt{3}}\right)=2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=2\frac{\pi}{6}=\frac{\pi}{3}$

and, $f'(1)=\frac{2}{1+1^2}=\frac{2}{2}=1$

ILLUSTRATION 3.66

Let $f: R-\{0\}\to R$ be a function which is differentiable in its domain and satisfying the equation $f(x+y)=f(x)+f(y)+\frac{x+y}{xy}-\frac{1}{x+y}$, also $f'(1)=2$. Then find the function.

Sol. $f(x+y)=f(x)+f(y)+\frac{1}{x}+\frac{1}{y}-\frac{1}{x+y}$ ...(1)

Differentiating w.r.t. to x, keeping y as constant.

$f'(x+y)=f'(x)-\frac{1}{x^2}+\frac{1}{(x+y)^2}$

Putting $x=1$

$f'(1+y)=2-1+\frac{1}{(1+y)^2}=1+\frac{1}{(1+y)^2}$

$\Rightarrow$ $f'(x)=1+\frac{1}{x^2}$

$\Rightarrow$ $f(x)=x-\frac{1}{x}+C$ ...(2)

Put $x=y=1$ in (1), we get

$f(2)-2f(1)=2-\frac{1}{2}=\frac{3}{2}$

Also from (2), $f(2)=2-\frac{1}{2}-C$

$\Rightarrow$ $C=0$

$\Rightarrow$ $f(x)=x-\frac{1}{x}$

ILLUSTRATION 3.67

Find function $f(x)$ which is differentiable and satisfy the relation $f(x+y)=f(x)+f(y)+(e^x-1)(e^y-1)\ \forall\ x,y\in R$, and $f'(0)=2$.

Sol. We have

$f(x+y)=f(x)+f(y)+(e^x-1)(e^y-1)\ \forall\ x,y\in R$ ...(1)

Differentiating w.r.t. x, keeping y as constant, we get

$f'(x+y)=f'(x)+e^x(e^y-1)$

Putting $x=0$

$f'(y)=f'(0)+e^y-1$

$\therefore$ $f'(y)=e^y+1$ (as $f'(0)=2$)

$\therefore$ $f(y)=e^y+y+c$ ...(2)

Putting $x=y=0$ in (1), we get

$f(0)=f(0)+f(0)+0$

$\therefore$ $f(0)=0$

In (2), putting $y=0$, we get

$0=1+0+c$

$\Rightarrow$ $c=-1$

$\therefore$ $f(x)=e^x+x-1$

ILLUSTRATION 3.68

Find function $f(x)$ which satisfy the relation

$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}\ \forall\ x,y\in R;\ y\neq 0, f(y)\neq 0$ and $f'(1)=2$.

Sol. We have $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$ ...(1)

Differentiating w.r.t. x, keeping y as constant, we get

$f'\left(\frac{x}{y}\right)\frac{1}{y}=\frac{f'(x)}{f(y)}$

Putting $x = y$, we get

$$f'(1)\ \frac{1}{x} = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \frac{2}{x}$$

Integrating both sides, we get

$$\log_e f(x) = 2\log_e x + \log c$$

$$\Rightarrow \quad f(x) = cx^2$$

In (1), putting $x = y = 1$, we get $f(1) = 1$

$$\therefore \quad f(x) = x^2$$

ILLUSTRATION 3.69

Let $f: R \to R$ be a function satisfying condition $f(x + y^3) = f(x) + [f(y)]^3$ for all $x, y \in R$. If $f'(0) \geq 0$, find $f(10)$.

Sol. Given $f(x + y^3) = f(x) + [f(y)]^3$ (1)

and $f'(0) \geq 0$ (2)

Replacing x, y by 0, we get

$$f(0) = f(0) + f(0)^3 \text{ or } f(0) = 0 \quad (3)$$

Also, $$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} \quad (4)$$

Let $$I = f'(0) = \lim_{h\to 0} \frac{f(0 + (h^{1/3})^3) - f(0)}{(h^{1/3})^3}$$

$$= \lim_{h\to 0} \frac{f((h^{1/3}))^3}{(h^{1/3})^3} = \lim_{h\to 0}\left(\frac{f(h^{1/3})}{(h^{1/3})}\right)^3 = I^3$$

or $I = I^3$

or $I = 0, 1, -1$ as $f'(0) \geq 0$

$\therefore$ $f'(0) = 0, 1$ (5)

Thus, $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x + (h^{1/3})^3) - f(x)}{(h^{1/3})^3}$$

$$= \lim_{h\to 0} \frac{f(x) + (f(h^{1/3}))^3 - f(x)}{(h^{1/3})^3} \quad \text{[using (1)]}$$

$$= \lim_{h\to 0}\left(\frac{f(h^{1/3})}{(h^{1/3})}\right)^3 = (f'(0))^3$$

$= 0, 1$ [As $f'(0) = 0, 1$ using (5)]

Integrating both sides, we get

$$f(x) = c \text{ or } x + c$$

or $f(x) = 0$ or x $(\because f(0) = 0)$

Thus, $f(10) = 0$ or 10

ILLUSTRATION 3.70

Let $f(x + y) = f(x) + f(y) + 2xy - 1$ for all real x and y and $f(x)$ be a differentiable function. If $f'(0) = \cos\alpha$, then prove that $f(x) > 0\ \forall\ x \in R$.

Sol. We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x) + f(h) + 2hx - 1 - f(x)}{h}$$

(Using the given definition)

$$= \lim_{h\to 0}\left(2x + \frac{f(h) - 1}{h}\right).$$

Now substituting $x = y = 0$ in the given functional relation, we get

$$f(0) = f(0) + f(0) + 0 - 1 \text{ or } f(0) = 1$$

$$\therefore \quad f'(x) = 2x + \lim_{h\to 0} \frac{f(h) - f(0)}{h} = 2x + f'(0)$$

$$= 2x + \cos\alpha$$

Integrating, we get $f(x) = x^2 + x\cos\alpha + C$.

Since $f(0) = 1$,

$1 = C$

or $f(x) = x^2 + x\cos\alpha + 1$

It is a quadratic in x with discriminant

$$D = \cos^2\alpha - 4 < 0$$

and coefficient of $x^2 = 1 > 0$. Therefore,

$$f(x) > 0\ \forall\ x \in R$$

Alternative method:

$$f(x + y) = f(x) + f(y) + 2xy - 1 \quad (1)$$

Differentiate w.r.t. x keeping y as constant

$$f'(x + y) = f'(x) + 2y$$

Put $x = 0$ and $y = x$. Then

$$f'(x) = f'(0) + 2x$$

$$= \cos\alpha + 2x$$

$$\therefore \quad f(x) = x\cos\alpha + x^2 + c \quad (2)$$

Put $x = y = 0$ in (1). Then $f(0) = f(0) + f(0) - 1$ or $f(0) = 1$.

Then from (2), we get $f(x) = x^2 + (\cos\alpha)x + 1$.

CONCEPT APPLICATION EXERCISE 3.9

1. Let $f(x + y) = f(x) \cdot f(y)$ for all x and y. Suppose $f(5) = 2$ and $f'(0) = 3$. Find $f'(5)$.
2. Let $f(xy) = f(x)f(y)\ \forall\ x, y \in R$ and f is differentiable at $x = 1$ such that $f'(1) = 1$. Also, $f(1) \neq 0$, $f(2) = 3$. Then find $f'(2)$.
3. Let f be a function such that $f(x + y) = f(x) + f(y)$ for all x and y and $f(x) = (2x^2 + 3x)\, g(x)$ for all x, where $g(x)$ is continuous and $g(0) = 3$. Then find $f'(x)$.
4. Let $g: R \to R$ be a differentiable function satisfying $g(x) = g(y)\, g(x - y)\ \forall\ x, y \in R$ and $g'(0) = a$ and $g'(3) = b$. Then find the value of $g'(-3)$.
5. Let $f(x^m y^n) = mf(x) + nf(y)$ for all $x, y \in R^+$ and for all $m, n \in R$. If $f'(x)$ exists and has the value $\frac{e}{x}$, then find $\lim_{x\to 0} \frac{f(1+x)}{x}$.
6. If $f\left(\frac{x+2y}{3}\right) = \frac{f(x) + 2f(y)}{3}\ \forall\ x, y \in R$ and $f'(0) = 1$, $f(0) = 2$, then find $f(x)$.
7. Prove that $\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$ (without using L'Hopital's rule).

ANSWERS

1. 6	**2.** 3/2	**3.** 9	**4.** $\frac{a^2}{b}$
5. e	**6.** $x + 2$		

Solved Examples

Example 3.1

If $f(x) = \log_x (\log x)$, then find $f'(x)$ at $x = e$

Sol. Given $f(x) = \log_x (\log x) = \dfrac{\log_e(\log_e x)}{(\log_e x)}$

or $$f'(x) = \frac{\frac{1}{\log_e x}\frac{1}{x}\log_e x - \frac{1}{x}\log_e(\log_e x)}{(\log_e x)^2}$$

$$= \frac{\frac{1}{x}\left[1 - \log_e(\log_e x)\right]}{(\log_e x)^2}$$

At $x = e$, we get

$$f'(e) = \frac{\frac{1}{e}\left[1 - \log_e(\log_e e)\right]}{(\log_e e)^2} = \frac{\frac{1}{e}\left[1 - \log_e 1\right]}{(1)^2}$$

$$= \frac{1}{e}(1 - 0) = \frac{1}{e}$$

Example 3.2

Given that $\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\cdots = \frac{\sin x}{x}$. Then find the sum $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{4} + \cdots$.

Sol. We have $\cos\frac{x}{2}.\cos\frac{x}{4}.\cos\frac{x}{8}\cdots = \frac{\sin x}{x}$

Taking log on both sides, we get

$$\log\cos\frac{x}{2} + \log\cos\frac{x}{4} + \log\cos\frac{x}{8}\cdots + \cdots = \log\sin x - \log x$$

Differentiating both sides with respect to x, we get

$$-\frac{1}{2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} - \frac{1}{4}\frac{\sin\frac{x}{4}}{\cos\frac{x}{4}} - \frac{1}{8}\frac{\sin\frac{x}{8}}{\cos\frac{x}{8}}\cdots = \frac{\cos x}{\sin x} - \frac{1}{x}$$

or $$-\frac{1}{2}\tan\frac{x}{2} - \frac{1}{4}\tan\frac{x}{4} - \frac{1}{8}\tan\frac{x}{8} - \cdots = \cot x - \frac{1}{x}$$

Differentiating both sides with respect to x, we get

$$-\frac{1}{2^2}\sec^2\frac{x}{2} - \frac{1}{4^2}\sec^2\frac{x}{4} - \frac{1}{8^2}\sec^2\frac{x}{8} - \cdots = -\operatorname{cosec}^2 x + \frac{1}{x^2}$$

or $$\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{4^2}\sec^2\frac{x}{4} + \frac{1}{8^2}\sec^2\frac{x}{8} + \cdots = \operatorname{cosec}^2 x - \frac{1}{x^2}.$$

Example 3.3

If $y = f(a^x)$ and $f'(\sin x) = \log_e x$, then find $\frac{dy}{dx}$, if it exists, where $\frac{\pi}{2} < x < \pi$.

Sol. Given that $f'(\sin x) = \dfrac{df(\sin x)}{d(\sin x)} = \log_e x$

$$= \log_e(\pi - \sin^{-1}(\sin x))$$

$[\because \sin^{-1}\sin x = \pi - x \text{ for } x \in (\pi/2, \pi)]$

$\therefore$ $$\frac{df(t)}{dt} = \log_e(\pi - \sin^{-1} t)$$

or $$f'(t) = \log_e(\pi - \sin^{-1} t) \quad (1)$$

and $$y = f(a^x)$$

$\therefore$ $$\frac{dy}{dx} = f'(a^x)a^x \log_e a$$

$$= a^x \log_e a \log_e(\pi - \sin^{-1} a^x) \quad \text{[using (1)]}$$

Example 3.4

If P_n is the sum of a G.P. upto n terms $(n \ge 3)$, then prove that $(1-r)\frac{dP_n}{dr} = (1-n)P_n + nP_{n-1}$, where r is the common ratio of G.P.

Sol. Let the first term of G.P. be α. Then

$$P_n = \alpha\left[\frac{1-r^n}{1-r}\right]$$

$$\frac{dP_n}{dr} = \alpha\left[\frac{(1-r)(-nr^{n-1}) + (1-r^n)}{(1-r)^2}\right]$$

$\therefore$ $$(1-r)\frac{dP_n}{dr} = \alpha\left(\frac{-nr^{n-1} + nr^n}{1-r}\right) + \left(\frac{1-r^n}{1-r}\right)\alpha$$

$$= \alpha n\cdot\frac{(1 - r^{n-1} - 1 + r^n)}{1-r} + P_n$$

$$= n\cdot P_{n-1} - nP_n + P_n$$

$$= (1-n)P_n + nP_{n-1}$$

Example 3.5

If $g(x) = \dfrac{f(x)}{(x-a)(x-b)(x-c)}$, where $f(x)$ is a polynomial of degree < 3, then prove that

$$\frac{dg(x)}{dx} = \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}.$$

Sol. By partial fractions, we have

$$g(x) = \frac{f(a)}{(x-a)(a-b)(a-c)} + \frac{f(b)}{(b-a)(x-b)(b-c)} + \frac{f(c)}{(c-a)(c-b)(x-c)}$$

$$= \frac{1}{(a-b)(b-c)(c-a)} \times \left[\frac{f(a)(c-b)}{(x-a)} + \frac{f(b)(a-c)}{(x-b)} + \frac{f(c)(b-a)}{(x-c)}\right]$$

$$= \begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\therefore \quad \frac{dg(x)}{dx} = \begin{vmatrix} 1 & a & -f(a)(x-a)^{-2} \\ 1 & b & -f(b)(x-b)^{-2} \\ 1 & c & -f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

EXAMPLE 3.6

If $x = \operatorname{cosec}\theta - \sin\theta$ and $y = \operatorname{cosec}^n\theta - \sin^n\theta$, then show that $(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$.

Sol. As $x = \operatorname{cosec}\theta - \sin\theta$, we have

$$x^2 + 4 = (\operatorname{cosec}\theta - \sin\theta)^2 + 4 = (\operatorname{cosec}\theta + \sin\theta)^2 \quad (1)$$

and $$y^2 + 4 = (\operatorname{cosec}^n\theta - \sin^n\theta)^2 + 4 = (\operatorname{cosec}^n\theta + \sin^n\theta)^2 \quad (2)$$

Now,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{n(\operatorname{cosec}^{n-1}\theta)(-\operatorname{cosec}\theta\cot\theta) - n\sin^{n-1}\theta\cos\theta}{-\operatorname{cosec}\theta\cot\theta - \cos\theta}$$

$$= \frac{n(\operatorname{cosec}^n\theta\cot\theta + \sin^{n-1}\theta\cos\theta)}{(\operatorname{cosec}\theta\cot\theta + \cos\theta)}$$

$$= \frac{n\cot\theta(\operatorname{cosec}^n\theta + \sin^n\theta)}{\cot\theta(\operatorname{cosec}\theta + \sin\theta)}$$

$$= \frac{n(\operatorname{cosec}^n\theta + \sin^n\theta)}{(\operatorname{cosec}\theta + \sin\theta)} = \frac{n\sqrt{y^2+4}}{\sqrt{x^2+4}} \quad \text{[From (1) and (2)]}$$

Squaring both sides, we get $\left(\frac{dy}{dx}\right)^2 = \frac{n^2(y^2+4)}{(x^2+4)}$

or $$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$

EXAMPLE 3.7

If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$,

prove that $\frac{y'}{y} = \frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$.

Sol. $$y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \left(\frac{c+x-c}{x-c}\right)$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{x}{x-c}$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx + x(x-b)}{(x-b)(x-c)}$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{x^2}{(x-b)(x-c)}$$

$$= \frac{ax^2 + x^2(x-a)}{(x-a)(x-b)(x-c)}$$

$$= \frac{x^3}{(x-a)(x-b)(x-c)}$$

$$\therefore \quad \log y = \log\left\{\frac{x^3}{(x-a)(x-b)(x-c)}\right\}$$

or $$\log y = 3\log x - \{\log(x-a) + \log(x-b) + \log(x-c)\}$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \left\{\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}\right\}$$

or $$\frac{dy}{dx} = y\left\{\left(\frac{1}{x} - \frac{1}{x-a}\right) + \left(\frac{1}{x} - \frac{1}{x-b}\right) + \left(\frac{1}{x} - \frac{1}{x-c}\right)\right\}$$

$$= y\left\{-\frac{a}{x(x-a)} - \frac{b}{x(x-b)} - \frac{c}{x(x-c)}\right\}$$

$$= \frac{y}{x}\left\{\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{x-c}\right\}.$$

EXAMPLE 3.8

If $y = \frac{2}{\sqrt{a^2-b^2}}\left\{\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\right\}$, then show that $\frac{d^2y}{dx^2} = \frac{b\sin x}{(a+b\cos x)^2}$.

Sol. $y = A\tan^{-1}\left(B\tan\frac{x}{2}\right)$,

where $A = \frac{2}{\sqrt{a^2-b^2}}$, $B = \sqrt{\frac{a-b}{a+b}}$

$$AB = \frac{2}{\sqrt{(a-b)(a+b)}}\sqrt{\frac{a-b}{a+b}} = \frac{2}{a+b}.$$

$$\frac{dy}{dx} = \frac{AB\sec^2\frac{x}{2} \times \frac{1}{2}}{1 + B^2\tan^2\frac{x}{2}}$$

$$= \frac{1}{a+b} \cdot \frac{(a+b)}{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}}$$

$$= \frac{1}{a+b\cos x} \quad (1)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{b\sin x}{(a+b\cos x)^2}$$

EXAMPLE 3.9

If $y = (1/2^{n-1})\cos(n\cos^{-1}x)$, then prove that y satisfies the differential equation $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + n^2y = 0$.

Sol. $y = (1/2^{n-1})\cos(n\cos^{-1}x)$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2^{n-1}}\sin(n\cos^{-1}x)\left[\frac{-n}{\sqrt{(1-x^2)}}\right]$$

$$\text{or} \quad (1-x^2)\left(\frac{dy}{dx}\right)^2 = \frac{n^2}{2^{2n-2}}\sin^2(n\cos^{-1}x)$$

$$= \frac{n^2}{2^{2n-2}}[1-\cos^2(n\cos^{-1}x)]$$

$$= n^2\left[\frac{1}{2^{2n-2}} - y^2\right]$$

Differentiating both sides w.r.t. x, we get

$$(1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -2n^2y\frac{dy}{dx}$$

$$\text{or} \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0$$

EXAMPLE 3.10

Let $f(x)$ and $g(x)$ be two functions having finite nonzero third-order derivatives $f'''(x)$ and $g'''(x)$ for all $x \in R$. If $f(x)\,g(x) = 1$ for all $x \in R$, then prove that $\dfrac{f'''}{f'} - \dfrac{g'''}{g'} = 3\left(\dfrac{f''}{f} - \dfrac{g''}{g}\right)$.

Sol. We have $f(x)\,g(x) = 1$. Differentiating with respect to x, we get

$$f'g + fg' = 0 \quad (1)$$

Differentiating (1) w.r.t. x, we get

$$f''g + 2f'g' + fg'' = 0 \quad (2)$$

Differentiating (2) w.r.t. x, we get

$$f'''g + g'''f + 3f''g' + 3g''f' = 0$$

$$\text{or} \quad \frac{f'''}{f'}(fg') + \frac{g'''}{g'}(f'g) + \frac{3f''}{f}(f'g) + \frac{3g''}{g}(fg') = 0 \quad \text{[Using (1)]}$$

$$\text{or} \quad \left(\frac{f'''}{f'} + \frac{3g''}{g}\right)(f'g) = -\left(\frac{g'''}{g'} + \frac{3f''}{f}\right)(f'g)$$

$$\text{or} \quad -\left(\frac{f'''}{f'} + \frac{3g''}{g}\right)(fg') = \left(\frac{g'''}{g'} + \frac{3f''}{g}\right)fg' \quad \text{[Using (1)]}$$

$$\text{or} \quad \frac{f'''}{f'} + \frac{3g''}{g} = \frac{g'''}{g'} + \frac{3f''}{f}$$

$$\text{or} \quad \frac{f'''}{f'} - \frac{g'''}{g'} = 3\left(\frac{f''}{f} - \frac{g''}{g}\right)$$

EXAMPLE 3.11

If a curve is represented parametrically by the equations $x = f(t)$ and $y = g(t)$ then prove that $\dfrac{d^2y}{dx^2} = -\left[\dfrac{g'(t)}{f'(t)}\right]^3\left(\dfrac{d^2x}{dy^2}\right)$.

Sol. $\dfrac{dx}{dy} = \dfrac{1}{\dfrac{dy}{dx}}$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{1}{\frac{dy}{dx}}\right) = \frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right)\cdot\frac{dx}{dy}$$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)^2}\cdot\left(\frac{d^2y}{dx^2}\right)\left(\frac{dx}{dy}\right)$$

$$\therefore \quad \frac{d^2x}{dy^2} = \frac{-\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3} \quad \text{or} \quad \frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^3\cdot\frac{d^2x}{dy^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left(\frac{dy/dt}{dx/dt}\right)^3\cdot\frac{d^2x}{dy^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{g'(t)}{f'(t)}\right]^3\left(\frac{d^2x}{dy^2}\right)$$

EXAMPLE 3.12

If $f\left(\dfrac{x+y}{3}\right) = \dfrac{2+f(x)+f(y)}{3}$ for all real x and y and $f'(2) = 2$, then determine $y = f(x)$.

Sol. $$f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3} \quad (1)$$

Replacing x by $3x$ and y by 0, we get

$$f(x) = \frac{2+f(3x)+f(0)}{3}$$

$$\text{or} \quad f(3x) - 3f(x) + 2 = -f(0) \quad (2)$$

In (1) putting $x = 0$ and $y = 0$, we get

$$f(0) = 2 \quad (3)$$

Now, $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{2+f(3x)+f(3h)}{3} - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(3x)-3f(x)+f(3h)+2}{3h}$$

$$= \lim_{h\to 0} \frac{f(3h) - f(0)}{3h} \quad \text{[from (2)]}$$

$$= f'(0) = c \text{ (say)}$$

$\therefore \quad f'(x) = c$

At $\quad x = 2, f'(2) = c = 2$ (Given)

$\therefore \quad f'(x) = 2$

Integrating both sides, we get

$$f(x) = 2x + d \quad \text{[From (2)]}$$

$\therefore \quad d = 2$

Hence, $f(x) = 2x + 2$.

Alternative method:

$$f\left(\frac{x+y}{3}\right) = \frac{2 + f(x) + f(y)}{3} \quad (1)$$

Differentiating w.r.t. x keeping y as constant, we get

$$f'\left(\frac{x+y}{3}\right)\frac{1}{3} = \frac{f'(x)}{3}$$

Put $x = 2$ and $y = 3x - 2$. Then

$$f'(x) = 2$$

Integrating, we get $f(x) = 2x + c$.

Now, put $x = y = 0$ in (1). Then $3f(0) = 2 + 2f(0)$

or $\quad f(0) = 2$

Hence, $f(x) = 2x + 2$

EXAMPLE 3.13

If $f(xy) = \dfrac{f(x)}{y} + \dfrac{f(y)}{x}$ holds for all real x and y greater than 0 and $f(x)$ is a differentiable function for all $x > 0$ such that $f(e) = \dfrac{1}{e}$, then find $f(x)$.

Sol. We have

$$f(xy) = \frac{f(x)}{y} + \frac{f(y)}{x} \text{ for all } x, y > 0$$

or $\quad f(1) = f(1) + f(1)$ [Putting $x = y = 1$]

or $\quad f(1) = 0$

Now, $f(x)$ is differentiable for all $x > 0$. Therefore,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left\{x\left(1+\frac{h}{x}\right)\right\} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{f(x)}{1+\frac{h}{x}} + \dfrac{f\left(1+\frac{h}{x}\right)}{x} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{-\frac{h}{x}f(x)}{1+\frac{h}{x}} + \dfrac{f\left(1+\frac{h}{x}\right)}{x}}{h}$$

$$= -\frac{f(x)}{x} + \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{hx}$$

$$= -\frac{f(x)}{x} + \frac{1}{x^2} \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$$

$$= \frac{-f(x)}{x} + \frac{A}{x^2}, \text{ where } A = \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$$

$\therefore \quad \dfrac{d}{dx}(f(x)) = -\dfrac{f(x)}{x} + \dfrac{A}{x^2}$

or $\quad \dfrac{d}{dx}(f(x)) + \dfrac{f(x)}{x} = \dfrac{A}{x^2}$

or $\quad x\dfrac{d}{dx}(f(x)) + f(x) = \dfrac{A}{x}$

or $\quad \dfrac{d}{dx}[x f(x)] = \dfrac{A}{x}$

or $\quad x f(x) = A \log_e x + \log C$ [On integration]

Putting $x = 1$, we get

$$f(1) = A \log_e 1 + \log C$$

or $\quad 0 = \log C \quad [\because f(1) = 0]$

or $\quad xf(x) = A \log_e x$

Putting $x = e$, we get

$$ef(e) = A \log_e e$$

or $A = 1 \quad \left[\because f(e) = \dfrac{1}{e}\right]$

$\therefore \quad xf(x) = \log_e x$

or $\quad f(x) = \dfrac{\log_e x}{x}$

EXAMPLE 3.14

If $|a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx| \le |\sin x|$ for $x \in R$, then prove that $|a_1 + 2a_1 + 3a_3 + \cdots + na_n| \le 1$.

Sol. Clearly, we can get $a_1 + 2a_2 + 3a_3 + \cdots + na_n$, by differentiating $a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$ and then putting $x = 0$.

Thus, we have to prove that $|f'(0)| \le 1$.

where, $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$

$\therefore \quad f'(x) = a_1 \cos x + 2a_2 \cos 2x + \cdots + na_n \cos nx$

or $\quad f'(0) = a_1 + 2a_2 + \cdots + na_n$

Also, given $|f(x)| \le |\sin x|$ for $x \in R$ (1)

Put $x = 0$. Then $|f(0)| \le 0$ or $f(0) = 0$. Now,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} \quad [\text{as } f(0) = 0]$$

or $\quad |f'(0)| = \lim_{h\to 0} \left|\dfrac{f(h)}{h}\right| \le \lim_{h\to 0} \left|\dfrac{\sin h}{h}\right| = 1 \quad [\text{as } |f(x)| \le |\sin x|]$

Hence, $|f'(0)| \le 1$.

EXAMPLE 3.15

Suppose $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$, prove that $|a_1 + 2a_2 + \cdots + na_n| \le 1$.

Sol. Given $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

$\therefore$ $p'(x) = 0 + a_1 + 2a_2x + \cdots + na_nx^{n-1}$

or $p'(1) = a_1 + 2a_2 + \cdots + na_n$ (1)

Now, $|p(x)| \le |e^{x-1} - 1|$

$\therefore$ $|p(1)| \le 0$ $(\because |e^{1-1} - 1| = |e^0 - 1| = |1 - 1| = 0)$

or $p(1) = 0$ $[\therefore |p(1)| \ge 0]$

As $|p(x)| \le |e^{x-1} - 1|$, we get

$$|p(1+h)| \le |e^h - 1| \ \forall\ h > -1, h \ne 0$$

or $|p(1+h) - p(1)| \le |e^h - 1|$ $[\because p(1) = 0]$

or $$\left|\frac{p(1+h) - p(1)}{h}\right| \le \left|\frac{e^h - 1}{h}\right|$$

Taking limit as $h \to 0$, we get

$$\lim_{h\to 0}\left|\frac{p(1+h) - p(1)}{h}\right| \le \lim_{h\to 0}\left|\frac{e^h - 1}{h}\right|$$

or $|p'(1)| \le 1$

or $|a_1 + 2a_2 + \cdots + na_n| \le 1$ [From (1)]

Exercises

Single Correct Answer Type

1. If $y = a \sin x + b \cos x$, then $y^2 + \left(\frac{dy}{dx}\right)^2$ is a
   (1) function of x (2) function of y
   (3) function of x and y (4) constant

2. $\frac{d}{dx}\sqrt{\frac{1-\sin 2x}{1+\sin 2x}}$ is equal to, $(0 < x < \pi/4)$,
   (1) $\sec^2 x$ (2) $-\sec^2\left(\frac{\pi}{4}-x\right)$
   (3) $\sec^2\left(\frac{\pi}{4}+x\right)$ (4) $\sec^2\left(\frac{\pi}{4}-x\right)$

3. If $y = |\cos x| + |\sin x|$, then $\frac{dy}{dx}$ at $x = \frac{2\pi}{3}$ is
   (1) $\frac{1-\sqrt{3}}{2}$ (2) 0
   (3) $\frac{1}{2}(\sqrt{3}-1)$ (4) none of these

4. If $f(x) = |\log_e |x||$, then $f'(x)$ equals
   (1) $\frac{1}{|x|}$, where $x \neq 0$
   (2) $\frac{1}{x}$ for $|x| > 1$ and $-\frac{1}{x}$ for $|x| < 1$
   (3) $-\frac{1}{x}$ for $|x| > 1$ and $\frac{1}{x}$ for $|x| < 1$
   (4) $\frac{1}{x}$ for $x > 0$ and $-\frac{1}{x}$ for $x < 0$

5. If $f(x) = \sqrt{1-\sin 2x}$, then $f'(x)$ is equal to
   (1) $-(\cos x + \sin x)$, for $x \in (\pi/4, \pi/2)$
   (2) $\cos x + \sin x$, for $x \in (0, \pi/4)$
   (3) $-(\cos x + \sin x)$, for $x \in (0, \pi/4)$
   (4) $\cos x - \sin x$, for $x \in (\pi/4, \pi/2)$

6. Instead of the usual definition of derivative $Df(x)$, if we define a new kind of derivative $D^*F(x)$ by the formula $D^*(x) = \lim_{h\to 0}\frac{f^2(x+h)-f^2(x)}{h}$, where $f^2(x)$ means $[f(x)]^2$ and if $f(x) = x \log x$, then $D^*f(x)|_{x=e}$ has the value
   (1) e (2) $2e$
   (3) $4e$ (4) none of these

7. If $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ $(0 < x < \pi/2)$, then $\frac{dy}{dx} =$
   (1) $\frac{1}{2}$ (2) $\frac{2}{3}$ (3) 3 (4) 1

8. If $y = \frac{(a-x)\sqrt{a-x}-(b-x)\sqrt{x-b}}{\sqrt{a-x}+\sqrt{x-b}}$, then $\frac{dy}{dx}$ wherever it is defined is
   (1) $\frac{x+(a+b)}{\sqrt{(a-x)(x-b)}}$ (2) $\frac{2x-a-b}{2\sqrt{a-x}\sqrt{x-b}}$
   (3) $-\frac{(a+b)}{2\sqrt{(a-x)(x-b)}}$ (4) $\frac{2x+(a+b)}{2\sqrt{(a-x)(x-b)}}$

9. The derivative of $y = (1-x)(2-x)\cdots(n-x)$ at $x = 1$ is
   (1) 0 (2) $(-1)(n-1)!$
   (3) $n! - 1$ (4) $(-1)^{n-1}(n-1)!$

10. If $y = \sqrt{\frac{1-x}{1+x}}$, then $(1-x^2)\frac{dy}{dx}$ is equal to
    (1) y^2 (2) $1/y$ (3) $-y$ (4) $-y/x$

11. If $y = \frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$, then $\frac{dy}{dx}$ is equal to
    (1) $\frac{ay}{x\sqrt{a^2-x^2}}$ (2) $\frac{ay}{\sqrt{a^2-x^2}}$
    (3) $\frac{ay}{x\sqrt{x^2-a^2}}$ (4) none of these

12. Let $u(x)$ and $v(x)$ be differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to
    (1) 1 (2) 0 (3) 7 (4) -7

13. If $\sin^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \log a$, then $\frac{dy}{dx}$ is equal to
    (1) $\frac{x}{y}$ (2) $\frac{y}{x^2}$ (3) $\frac{x^2-y^2}{x^2+y^2}$ (4) $\frac{y}{x}$

14. Let $h(x)$ be differentiable for all x and let $f(x) = (kx + e^x)h(x)$, where k is some constant. If $h(0) = 5$, $h'(0) = -2$, and $f'(0) = 18$, then the value of k is
    (1) 5 (2) 4 (3) 3 (4) 2.2

15. If $\lim_{t\to x}\frac{e^t f(x)-e^x f(t)}{(t-x)(f(x))^2} = 2$ and $f(0) = \frac{1}{2}$, then find the value of $f'(0)$.
    (1) 4 (2) 2 (3) 0 (4) 1

16. If $f(0) = 0$, $f'(0) = 2$, then the derivative of $y = f(f(f(f(x))))$ at $x = 0$ is
    (1) 2 (2) 8 (3) 16 (4) 4

17. If $f(x) = \sqrt{1+\cos^2(x^2)}$, then $f'\left(\frac{\sqrt{\pi}}{2}\right)$ is
    (1) $\sqrt{\pi}/6$ (2) $-\sqrt{\pi/6}$ (3) $1/\sqrt{6}$ (4) $\pi/\sqrt{6}$

18. $\frac{d}{dx}\cos^{-1}\sqrt{\cos x}, 0<x<\frac{\pi}{2}$ is equal to

(1) $\frac{1}{2}\sqrt{1+\sec x}$ (2) $\sqrt{1+\sec x}$

(3) $-\frac{1}{2}\sqrt{1+\sec x}$ (4) $-\sqrt{1+\sec x}$

19. If $y=\tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$, then $\frac{dy}{dx}$ at $x=0$ is

(1) 1 (2) 2

(3) ln 2 (4) $-\frac{1}{10}\ln 2$

20. If $y=\left(x+\sqrt{x^2+a^2}\right)^n$, then $\frac{dy}{dx}$ is

(1) $\frac{ny}{\sqrt{x^2+a^2}}$ (2) $-\frac{ny}{\sqrt{x^2+a^2}}$

(3) $\frac{nx}{\sqrt{x^2+a^2}}$ (4) $-\frac{nx}{\sqrt{x^2+a^2}}$

21. If $y=\log_{\sin x}(\tan x)$, then $\left(\frac{dy}{dx}\right)_{\pi/4}$ is equal to

(1) $\frac{4}{\log 2}$ (2) $-4\log 2$

(3) $\frac{-4}{\log 2}$ (4) none of these

22. $\frac{d}{dx}\left[\sin^2\cot^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$ is equal to

(1) -1 (2) $\frac{1}{2}$ (3) $-\frac{1}{2}$ (4) 1

23. The differential coefficient of $f(\log_e x)$ with respect to x, where $f(x)=\log_e x$, is

(1) $\frac{x}{\log_e x}$ (2) $\frac{1}{x}\log_e x$

(3) $\frac{1}{x\log_e x}$ (4) none of these

24. If $f'(x)=\sqrt{2x^2-1}$ and $y=f(x^2)$, then $\frac{dy}{dx}$ at $x=1$ is

(1) 2 (2) 1

(3) -2 (4) none of these

25. If $u=f(x^3)$, $v=g(x^2)$, $f'(x)=\cos x$, and $g'(x)=\sin x$, then $\frac{du}{dv}$ is

(1) $\frac{3}{2}x\cos x^3\operatorname{cosec} x^2$ (2) $\frac{2}{3}\sin x^3\sec x^2$

(3) $\tan x$ (4) none of these

26. A function f, defined for all positive real numbers, satisfies the equation $f(x^2)=x^3$ for every $x>0$. Then the value of $f'(4)$ is

(1) 12 (2) 3

(3) 3/2 (4) cannot be determined

27. Let $f:(-5, 5)\to R$ be a differentiable function with $f(4)=1$, $f'(4)=1$, $f(0)=-1$ and $f'(0)=1$. If $g(x)=(f(2f^2(x)+2))^2$, then $g'(0)$ equals

(1) 4 (2) -4 (3) 8 (4) -8

28. The function $f(x)=e^x+x$, being differentiable and one-to-one, has a differentiable inverse $f^{-1}(x)$. The value of $\frac{d}{dx}(f^{-1})$ at $f(\log 2)$ is

(1) $\frac{1}{\ln 2}$ (2) $\frac{1}{3}$

(3) $\frac{1}{4}$ (4) none of these

29. If $f(x)=x+\tan x$ and f is the inverse of g, then $g'(x)$ equals

(1) $\frac{1}{1+[g(x)-x]^2}$ (2) $\frac{1}{2-[g(x)-x]^2}$

(3) $\frac{1}{2+[g(x)-x]^2}$ (4) none of these

30. If $f(x)=x^3+3x+4$ and g is the inverse function of $f(x)$, then the value of $\frac{d}{dx}\left(\frac{g(x)}{g(g(x))}\right)$ at $x=4$ equals

(1) $\frac{-1}{6}$ (2) 6

(3) $\frac{-1}{3}$ (4) non-existent

31. If $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, then $(1-x^2)\frac{dy}{dx}$ is equal to

(1) $x+y$ (2) $1+xy$ (3) $1-xy$ (4) $xy-2$

32. If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\cdots\infty}}}$, then $\frac{dy}{dx}$ is

(1) $\frac{x}{2y-1}$ (2) $\frac{x}{2y+1}$

(3) $\frac{1}{x(2y-1)}$ (4) $\frac{1}{x(1-2y)}$

33. For $0<x<\frac{1}{3}$, $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\sqrt{x}(3-x)}{1-3x}\right)\right]=$

(1) $\frac{1}{2(1+x)\sqrt{x}}$ (2) $\frac{3}{(1+x)\sqrt{x}}$

(3) $\frac{2}{(1+x)\sqrt{x}}$ (4) $\frac{3}{2(1+x)\sqrt{x}}$

34. Suppose the function $f(x)-f(2x)$ has the derivative 5 at $x=1$ and derivative 7 at $x=2$. The derivative of the function $f(x)-f(4x)$ at $x=1$ has the value equal to

(1) 19 (2) 9 (3) 17 (4) 14

35. If $y\sqrt{x^2+1}=\log(\sqrt{x^2+1}-x)$, then $(x^2+1)\frac{dy}{dx}+xy+1=$

(1) 0 (2) 1

(3) 2 (4) none of these

36. Let $e^y = \dfrac{\sqrt{1+\alpha}+\sqrt{1-\alpha}}{\sqrt{1+\alpha}-\sqrt{1-\alpha}}$ and $\tan\dfrac{x}{2} = \sqrt{\dfrac{1-\alpha}{1+\alpha}}$, $\alpha \in [-1, 0) \cup (0, 1]$. Then $\left(\dfrac{dy}{dx}\right)_{x=\frac{\pi}{3}} =$

(1) 1/2 (2) 1 (3) 2 (4) 1/3

37. The derivative of $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = 0$ is

(1) 1/8 (2) 1/4 (3) 1/2 (4) 1

38. If $\log_e((e-1)e^{xy} + x^2) = x^2 + y^2$, then $\dfrac{dy}{dx}$ at (1, 0) is equal to

(1) 0 (2) 1 (3) 2 (4) 3

39. If $y = x^{(x^x)}$, then $\dfrac{dy}{dx}$ is

(1) $y[x^x(\log ex)\log x + x^x]$
(2) $y[x^x(\log ex)\log x + x]$
(3) $y[x^x(\log ex)\log x + x^{x-1}]$
(4) $y[x^x(\log_e x)\log x + x^{x-1}]$

40. The first derivative of the function $\left[\cos^{-1}\left(\sin\sqrt{\dfrac{1+x}{2}}\right)+x^x\right]$ with respect to x at $x = 1$ is

(1) 3/4 (2) 0 (3) 1/2 (4) −1/2

41. Let $f(x) = x^x, x \in (0, \infty)$ and let $g(x)$ be inverse of $f(x)$, then $g'(x)$ must be

(1) $x(1 + \log x)$ (2) $x(1 + \log(x))$
(3) $\dfrac{1}{x(1+\log g(x))}$ (4) non-existent

42. If $y = ax^{n+1} + bx^{-n}$, then $x^2\dfrac{d^2y}{dx^2}$ is equal to

(1) $n(n-1)y$ (2) $n(n+1)y$
(3) ny (4) n^2y

43. If $y = ae^{mx} + be^{-mx}$, then $\dfrac{d^2y}{dx^2} - m^2y$ is equal to

(1) $m^2(ae^{mx} - be^{-mx})$ (2) 1
(3) 0 (4) none of these

44. Suppose $f(x) = e^{ax} + e^{bx}$, where $a \neq b$, and that $f''(x) - 2f'(x) - 15f(x) = 0$ for all x. Then the product ab is

(1) 25 (2) 9 (3) −15 (4) −9

45. $\dfrac{d^{20}y}{dx^{20}}(2\cos x\cos 3x)$ is equal to

(1) $2^{20}(\cos 2x - 2^{20}\cos 3x)$
(2) $2^{20}(\cos 2x + 2^{20}\cos 4x)$
(3) $2^{20}(\sin 2x + 2^{20}\sin 4x)$
(4) $2^{20}(\sin 2x - 2^{20}\sin 4x)$

46. $\dfrac{d^n}{dx^n}(\log x) =$

(1) $\dfrac{(n-1)!}{x^n}$ (2) $\dfrac{n!}{x^n}$
(3) $\dfrac{(n-2)!}{x^n}$ (4) $(-1)^{n-1}\dfrac{(n-1)!}{x^n}$

47. The nth derivative of the function $f(x) = \dfrac{1}{1-x^2}$ [where $x \in (-1, 1)$] at the point $x = 0$ where n is even is

(1) 0 (2) $n!$ (3) $^nC_2$ (4) 2^nC_2

48. If $y = x\log\left(\dfrac{x}{a+bx}\right)$, then $x^3\dfrac{d^2y}{dx^2} =$

(1) $x\dfrac{dy}{dx} - y$ (2) $\left(x\dfrac{dy}{dx} - y\right)^2$
(3) $y\dfrac{dy}{dx} - x$ (4) $\left(y\dfrac{dy}{dx} - x\right)^2$

49. If $ax^2 + 2hxy + by^2 = 1$, then $\dfrac{d^2y}{dx^2}$ is

(1) $\dfrac{h^2 - ab}{(hx + by)^3}$ (2) $\dfrac{ab - h^2}{(hx + by)^2}$
(3) $\dfrac{h^2 + ab}{(hx + by)^2}$ (4) none of these

50. If $y^{1/m} = (x + \sqrt{1+x^2})$, then $(1 + x^2)y_2 + xy_1$ is (where y_r represents the rth derivative of y w.r.t. x)

(1) m^2y (2) my^2
(3) m^2y^2 (4) none of these

51. If $(\sin x)(\cos y) = 1/2$, then d^2y/dx^2 at $(\pi/4, \pi/4)$ is

(1) −4 (2) −2 (3) −6 (4) 0

52. A function f satisfies the condition $f(x) = f'(x) + f''(x) + f'''(x) + \cdots$, where $f(x)$ is a differentiable function indefinitely and dash denotes the order of derivative. If $f(0) = 1$, then $f(x)$ is

(1) $e^{x/2}$ (2) e^x (3) e^{2x} (4) e^{4x}

53. Let $f(x)$ be a polynomial of degree 3 such that $f(3) = 1$, $f'(3) = -1$, $f''(3) = 0$, and $f'''(3) = 12$. Then the value of $f'(1)$ is

(1) 12 (2) 23 (3) −13 (4) none of these

54. If $y^2 = ax^2 + bx + c$, then $y^3\dfrac{d^2y}{dx^2}$ is

(1) a constant (2) a function of x only
(3) a function of y only (4) a function of x and y

55. If $y = \sin x + e^x$, then $\dfrac{d^2x}{dy^2} =$

(1) $(-\sin x + e^x)^{-1}$ (2) $\dfrac{\sin x - e^x}{(\cos x + e^x)^2}$
(3) $\dfrac{\sin x - e^x}{(\cos x + e^x)^3}$ (4) $\dfrac{\sin x + e^x}{(\cos x + e^x)^3}$

56. If $y = \sin px$ and y_n is the nth derivative of y, then $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ is

(1) 1 (2) 0
(3) −1 (4) none of these

57. If $f''(x) = -f(x)$ and $g(x) = f'(x)$ and

$$F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$$

and given that $F(5) = 5$, then $F(10)$ is

(1) 5 (2) 10 (3) 0 (4) 15

58. Let $y = \ln(1 + \cos x)^2$. Then the value of $\frac{d^2y}{dx^2} + \frac{2}{e^{y/2}}$ equals

(1) 0 (2) $\frac{2}{1+\cos x}$

(3) $\frac{4}{1+\cos x}$ (4) $\frac{-4}{(1+\cos x)^2}$

59. $x = t\cos t$, $y = t + \sin t$. Then $\frac{d^2x}{dy^2}$ at $t = \frac{\pi}{2}$ is

(1) $\frac{\pi+4}{2}$ (2) $-\frac{\pi+4}{2}$

(3) -2 (4) none of these

60. If $f(x) = (x-1)^4 (x-2)^3 (x-3)^2 (x-4)$, then the value of $f'''(1) + f''(2) + f'(3) + f'(4)$ equals

(1) 0 (2) 50 (3) 324 (4) 648

61. Let $y = t^{10} + 1$ and $x = t^8 + 1$. Then $\frac{d^2y}{dx^2}$ is

(1) $\frac{5}{2}t$ (2) $20t^8$

(3) $\frac{5}{16t^6}$ (4) none of these

62. If $x = \log p$ and $y = \frac{1}{p}$, then

(1) $\frac{d^2y}{dx^2} - 2p = 0$ (2) $\frac{d^2y}{dx^2} + y = 0$

(3) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ (4) $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$

63. If $x = \phi(t)$, $y = \psi(t)$, then $\frac{d^2y}{dx^2}$ is

(1) $\frac{\phi'\psi'' - \psi'\phi''}{(\phi')^2}$ (2) $\frac{\phi'\psi'' - \psi'\phi''}{(\phi')^3}$

(3) $\frac{\phi''}{\psi''}$ (4) $\frac{\psi''}{\phi''}$

64. If $f(x) = x^4\tan(x^3) - x\ln(1 + x^2)$, then the value of $\frac{d^4(f(x))}{dx^4}$ at $x = 0$ is

(1) 0 (2) 6 (3) 12 (4) 24

65. If graph of $y = f(x)$ is symmetrical about the y-axis and that of $y = g(x)$ is symmetrical about the origin and if $h(x) = f(x)\cdot g(x)$, then $\frac{d^3h(x)}{dx^3}$ at $x = 0$ is

(1) cannot be determined (2) $f(0)\cdot g(0)$
(3) 0 (4) none of these

66. Let $g(x)$ be the inverse of an invertible function $f(x)$, which is differentiable for all real x. Then $g''(f(x))$ equals

(1) $-\frac{f''(x)}{(f'(x))^3}$

(2) $\frac{f'(x)f''(x) - (f'(x))^3}{f'(x)}$

(3) $\frac{f'(x)f''(x) - (f'(x))^2}{(f'(x))^2}$

(4) none of these

67. $f(x) = e^x - e^{-x} - 2\sin x - \frac{2}{3}x^3$. Then the least value of n for which $\left.\frac{d^n}{dx^n}f(x)\right|_{x=0}$ is nonzero is

(1) 5 (2) 6 (3) 7 (4) 8

68. Let $y = f(x)$ and $x = \frac{1}{z}$. If $\frac{d^2y}{dx^2} = \lambda z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}$, then the value of λ is

(1) 1 (2) 2

(3) $\frac{1}{2}$ (4) $\frac{1}{4}$

69. Let $x = f(t)$ and $y = g(t)$, where x and y are twice differentiable function. If $f'(0) = g'(0) = f''(0) = 2$, $g''(0) = 6$, then the value of $\left(\frac{d^2y}{dx^2}\right)_{t=0}$ is equal to

(1) 0 (2) 1
(3) 2 (4) 3

70. If $f(x)$ satisfies the relation

$$f\left(\frac{5x-3y}{2}\right) = \frac{5f(x) - 3f(y)}{2} \; \forall\, x, y \in R,$$

and $f(0) = 3$ and $f'(0) = 2$, then the period of $\sin(f(x))$ is

(1) 2π (2) π (3) 3π (4) 4π

71. A function $f: R \to R$ satisfies $\sin x \cos y\,(f(2x + 2y) - f(2x - 2y)) = \cos x \sin y\,(f(2x + 2y) + f(2x - 2y))$. If $f'(0) = \frac{1}{2}$, then

(1) $f''(x) - f(x) = 0$
(2) $4f''(x) + f(x) = 0$
(3) $f''(x) + f(x) = 0$
(4) $4f''(x) - f(x) = 0$

Multiple Correct Answers Type

1. If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$, then $\frac{dy}{dx}$ is equal to

(1) $\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2\sqrt{x}}$ (2) $\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2x}$

(3) $\frac{1}{2\sqrt{x}}\sqrt{y^2 - 4}$ (4) $\frac{1}{2\sqrt{x}}\sqrt{y^2 + 4}$

2. Let $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \infty}}}$, then $\frac{dy}{dx}$ is equal to

(1) $\frac{1}{2y - 1}$ (2) $\frac{x}{x + 2y}$

(3) $\frac{1}{\sqrt{1 + 4x}}$ (4) $\frac{y}{2x + y}$

3. If $f(\theta) = \tan\left(\sin^{-1}\sqrt{\frac{2}{3 + \cos 2\theta}}\right)$, then

(1) $f\left(\frac{\pi}{4}\right) = 1$

(2) $f\left(\frac{\pi}{4}\right) = \sqrt{2}$

(3) $\frac{d(f(\theta))}{d(\cos\theta)}$ at $\theta = \frac{\pi}{4}$ is -2

(4) $f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

4. $f(x) = |x^2 - 3|x| + 2|$. Then which of the following is/are true?

(1) $f'(x) = 2x - 3$ for $x \in (0, 1) \cup (2, \infty)$

(2) $f'(x) = 2x + 3$ for $x \in (-\infty, -2) \cup (-1, 0)$

(3) $f'(x) = -2x - 3$ for $x \in (-2, -1)$

(4) None of these

5. Let $f(x) = \frac{\sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1} - 1}x$. Then

(1) $f'(10) = 1$

(2) $f'(3/2) = -1$

(3) domain of $f(x)$ is $x \geq 1$

(4) range of $f(x)$ is $(-2, -1] \cup (2, \infty)$.

6. If $x^3 - 2x^2y^2 + 5x + y - 5 = 0$ and $y(1) = 1$, then

(1) $y'(1) = 4/3$ (2) $y''(1) = -4/3$

(3) $y''(1) = -8\frac{22}{27}$ (4) $y'(1) = 2/3$

7. If $y = x^{(\log x)^{\log(\log x)}}$, then $\frac{dy}{dx}$ is

(1) $\frac{y}{x}((\ln x^{x-1}) + 2\ln x \ln(\ln x))$

(2) $\frac{y}{x}(\log x)^{\log(\log x)}(2\log(\log x) + 1)$

(3) $\frac{y}{x\ln x}[(\ln x)^2 + 2\ln(\ln x)]$

(4) $\frac{y}{x}\frac{\log y}{\log x}[2\log(\log x) + 1]$

8. If $f(x - y), f(x)\ f(y)$, and $f(x + y)$ are in A.P. for all x, y, and $f(0) \neq 0$, then

(1) $f(4) = f(-4)$ (2) $f(2) + f(-2) = 0$

(3) $f'(4) + f'(-4) = 0$ (4) $f'(2) = f'(-2)$

9. If $y = \cos^{-1}\left(\frac{2x}{1 + x^2}\right)$, then $\frac{dy}{dx}$ is

(1) $\frac{-2}{1 + x^2}$ for all x (2) $\frac{-2}{1 + x^2}$ for all $|x| < 1$

(3) $\frac{2}{1 + x^2}$ for $|x| > 1$ (4) none of these

10. $f_n(x) = e^{f_{n-1}(x)}$ for all $n \in N$ and $f_0(x) = x$, then $\frac{d}{dx}\{f_n(x)\}$ is

(1) $f_n(x)\frac{d}{dx}\{f_{n-1}(x)\}$

(2) $f_n(x)\ f_{n-1}(x)$

(3) $f_n(x) f_{n-1}(x) \cdots f_2(x) . f_1(x)$

(4) None of these

11. Suppose f and g are functions having second derivatives f'' and g'' everywhere. If $f(x) \cdot g(x) = 1$ for all x and f' and g' are never zero, then $\frac{f''(x)}{f'(x)} - \frac{g''(x)}{g'(x)}$ equals

(1) $\frac{-2f'(x)}{f(x)}$ (2) $-\frac{2g'(x)}{g(x)}$

(3) $\frac{-f'(x)}{f(x)}$ (4) $\frac{2f'(x)}{f(x)}$

12. If $y = e^{-x}\cos x$ and $y_n + k_n y = 0$, where $y_n = \frac{d^n y}{dx^n}$ and k_n are constants $\forall\ n \in N$, then

(1) $k_4 = 4$ (2) $k_8 = -16$ (3) $k_{12} = 20$ (4) $k_{16} = -24$

13. If a function is represented parametrically by the equations $x = \frac{1 + \log_e t}{t^2}$; $y = \frac{3 + 2\log_e t}{t}$, then which of the following statements are true?

(1) $y''(x - 2xy') = y$ (2) $yy' = 2x(y')^2 + 1$

(3) $xy' = 2y(y')^2 + 2$ (4) $y''(y - 4xy') = (y')^2$

14. If $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2 + 1} + \log_e\sqrt{x + \sqrt{x^2 + 1}}$, then

(1) $y' = x + \sqrt{x^2 + 1}$ (2) $y' = \frac{1}{x + \sqrt{x^2 + 1}}$

(3) $2y = xy' + \log_e y'$ (4) $2y = xy' - \log_e y'$

15. A curve parametrically given by $x = t + t^3$ and $y = t^2$, where $t \in R$. For what value(s) of t is $\frac{dy}{dx} = \frac{1}{2}$?

(1) 1/3 (2) 2 (3) 3 (4) 1

16. If $e^{\sin(x^2+y^2)} = \tan\frac{y^2}{4} + \sin^{-1} x$, then $y'(0)$ can be

(1) $\frac{1}{3\sqrt{\pi}}$ (2) $-\frac{1}{3\sqrt{\pi}}$ (3) $-\frac{1}{5\sqrt{\pi}}$ (4) $-\frac{1}{3\sqrt{5\pi}}$

Linked Comprehension Type

For Problems 1–3

$f(x)$ is a polynomial function, $f: R \to R$, such that $f(2x) = f'(x)f''(x)$.

1. The value of $f(3)$ is
   (1) 4 (2) 12 (3) 15 (4) none of these
2. $f(x)$ is
   (1) one-one and onto (2) one-one and into
   (3) many-one and onto (4) many-one and into
3. Equation $f(x) = x$ has
   (1) three real and positive roots
   (2) three real and negative roots
   (3) one real root
   (4) three real roots such that sum of roots is zero

For Problems 4–6

$f: R \to R, f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in R$.

4. The value of $f(1)$ is
   (1) 2 (2) 3 (3) –1 (4) 4
5. $f(x)$ is
   (1) one-one and onto (2) one-one and into
   (3) many-one and onto (4) many-one and into
6. The value of $f'(1) + f''(2) + f'''(3)$ is
   (1) 0 (2) –1 (3) 2 (4) 3

For Problems 7–9

Repeated roots : If equation $f(x) = 0$, where $f(x)$ is a polynomial function, has roots $\alpha, \alpha, \beta, \cdots$ or α root is repeated root, then $f(x) = 0$ is equivalent to $(x-\alpha)^2(x-\beta)\cdots = 0$, from which we can conclude that $f'(x) = 0$ or $2(x-\alpha)[(x-\beta)\cdots] + (x-\alpha)^2[(x-\beta)\cdots]' = 0$ or $(x-\alpha)[2\{(x-\beta)\cdots\} + (x-\alpha)\{(x-\beta)\cdots\}'] = 0$ has root α. Thus, if α root occurs twice in the, equation, then it is common in equations $f(x) = 0$ and $f'(x) = 0$.

Similarly, if α root occurs thrice in equation, then it is common in the equations $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$.

7. If $x - c$ is a factor of order m of the polynomial $f(x)$ of degree n $(1 < m < n)$, then $x = c$ is a root of the polynomial [where $f^r(x)$ represent rth derivative of $f(x)$ w.r.t. x]
   (1) $f^m(x)$ (2) $f^{m-1}(x)$
   (3) $f^n(x)$ (4) none of these
8. If $a_1x^3 + b_1x^2 + c_1x + d_1 = 0$ and $a_2x^3 + b_2x^2 + c_2x + d_2 = 0$ have a pair of repeated roots common, then
   $$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1 - a_1b_2 & c_1a_2 - c_2a_1 & d_1a_2 - d_2a_1 \end{vmatrix} =$$
   (1) 0 (2) 1 (3) –1 (4) 2
9. If α root occurs p times and β root occurs q times in polynomial equation $f(x) = 0$ of degree n $(1 < p, q < n)$, then which of the following is not true [where $f^r(x)$ represents rth derivative of $f(x)$ w.r.t. x]?
   (1) If $p < q < n$, then α and β are two of the roots of the equation $f^{p-1}(x) = 0$.
   (2) If $q < p < n$, then α and β are two of the roots of the equation $f^{q-1}(x) = 0$.
   (3) If $p < q < n$, then equations $f(x) = 0$ and $f^p(x) = 0$ have exactly one root common.
   (4) If $q < p < n$, then equations $f^q(x) = 0$ and $f^p(x) = 0$ have exactly two roots common.

For Problems 10–12

Equation $x^n - 1 = 0$, $n > 1$, $n \in N$, has roots $1, a_1, a_2, \ldots, a_{n-1}$.

10. The value of $(1-a_1)(1-a_2)\cdots(1-a_{n-1})$ is
    (1) $n^2/2$ (2) n
    (3) $(-1)^n n$ (4) none of these
11. The value of $\sum_{r=1}^{n-1} \frac{1}{2-a_r}$ is
    (1) $\frac{2^{n-1}(n-2)+1}{2^n-1}$ (2) $\frac{2^n(n-2)+1}{2^n-1}$
    (3) $\frac{2^{n-1}(n-1)-1}{2^n-1}$ (4) none of these
12. The value of $\sum_{r=1}^{n-1} \frac{1}{1-a_r}$ is
    (1) $\frac{n}{4}$ (2) $\frac{n(n-1)}{2}$
    (3) $\frac{n-1}{2}$ (4) none of these

For Problems 13–15

$f(x) = x^2 + xg'(1) + g''(2)$ and $g(x) = f(1)x^2 + xf'(x) + f''(x)$.

13. The value of $f(3)$ is
    (1) 1 (2) 0 (3) –1 (4) –2
14. The value of $g(0)$ is
    (1) 0 (2) –3
    (3) 2 (4) none of these
15. The domain of the function $\sqrt{\frac{f(x)}{g(x)}}$ is
    (1) $(-\infty, 1] \cup (2, 3]$ (2) $(-2, 0] \cup (1, \infty)$
    (3) $(-\infty, 0] \cup (2/3, 3]$ (4) none of these

For Problems 16–18

$g(x + y) = g(x) + g(y) + 3xy(x + y)\ \forall\ x, y \in R$ and $g'(0) = -4$.

16. Number of real roots of the equation $g(x) = 0$ is
    (1) 2 (2) 0 (3) 1 (4) 3
17. For which of the following values of x is $\sqrt{g(x)}$ not defined?
    (1) $[-2, 0]$ (2) $[2, \infty)$
    (3) $[-1, 1]$ (4) none of these

18. The value of $g'(1)$ is

(1) 0 (2) 1
(3) −1 (4) none of these

For Problems 19–21

A curve is represented parametrically by the equations $x = f(t) = a^{\ln(b^t)}$ and $y = g(t) = b^{-\ln(a^t)}$ $a, b > 0$ and $a \neq 1, b \neq 1$ where $t \in R$.

19. Which of the following is not a correct expression for $\frac{dy}{dx}$?

(1) $\frac{-1}{f(t)^2}$ (2) $-(g(t))^2$
(3) $\frac{-g(t)}{f(t)}$ (4) $\frac{-f(t)}{g(t)}$

20. The value of $\frac{d^2y}{dx^2}$ at the point where $f(t) = g(t)$ is

(1) 0 (2) $\frac{1}{2}$ (3) 1 (4) 2

21. The value of $\frac{f(t)}{f'(t)} \cdot \frac{f''(-t)}{f'(-t)} + \frac{f(-t)}{f'(-t)} \cdot \frac{f''(t)}{f'(t)}$ $\forall\, t \in R$ is equal to

(1) −2 (2) 2 (3) −4 (4) 4

For Problems 22 and 23

Let $f: R \to R$ be a differentiable function satisfying $f(x + y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. If $\lim_{x\to 0} \frac{f(x)}{x} = 1$, then

22. The value of $f'(3)$ is

(1) 8 (2) 10
(3) 12 (4) 18

23. The value of $f(9)$ is

(1) 240 (2) 356
(3) 252 (4) 730

Matrix Match Type

1. Match the following lists:

List I	List II
a. Differentiable function $f(x)$ satisfies the relation $f(1 - x) = f(1 + x)$ for all $x \in R$.	**p.** Graph of $f'(x)$ is symmetrical about point (1, 0).
b. Differentiable function $f(x)$ satisfies the relation $f(2 - x) + f(x) = 0$ for all $x \in R$.	**q.** Graph of $f'(x)$ is symmetrical about line $x = 1$.
c. Differentiable function $f(x)$ satisfies the relation $f(x + 2) + f(x) = 0$ for all $x \in R$.	**r.** $f'(-1) = f'(3)$.
d. Differentiable function $f(x)$ satisfies the relation $f(x) + f(y) + f(x) \cdot f(y) = 1$ for all x, y and $f(x) > 0$.	**s.** $f'(x)$ has period 4.

2. Match the following lists:

List I	List II
a. Let $y = f(x)$ be given by $x = t^5 - 5t^3 - 20t + 7$ and $y = 4t^3 - 3t^2 - 18t + 3$. Then $-5 \times \frac{dy}{dx}$ at $t = 1$ is	**p.** 0
b. Let $P(x)$ be a polynomial of degree 4, with $P(2) = -1$, $P'(2) = 0$, $P''(2) = 2$, $P'''(2) = -12$, and $P^{iv}(2) = 24$. Then $P''(3)$ is	**q.** −2
c. $y = \frac{1}{x}$. Then $\frac{\frac{dy}{\sqrt{1+y^4}}}{\frac{dx}{\sqrt{1+x^4}}}$ is	**r.** 2
d. $f\left(\frac{2x+3y}{5}\right) = \frac{2f(x)+3f(y)}{5}$ and $f'(0) = p$ and $f(0) = q$. Then $f''(0)$ is	**s.** −1

3. Match the following lists:

List I	List II
a. $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$. Then $\frac{dy}{dx} = -\frac{1}{1+x^2}$	**p.** for $x < 0$
b. $y = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$. Then $\frac{dy}{dx} = -\frac{1}{1+x^2}$	**q.** for $x > 1$
c. $y = \lvert e^{\lvert x\rvert} - e\rvert$. Then $\frac{dy}{dx} > 0$	**r.** for $x < -1$
d. $u = \log\lvert 2x\rvert$, $v = \lvert\tan^{-1}x\rvert$. Then $du/dv > 2$	**s.** for $-1 < x < 0$

4. Match the values of x in List II where derivative of the function in List I is negative.

List I	List II
a. $y = \lvert x^2 - 2\lvert x\rvert\rvert$	**p.** (1, 2)
b. $y = \lvert\log_e\lvert x\rvert\rvert$	**q.** (−3, −2)
c. $y = x[x/2]$, where [·] represents greatest integer function	**r.** (−1, 0)
d. $y = \lvert\sin x\rvert$	**s.** (0, 1)

5. Match the relation for derivatives given in List II with the relation given in List I and then choose the correct code.

List I	List II
a. $xy - \log y = 1$	**p.** $xy_2 - \frac{1}{2}y_1 - \frac{1}{4}y = 0$
b. $y = (\sin^{-1}x)/\sqrt{1-x^2}$	**q.** $(1 - x^2)y_1 - xy = 1$
c. $y = \sqrt{2x - x^2}$	**r.** $y^2 + (xy - 1)\,y_1 = 0$
d. $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$	**s.** $y^3y_2 + 1 = 0$

Codes:

	a	b	c	d
(1)	q	p	s	r
(2)	s	p	q	r
(3)	r	q	s	p
(4)	q	p	r	s

Numerical Value Type

1. $f'(x) = \phi(x)$ and $\phi'(x) = f(x)$ for all x. Also, $f(3) = 5$ and $f'(3) = 4$. Then the value of $[f(10)]^2 - [\phi(10)]^2$ is______.

2. If $y = f(x)$ is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = -2$, then $f'(-3)$ equals______.

3. If $x^3 + 3x^2 - 9x + c$ is of the form $(x - \alpha)^2 (x - \beta)$, then the positive value of c is______.

4. If graph of $y = f(x)$ is symmetrical about the point (5, 0) and $f'(7) = 3$, then the value of $f'(3)$ is______.

5. Let $g(x) = f(x) \sin x$, where $f(x)$ is a twice differentiable function on $(-\infty, \infty)$ such that $f'(-\pi) = 1$. The value of $g''(-\pi)$ equals______.

6. Let $f(x) = (x-1)(x-2)(x-3) \cdots (x-n)$, $n \in N$, and $f'(n) = 5040$. Then the value of n is______.

7. $y = f(x)$, where f satisfies the relation $f(x+y) = 2f(x) + xf(y) + y\sqrt{f(x)}$ $\forall x, y \in R$ and $f'(0) = 0$. Then $f(6)$ is equal to______.

8. If function f satisfies the relation $f(x) \times f'(-x) = f(-x) \times f'(x)$ for all x, and $f(0) = 3$, and if $f(3) = 3$, then the value of $f(-3)$ is______.

9. If $y = \dfrac{a + bx^{3/2}}{x^{5/4}}$ and $y' = 0$ at $x = 5$, then the value of a^2/b^2 is______.

10. Let $y = \dfrac{2^{\log_{2^{1/4}} x} - 3^{\log_{27}(x^2+1)^3} - 2x}{7^{4\log_{49} x} - x - 1}$ and $\dfrac{dy}{dx} = ax + b$, then the value of $a + b$ is______.

11. $\lim\limits_{h \to 0} \dfrac{(e+h)^{\ln(e+h)} - e}{h}$ is______.

12. If the function $f(x) = -4e^{\frac{1-x}{2}} + 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3}$ and $g(x) = f^{-1}(x)$, then the reciprocal of $g'\left(\dfrac{-7}{6}\right)$ is______.

13. Suppose that $f(0) = 0$ and $f'(0) = 2$, and let $g(x) = f(-x + f(f(x)))$. The value of $g'(0)$ is equal to______.

14. $f(x)$ is a polynomial of degree 4 with real coefficients such that $f(x) = 0$ satisfied by $x = 1, 2, 3$ only then $f'(1) f'(2) f'(3)$ is equal to______.

15. A nonzero polynomial with real coefficients has the property that $f(x) = f'(x) \cdot f''(x)$. If a is the leading coefficient of $f(x)$, then the value of $1/a$ is______.

16. A function is represented parametrically by the equations $x = \dfrac{1+t}{t^3}$; $y = \dfrac{3}{2t^2} + \dfrac{2}{t}$. Then the value of $\left|\dfrac{dy}{dx} - x\left(\dfrac{dy}{dx}\right)^3\right|$ is______.

17. Let $z = (\cos x)^5$ and $y = \sin x$. Then the value of $\dfrac{d^2z}{dy^2}$ at $x = \dfrac{2\pi}{9}$ is______.

18. Let $g(x) = \begin{cases} \dfrac{x^2 + x\tan x - x\tan 2x}{ax + \tan x - \tan 3x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

If $g'(0)$ exists and is equal to nonzero value b, then $52\dfrac{b}{a}$ is equal to______.

19. Let $f(x) = x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + \cdots \infty}}}$.

Then the value of $f(50) \cdot f'(50)$ is______.

20. Let $F(x) = f(x)\, g(x)\, h(x)$ for all real x, where $f(x)$, $g(x)$, and $h(x)$ are differentiable functions. At some point x_0, $F'(x_0) = 21F(x_0)$, $f'(x_0) = 4f(x_0)$, $g'(x_0) = -7g(x_0)$, and $h'(x_0) = kh(x_0)$. Then k =______.

21. If $y = \dfrac{\sqrt[3]{1+3x}\,\sqrt[4]{1+4x}\,\sqrt[5]{1+5x}}{\sqrt[7]{1+7x}\,\sqrt[8]{1+8x}}$, then $y'(0)$ is equal to______.

Archives

JEE MAIN

Single Correct Answer Type

1. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals

(1) -1 (2) 1 (3) log 2 (4) $-\log 2$

(AIEEE 2009)

2. Let $f: (-1, 1) \to R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0)$ is equal to

(1) -2 (2) 4
(3) -4 (4) 0

(AIEEE 2010)

3. $\dfrac{d^2x}{dy^2}$ equals

(1) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$ (2) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$

(3) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$ (4) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$

(AIEEE 2011)

4. If $y = \sec(\tan^{-1} x)$, then $\dfrac{dy}{dx}$ at $x = 1$ is equal to

(1) 1/2 (2) 1 (3) $\sqrt{2}$ (4) $1/\sqrt{2}$

(AIEEE 2011)

5. If g is the inverse of a function f and $f'(x) = \dfrac{1}{1+x^5}$, then $g'(x)$ is equal to

(1) $1 + x^5$ (2) $5x^4$

(3) $\dfrac{1}{1+\{g(x)\}^5}$ (4) $1 + \{g(x)\}^5$

(JEE Main 2014)

6. If for $x \in \left(0, \dfrac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\dfrac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x} \cdot g(x)$, then $g(x)$ equals

(1) $\dfrac{3}{1+9x^3}$ (2) $\dfrac{9}{1+9x^3}$ (3) $\dfrac{3x\sqrt{x}}{1-9x^3}$ (4) $\dfrac{3x}{1-9x^3}$

(JEE Main 2017)

JEE ADVANCED

Multiple Correct Answers Type

1. Let $f(x) = x \sin \pi x$, $x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at

(1) a unique point in the interval $\left(n, n + \dfrac{1}{2}\right)$

(2) a unique point in the interval $\left(n + \dfrac{1}{2}, n + 1\right)$

(3) a unique point in the interval $(n, n + 1)$

(4) two points in the interval $(n, n + 1)$

(JEE Advanced 2013)

2. Let $f : (0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \dfrac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then

(1) $\lim\limits_{x \to 0^+} f'\left(\dfrac{1}{x}\right) = 1$ (2) $\lim\limits_{x \to 0^+} x f\left(\dfrac{1}{x}\right) = 2$

(3) $\lim\limits_{x \to 0^+} x^2 f'(x) = 0$ (4) $|f(x)| \le 2$ for all $x \in (0, 2)$

(JEE Advanced 2016)

3. Let $f : R \to R$, $g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then

(1) $g'(2) = \dfrac{1}{15}$ (2) $h'(1) = 666$

(3) $h(0) = 16$ (4) $h(g(3)) = 36$

(JEE Advanced 2016)

Matrix Match Type

1. Match the statements/expressions given in List I with the values given in List II.

List I	List II
a. Let $y(x) = \cos(3\cos^{-1} x)$, $x \in [-1, 1]$, $x\ \pi \pm \dfrac{\sqrt{3}}{2}$. Then $\dfrac{1}{y(x)}\left\{(x^2-1)\dfrac{d^2y(x)}{dx^2} + x\dfrac{dy(x)}{dx}\right\}$ equals	**p.** 1
b. Let $A_1, A_2, \ldots, A_n$ $(n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a}_k$ be the position vector of the point A_k, $k = 1, 2, \ldots n$. If $\left\lvert\sum_{k=1}^{n-1}(\vec{a}_k \times \vec{a}_{k+1})\right\rvert = \left\lvert\sum_{k=1}^{n-1}(\vec{a}_k \cdot \vec{a}_{k+1})\right\rvert$, then the minimum value of n is	**q.** 2
c. If the normal from the point $P(h, 1)$ on the ellipse $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$ is perpendicular to the line $x + y = 8$, then the value of h is	**r.** 8
d. Number of positive solutions satisfying the equation $\tan^{-1}\left(\dfrac{1}{2x+1}\right) + \tan^{-1}\left(\dfrac{1}{4x+1}\right) = \tan^{-1}\left(\dfrac{2}{x^2}\right)$ is	**s.** 9

Codes:

	a	b	c	d
(1)	s	r	q	p
(2)	q	s	r	p
(3)	s	r	p	q
(4)	q	s	p	r

(JEE Advanced 2014)

Numerical Value Type

1. Let $f(\theta) = \sin\left(\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4}$.

Then the value of $\dfrac{d}{d(\tan\theta)}(f(\theta))$ is _____.

(IIT-JEE, 2011)

2. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point $(1, 3)$ is **(JEE Advanced 2014)**

3. Let $f : R \to R$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$f(x + y) = f(x)f'(y) + f'(x)f(y)$ for all $x, y \in R$.

Then, the value of $\log_e(f(4))$ is __________ .

(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (4)	2. (2)	3. (3)	4. (2)	5. (3)
6. (3)	7. (1)	8. (2)	9. (2)	10. (3)
11. (1)	12. (1)	13. (4)	14. (3)	15. (3)
16. (3)	17. (2)	18. (1)	19. (4)	20. (1)
21. (3)	22. (2)	23. (3)	24. (1)	25. (1)
26. (2)	27. (4)	28. (2)	29. (3)	30. (3)
31. (2)	32. (3)	33. (4)	34. (1)	35. (1)
36. (3)	37. (2)	38. (3)	39. (3)	40. (1)
41. (3)	42. (2)	43. (3)	44. (3)	45. (2)
46. (4)	47. (2)	48. (2)	49. (1)	50. (1)
51. (1)	52. (1)	53. (2)	54. (1)	55. (3)
56. (2)	57. (1)	58. (1)	59. (2)	60. (4)
61. (3)	62. (3)	63. (2)	64. (1)	65. (1)
66. (1)	67. (3)	68. (2)	69. (2)	70. (2)
71. (2)				

Multiple Correct Answers Type

1. (1), (3)	2. (1), (3), (4)
3. (2), (3), (4)	4. (1), (2), (3)
5. (1), (2), (4)	6. (1), (3)
7. (2), (4)	8. (1), (3)
9. (2), (3)	10. (1), (3)
11. (2), (4)	12. (1), (2)
13. (2), (4)	14. (1), (3)
15. (1), (4)	16. (1), (2), (4)

Linked Comprehension Type

1. (2)	2. (1)	3. (4)	4. (4)	5. (3)
6. (4)	7. (2)	8. (1)	9. (4)	10. (2)
11. (1)	12. (3)	13. (2)	14. (3)	15. (3)
16. (4)	17. (3)	18. (3)	19. (4)	20. (4)
21. (2)	22. (2)	23. (3)		

Matrix Match Type

1. a → p; b → q, r; c → s, r; d → q, r
2. a → q; b → r; c → s; d → p
3. a → q, r; b → p, r, s; c → q, s; d → q, r
4. a → p, q, r; b → q, s; c → q, r; d → r
5. (3)

Numerical Value Type

1. (9)	2. (−2)	3. (5)	4. (3)	5. (−2)
6. (8)	7. (9)	8. (3)	9. (5)	10. (3)
11. (2)	12. (5)	13. (6)	14. (0)	15. (18)
16. (1)	17. (2.5)	18. (7)	19. (50)	20. (24)
21. (1)				

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (1)	2. (3)	3. (1)	4. (4)	5. (4)
6. (2)				

JEE ADVANCED

Multiple Correct Answers Type

1. (2), (3)   2. (1)   3. (2), (3)

Matrix Match Type

1. (1)

Numerical Value Type

1. (1)   2. (8)   3. (2)

4 Continuity and Differentiability

CONTINUITY OF FUNCTION

CONTINUITY

The graph of the functions like x^2, x^3, $\sin x$, e^x and $\log_e x$ etc. can be drawn on the paper continuously in one go without lifting the pen. Such functions are called continuous functions which have no breaks in their graphs.

On the contrary, graph of signum function (sgn(x)) is broken at $x = 0$ and graph of greatest integer function ($[x]$) is broken at all integers. Such functions are not continuous in their domains.

Mathematically, if function $f(x)$ is continuous at $x = a$, then its left hand limit and right hand limit at $x = a$ must have the same value that must be equal to the value of the function at that point, i.e., equal to $f(a)$.

Unlike limits, for continuity, it is essential for the function to be defined at that particular point and the limiting value of the function should be equal to the value of the function at that point. Thus, function $f(x)$ is said to be continuous at $x = a$ if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a).$$

e.g., for $f(x) = \sin x$, we have

$$\underbrace{\lim_{x\to\pi^-} \sin x}_{\text{L.H.L. at } x=\pi} = \underbrace{\lim_{x\to\pi^+} \sin x}_{\text{R.H.L. at } x=\pi} = \underbrace{\sin\pi}_{f(\pi)} = 0$$

So, $\sin x$ is continuous at $x = \pi$.

Similarly, we find that $\sin x$ is continuous for all real x.

ILLUSTRATION 4.1

A function $f(x)$ satisfies the functional relation $f(x + y) = f(x) \times f(y)$ for all real x and y. Show that the function is continuous for all values of x if it is continuous at $x = 1$.

Sol. It is given that $f(x)$ is continuous at $x = 1$.

$\therefore \quad \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$

$\Rightarrow \quad \lim_{h\to0} f(1-h) = \lim_{h\to0} f(1+h) = f(1)$

$\Rightarrow \quad \lim_{h\to0} f(1)\cdot f(-h) = \lim_{h\to0} f(1)\cdot f(h) = f(1)$ (Using $f(x+y) = f(x)\cdot f(y)$)

$\Rightarrow \quad \lim_{h\to0} f(-h) = \lim_{h\to0} f(h) = 1 \qquad (1)$

Now consider some arbitrary point $x = a$.

$$\begin{aligned}\text{L.H.L.} &= \lim_{h\to0} f(a-h)\\ &= \lim_{h\to0} f(a).f(-h)\\ &= f(a)\lim_{h\to0} f(-h) = f(a) \qquad \text{[Using (1)]}\end{aligned}$$

$$\begin{aligned}\text{R.H.L.} &= \lim_{h\to0} f(a+h)\\ &= \lim_{h\to0} f(a).f(h)\\ &= f(a)\lim_{h\to0} f(h) = f(a) \qquad \text{[Using (1)]}\end{aligned}$$

Hence, at $x = a$, we have L.H.L. = R.H.L. = $f(a)$

Thus, function is continuous for all values of x.

ILLUSTRATION 4.2

If $|f(x)| \le |x|$ for all $x \in R$ then prove that $f(x)$ is continuous at 0.

Sol. We have $|f(x)| \le |x|$

or $\quad 0 \le |f(x)| \le |x|$

$|f(0)| \le 0 \quad \Rightarrow f(0) = 0$

Also, using Sandwich theorem of limits,

$$\lim_{x\to0} 0 \le \lim_{x\to0} |f(x)| \le \lim_{x\to0} |x|$$

$\Rightarrow \quad \lim_{x\to0} |f(x)| = 0$

$\Rightarrow \quad \lim_{x\to0} f(x) = 0 = f(0)$

$\therefore \quad f(x)$ is continuous at $x = 0$.

ILLUSTRATION 4.3

Let $f(x) = \dfrac{\log_e(1+x)^{1+x} - x}{x^2}$, then find the value of $f(0)$ so that the $f(x)$ is continuous at $x = 0$.

Sol. Since $f(x)$ is continuous at $x = 0$, we must have

$$\begin{aligned}f(0) &= \lim_{x\to0} f(x)\\ &= \lim_{x\to0} \frac{(1+x)\log_e(1+x) - x}{x^2} \qquad \left(\text{form } \frac{0}{0}\right)\\ &= \lim_{x\to0} \frac{\log_e(1+x) + 1 - 1}{2x} \qquad \text{(Using L'Hospital Rule)}\\ &= \frac{1}{2}\lim_{x\to0} \frac{\log_e(1+x)}{x}\\ &= \frac{1}{2}\end{aligned}$$

ILLUSTRATION 4.4

If $f(x)$ be continuous function for all real values of x and satisfies, $x^2 + \{f(x) - 2\}x + 2\sqrt{3} - 3 - \sqrt{3}f(x) = 0$, $\forall\, x \in R$. Then find the value of $f(\sqrt{3})$.

Sol. From the given equation, we have

$$f(x) = \frac{x^2 - 2x + 2\sqrt{3} - 3}{\sqrt{3} - x}, x \neq \sqrt{3}$$

As $f(x)$ is continuous for all $x \in R$, we must have

$$f(\sqrt{3}) = \lim_{x \to \sqrt{3}} f(x)$$

$$= \lim_{x \to \sqrt{3}} \frac{x^2 - 2x + 2\sqrt{3} - 3}{\sqrt{3} - x}$$

$$= \lim_{x \to \sqrt{3}} \frac{(2 - \sqrt{3} - x)(\sqrt{3} - x)}{(\sqrt{3} - x)}$$

$$= 2(1 - \sqrt{3})$$

Hence, $f(\sqrt{3}) = 2(1 - \sqrt{3})$

ILLUSTRATION 4.5

Let $f(x) = \begin{cases} \dfrac{a + 3\cos x}{x^2}, & x < 0 \\ b\tan\left(\dfrac{\pi}{[x+3]}\right), & x \geq 0 \end{cases}$

If $f(x)$ is continuous at $x = 0$, then find a and b, where $[\cdot]$ denotes the greatest integer function.

Sol. $f(x)$ is continuous at $x = 0$

Now $f(0^+) = \lim_{x \to 0^+} f(x)$

$$= \lim_{x \to 0^+} b\tan\left(\frac{\pi}{[x+3]}\right)$$

$$= \lim_{h \to 0} b\tan\left(\frac{\pi}{[0+h+3]}\right)$$

$$= b\tan\left(\frac{\pi}{3}\right) = \sqrt{3}b$$

$$f(0^-) = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} \frac{a + 3\cos x}{x^2}$$

For this limit to exists finitely, we must have $\lim_{x \to 0^-}(a + 3\cos x) = 0$

$\therefore \quad a + 3\cos 0 = 0 \Rightarrow a + 3 = 0 \Rightarrow a = -3$

$\therefore \quad f(0^-) = \lim_{x \to 0^-} \dfrac{-3 + 3\cos x}{x^2}$

$$= -3\lim_{x \to 0^-} \frac{1 - \cos x}{x^2}$$

$$= -3\lim_{x \to 0^-} \frac{2\sin^2 \frac{x}{2}}{4\frac{x^2}{4}}$$

$$= -\frac{3}{2}$$

Since $f(x)$ is continuous at $x = 0$, we have

$$\sqrt{3}b = \frac{-3}{2} \Rightarrow b = \frac{-\sqrt{3}}{2}$$

ILLUSTRATION 4.6

Determine the values of a, b, c for which the function

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}; & x < 0 \\ c; & x = 0 \\ \dfrac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}; & x > 0 \end{cases}$$

is continuous at $x = 0$.

Sol. We have $f(0) = c$ ...(1)

L.H.L. $= \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0^-} \frac{\sin(a+1)x + \sin x}{x}$$

$$= \lim_{x \to 0^-} \left[(a+1)\frac{\sin(a+1)x}{(a+1)x} + \frac{\sin x}{x}\right]$$

$$= (a + 1) + 1$$

$$= a + 2 \quad \text{...(2)}$$

R.H.L. $= \lim_{x \to 0^+} f(x)$

$$= \lim_{x \to 0^+} \frac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$$

$$= \lim_{x \to 0^+} \frac{(1 + bx)^{1/2} - 1}{bx}$$

$$= \frac{1}{2} \quad \left(\because \lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n\right) \quad \text{...(3)}$$

Since $f(x)$ is continuous at $x = 0$, we have

L.H.L. = R.H.L. = $f(0)$

$$\Rightarrow \quad c = a + 2 = \frac{1}{2}$$

$$\Rightarrow \quad a = -\frac{3}{2}, b \in R, c = \frac{1}{2}$$

ILLUSTRATION 4.7

If the function $g(x) = \begin{cases} \dfrac{e^{px} + \log_e(1 + 4x) + q}{x^3}, & x \neq 0 \\ r, & x = 0 \end{cases}$ is continuous at $x = 0$, then find the values of p, q and r.

Sol. Since $f(x)$ is continuous at $x = 0$, we must have

$$r = \lim_{x \to 0} \frac{e^{px} + \log_e(1 + 4x) + q}{x^3}$$

$$= \lim_{x \to 0} \frac{\left(1 + (px) + \frac{p^2x^2}{2!} + \frac{p^3x^3}{3!} + \ldots\right) + \left(4x - \frac{16x^2}{2} + \frac{64}{3}x^3 + \ldots\right) + q}{x^3}$$

$$= \lim_{x \to 0} \frac{(1+q)+(p+4)x+\left(\frac{p^2}{2}-8\right)x^2+\left(\frac{p^3}{6}+\frac{64}{3}\right)x^3+\ldots}{x^3}$$

$\therefore \quad 1+q=0, p+4=0, \frac{p^2}{2}-8=0$ and $\frac{p^3}{6}+\frac{64}{3}=r$

$\therefore \quad q=-1, p=-4, r=\frac{32}{3}$

DISCONTINUITY AND ITS TYPES

The strict condition for function $f(x)$ is to be continuous at $x = a$ is $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$. If at least one of these three value differ from others or at least one of them does not exist we have discontinuity.

So, $f(x)$ is said to be discontinuous at $x = a$ if

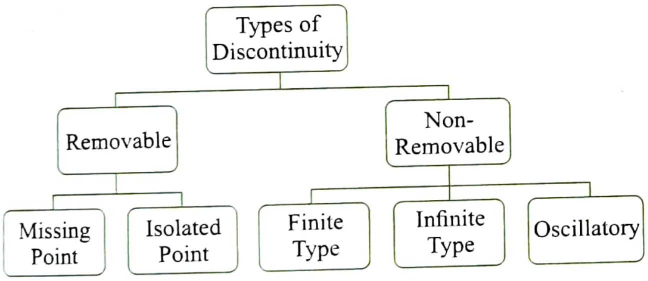
(i) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists but are not equal

(ii) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists and are equal but not equal to $f(a)$.

(iii) $f(a)$ is not defined.

(iv) At least one of the limits does not exists.

There are various types of discontinuities which are classified as shown in the following chart.



Removable Discontinuity

In case of removable discontinuity, the limit of the function $\lim_{x\to a} f(x)$ necessarily exists but it is either not equal to the value of the function at 'a' i.e. $f(a)$ or $f(a)$ is not defined. However, it is possible to redefine the function in such a way that the limit equals the value of the function at that particular point i.e. $\lim_{x\to a} f(x) = f(a)$.

Removable discontinuity can further be divided into missing point discontinuity and isolated point discontinuity.

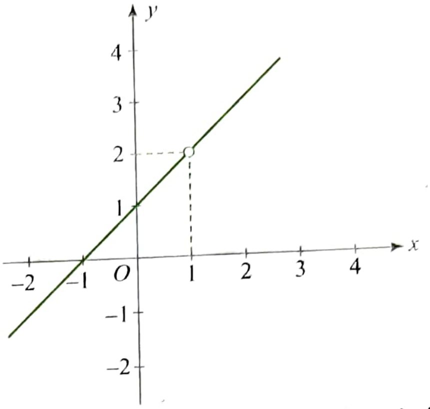
I. Missing Point Discontinuity

The discontinuity is said to be of missing point type if the limit of the function exists at point 'a' but the function is not defined at that point, i.e. $\lim_{x\to a} f(x)$ exists but $f(a)$ is not defined.

Consider the function $f(x) = \frac{x^2-1}{x-1}$, where $x \neq 1$.

$\therefore \quad f(x) = x + 1, x \neq 1.$

Graph of the function is as shown in the following figure.



In the above graph we observe that graph has 'hole'(missing point) at $x = 1$, which makes it discontinuous at $x = 1$. Here function is not defined at $x = 1$, i.e., $f(1)$ is not defined. So, function has missing point of discontinuity at $x = 1$.

However, we can redefine the function as

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$$, which is continuous at $x = 1$.

Since we are able to remove discontinuity at $x = 1$ by redefining the function $f(x)$, we say that $f(x)$ has removable type of discontinuity at $x = 1$.

II. Isolated Point Discontinuity

In this type of discontinuity, the limit of the function exists at the particular point say at $x = a$ and also function is defined at that point, but both are not equal, i.e. $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} f(x) \neq f(a)$.

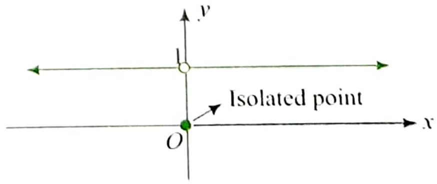
Consider the function $f(x) = |\text{sgn}(x)|$, where sgn represents signum function.

We know that $\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

$\therefore \quad f(x) = |\text{sgn}(x)| = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Clearly, function is discontinuous at $x = 0$ as $\lim_{x\to 0} f(x) = 1$ but $f(0) = 0$.

The graph of the function is as shown in the following figure.



From the graph it is evident that $f(x)$ has isolated point discontinuity at $x = 0$.

In this case also we can redefine the function $f(x)$ as

$$f(x) = \begin{cases} |\text{sgn}(x)|, & x \neq 0 \\ 1, & x = 0 \end{cases}$$, which is continuous at $x = 0$.

Thus, $f(x)$ has removable type of discontinuity at $x = 0$.

Nonremovable Discontinuity

A function is said to have nonremovable discontinuity at $x = a$ if the limit of the function does not exist. Since $\lim_{x \to a} f(x)$ does not exist, it is not possible to redefine the function in any way so as to make it continuous at $x = a$. The nonremovable discontinuities can further be classified into following three categories.

I. Finite Type (Jump Type)

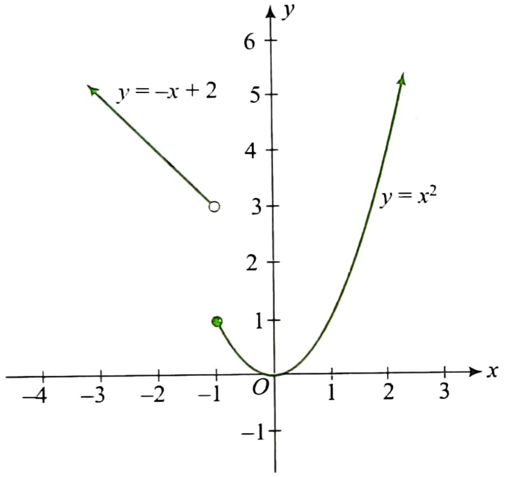
In this type of discontinuity, both the left hand and the right-hand limits exist but are unequal.

e.g. $$f(x) = \begin{cases} -x+2, & x < -1 \\ x^2, & x \ge -1 \end{cases}$$

Here $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-x+2) = 3$

and $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x^2 = 1$

Thus, at $x = -1$ both L.H.L. and R.H.L. exists but are not equal.



The value of function jumps by '2' units at $x = -1$, when we see from right to left.

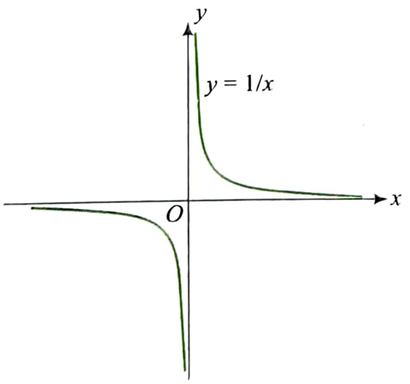
The non-negative difference between the two limits at the point of discontinuity is called the **Jump of Discontinuity**.

This type of discontinuity is also called Jump discontinuity.

II. Infinite Type (Asymptotic type)

The discontinuity is said to be of infinite type if at least one of the two limits is infinity.

Consider the function $f(x) = \frac{1}{x}$



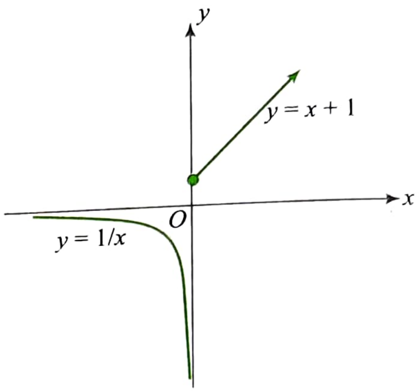
Here L.H.L. = $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and R.HL. = $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

So, function has infinite type of discontinuity at $x = 0$.

Also, $f(x)$ has vertical asymptote (y-axis) at $x = 0$.

So, this type of discontinuity is also called asymptotic discontinuity.

Now consider function $f(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ x+1, & x \ge 0 \end{cases}$



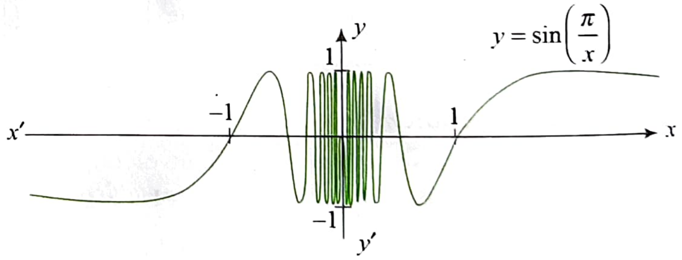
Here L.H.L. = $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and R.H.L. = $\lim_{x \to 0^+} (x+1) = 1$.

Since, L.H.L. is negative infinity, function has infinite type of discontinuity.

III. Oscillatory type

The discontinuity is said to be oscillatory when the limits oscillate between two finite quantities.

Consider function $f(x) = \sin \frac{\pi}{x}$. When $x \to 0$, $\frac{\pi}{x} \to \infty$ and $\sin(\to\infty)$ can take any between – 1 to 1 or we can say when $x \to 0$, $\sin \frac{\pi}{x}$ oscillates between –1 and 1 as shown in the following figure.



In all above cases (I, II and III) function cannot be made continuous by assigning some value to function at the point of discontinuity.

Note:

The notion of continuity at x_0 only applies to points of the domain. By definition, a function is said to be continuous if it is continuous at each point of its domain. The notion of discontinuity does not apply only to points of the domain but also to limit point that does not belong to the domain. So, a continuous function can have discontinuities, but not at points in the domain.

For example, the function

$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \ge 0 \end{cases}$$

is not continuous because the limit $\lim_{x \to 0} f(x)$ does not exist.
But the function

$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

is continuous, because it is continuous at each point of its domain $R - \{0\}$.
This function has however a point of discontinuity at $x = 0$.
All trigonometric functions, inverse trigonometric functions, polynomial function, logarithmic function, exponential function, rational function are continuous in their respective domains.

DIRECTIONAL CONTINUITY

A function may happen to be continuous in only one direction, either from the "left" or from the "right".

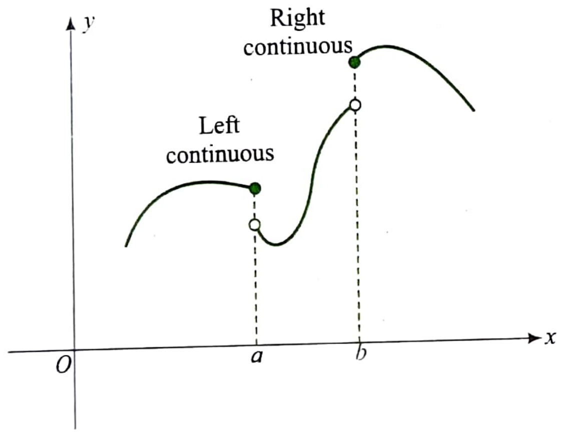
A function $y = f(x)$ is right-continuous at $x = a$ if

$$f(a) = \lim_{x \to a^+} f(x) \text{ or } f(a) = \lim_{h \to 0} f(a+h).$$

i.e. $f(x) = [x]$ is right-continuous at $x = 2$, $f(2) = \lim_{x \to 2^+} [x] = 2$.

A function $y = f(x)$ is left-continuous at $x = a$ if

$$f(a) = \lim_{x \to a^-} f(x) \text{ or } f(a) = \lim_{h \to 0} f(a-h).$$



A function which is continuous at $x = a$ is both right-continuous and left-continuous.

CONTINUITY IN INTERVAL

A function is said to be continuous in the open interval (a, b) if $f(x)$ is continuous at each and every point $\in (a, b)$. For any $c \in (a, b)$ $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$.

i.e. $f(x) = \tan x$ is continuous in $(-\pi/2, \pi/2)$ and $g(x) = [x]$, where $[\cdot]$ represents the greatest integer function is continuous in intervals $(1, 2)$, $(2, 3)$, $(-3, -2)$ etc. However $[x]$ is discontinuous in $(0, 2)$ as it is discontinuous at $x = 1$.

A function $f(x)$ is said to be continuous in the closed interval $[a, b]$ if

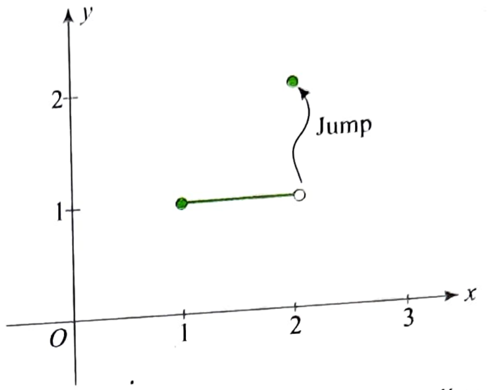
(i) it is continuous at every point in in the interval (a, b)

(ii) it is right continuous at $x = a$, i.e. $f(a) = \lim_{x \to a^+} f(x)$ (here L.H.L. should not be evaluated)

(iii) it is left continuous at $x = b$, i.e. $f(b) = \lim_{x \to b^-} f(x)$ (here R.H.L. should not be evaluated)

i.e., $f(x) = [x]$ is not continuous in $[1, 2]$ as $f(2) = 2$ and $\lim_{x \to 2^-} [x] = \lim_{h \to 0} [2 - h] = 1$.

So, it is discontinuous as at $x = 2$.



Thus, if function has at least one point of discontinuity in the given interval it is called discontinuous function in that interval.

Note:
If a function f is continuous on a closed interval $[a, b]$ then it is bounded, i.e., a continuous function whose domain is some closed interval must have its range also in closed interval

ILLUSTRATION 4.8

Find the points of discontinuity of the following functions over R.

(i) $f(x) = \dfrac{1}{2\sin x - 1}$ (ii) $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

(iii) $f(x) = \dfrac{1}{x^4 + x^2 + 1}$

Sol.

(i) $f(x) = \dfrac{1}{2\sin x - 1}$

$f(x)$ is discontinuous when $2 \sin x - 1 = 0$

or $\sin x = \dfrac{1}{2}$

$\Rightarrow x = n\pi + (-1)^n \dfrac{\pi}{6}, n \in Z$

$f(x)$ has infinity type (non-removable) of discontinuity at all these points.

(ii) $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

$f(x)$ is discontinuous when $x^2 - 3|x| + 2 = 0$

or $|x|^2 - 3|x| + 2 = 0$

$\Rightarrow (|x| - 1)(|x| - 2) = 0$

$\Rightarrow |x| = 1, 2$

$\Rightarrow x = \pm 1, \pm 2$

$f(x)$ has infinity type (non-removable) of discontinuity at all these points.

(iii) $f(x) = \dfrac{1}{x^4 + x^2 + 1} = \dfrac{1}{\left(x^2 + \frac{1}{2}\right)^2 + \frac{3}{4}}$

Now $\left(x^2 + \frac{1}{2}\right)^2 + \frac{3}{4} \geq 1 \; \forall \, x \in R$

So, $f(x)$ is continuous $\forall \, x \in R$

ILLUSTRATION 4.9

What kind of discontinuity following functions have? If possible, redefine the function to make it continuous.

(a) $f(x) = [x]$ at $x = 0$, where $[x]$ represents greatest integer function

(b) $f(x) = [x] + [-x]$ at $x = 1$, where $[x]$ represents greatest integer function.

(c) $f(x) = e^{\frac{1}{x}}$ at $x = 0$.

Sol.

(a) $f(x) = [x]$ at $x = 0$

L.H.L. $= \lim\limits_{x \to 0^-} [x] = -1$ and R.H.L. $= \lim\limits_{x \to 0^+} [x] = 0$

Here L.H.L. $\neq$ R.H.L..

$f(x)$ has finite type (Jump type) of discontinuity, which is non-removable type.

(b) $f(x) = [x] + [-x]$ at $x = 1$

$$\text{L.H.L.} = \lim_{x \to 1^-} ([x] + [-x])$$

$$= \lim_{h \to 0} ([1 - h] + [-(1 - h)])$$

$$= \lim_{h \to 0} ([1 - h] + [h - 1])$$

$$= 0 - 1 = -1$$

$$\text{R.H.L.} = \lim_{x \to 1^+} ([x] + [-x])$$

$$= \lim_{h \to 0} ([1 + h] + [-(1 + h)])$$

$$= 1 - 2 = -1$$

Also $f(1) = [1] + [-1] = 0$

Thus L.H.L. = R.H.L. but not equal to $f(1)$.

So, $f(x)$ has isolated point discontinuity.

We can redefine function as $\begin{cases} [x] + [-x], & x \neq 1 \\ -1, & x = 1 \end{cases}$, which is continuous at $x = 1$.

(c) $f(x) = e^{\frac{1}{x}}$ at $x = 0$.

$$\text{L.H.L.} = \lim_{x \to 0^-} e^{\frac{1}{x}} = \lim_{h \to 0} e^{\frac{1}{-h}} = e^{-\infty} = 0$$

$$\text{L.H.L.} = \lim_{x \to 0^+} e^{\frac{1}{x}} = \lim_{h \to 0} e^{\frac{1}{h}} = e^{\infty} = \infty$$

Since L.H.L. is infinity, $f(x)$ has infinity type of discontinuity at $x = 0$, which is nonremovable type.

ILLUSTRATION 4.10

What type of discontinuity does $f(x) = \dfrac{1}{\log_e |x^2 - 4|}$ have?

Sol. We have

$$f(x) = \frac{1}{\log_e |x^2 - 4|}$$

Clearly $f(x)$ is defined if $\log_e |x^2 - 4| > 0$ and $|x^2 - 4| \neq 1$

$\therefore \quad x^2 - 4 \neq 0$ and $x^2 - 4 \neq 1, -1$

$\Rightarrow \quad x^2 \neq 4$ and $x^2 \neq 3, 5$

$\Rightarrow \quad x \neq \pm 2$ and $x \neq \pm \sqrt{5}, \pm \sqrt{3}$

Now $\lim\limits_{x \to \pm 2} \dfrac{1}{\log_e |x^2 - 4|} = \lim\limits_{y \to 0} \dfrac{1}{\log_e y} = 0$

Thus, $f(x)$ has missing point type of discontinuity at $x = \pm 2$ which is removable type.

$$\lim_{x \to \sqrt{3}^+} \frac{1}{\log_e |x^2 - 4|} = \lim_{y \to 1^-} \frac{1}{\log_e y} = -\infty$$

Thus, $f(x)$ has infinity type of discontinuity at $x = \sqrt{3}$, which is nonremovable type.

Also $\lim\limits_{x \to \sqrt{5}^+} \dfrac{1}{\log_e |x^2 - 4|} = \lim\limits_{y \to 1^+} \dfrac{1}{\log_e y} = \infty$

Thus, $f(x)$ has infinity type of discontinuity at $x = \sqrt{5}$, which is nonremovable type.

$f(x)$ can be redefined at $x = 0$ as,

$$f(x) = \begin{cases} \dfrac{1}{\log_e |x^2 - 4|}, & x \neq \pm 2, \pm\sqrt{3}, \pm\sqrt{5} \\ 0, & x = \pm 2 \end{cases},$$

which is continuous at $x = \pm 2$.

ILLUSTRATION 4.11

What type of discontinuity does $f(x) = \cos^{-1}\left(\dfrac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)$ have?

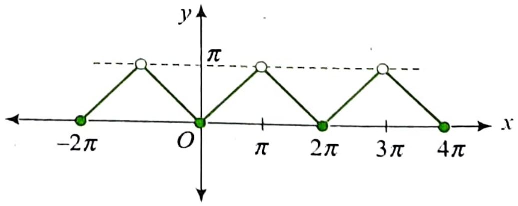
Sol. We have

$$f(x) = \cos^{-1}\left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)$$

Clearly domain of the function is $R - \{(2n + 1)\pi, n \in Z\}$

So, $\quad f(x) = \cos^{-1} (\cos x), x \in R - \{(2n + 1)\pi, n \in Z\}$.

Graph of $f(x)$ is as shown in the following figure.



We observe that $f(x)$ has missing point type of discontinuity at $x = (2n+1)\pi$, $n \in Z$.

$f(x)$ can be redefined as

$$f(x) = \begin{cases} \cos^{-1}(\cos x), & x \neq (2n+1)\pi,\ n \in Z \\ \pi, & x = (2n+1)\pi,\ n \in Z \end{cases},$$

which is continuous for all real x.

ILLUSTRATION 4.12

Let $f(x)$ be a function defined as

$$f(x) = \begin{cases} \dfrac{x^2-1}{x^2-2|x-1|-1}, & x \neq 1 \\ \dfrac{1}{2}, & x = 1 \end{cases}.$$

Discuss the continuity of the function at $x = 1$.

Sol. R.H.L. at $x = 1$ is

$$f(1^+) = \lim_{x\to 1^+} \frac{x^2-1}{x^2-2|x-1|-1}$$

$$= \lim_{x\to 1^+} \frac{x^2-1}{x^2-2(x-1)-1}$$

$$= \lim_{x\to 1^+} \frac{x+1}{(x+1)-2} \to \infty$$

Since, R.H.L. is infinity, $f(x)$ is discontinuous at $x = 1$.

ILLUSTRATION 4.13

If $f(x) = \begin{cases} \cos^{-1}\{\cot x\}, & x < \dfrac{\pi}{2} \\ \pi[x]-1, & x \geq \dfrac{\pi}{2} \end{cases}$. where $[\cdot]$ represents greatest function and $\{\cdot\}$ represents fractional part function, then discuss the continuity of $f(x)$ at $x = \dfrac{\pi}{2}$.

Sol. We have L.H.L. at $x = \dfrac{\pi}{2}$,

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-} \cos^{-1}\{\cot x\}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \cos^{-1}(\cot x - [\cos x])$$

$$= \lim_{h\to 0^+} \cos^{-1}(h-[h]) \quad \text{(Putting } \cot x = h\text{)}$$

$$= \lim_{h\to 0^+} \cos^{-1} h$$

$$= \frac{\pi}{2}$$

R.H.L. at $x = \dfrac{\pi}{2}$,

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+} (\pi[x]-1) = \pi - 1$$

Thus, L.H.L. $\neq$ R.H.L., $f(x)$ has jump type of non-removable discontinuity at $x = \dfrac{\pi}{2}$

ILLUSTRATION 4.14

Test the continuity of $f(x)$ at $x = 0$ if

$$f(x) = \begin{cases} (x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

Sol. We have $f(0) = 0$

L.H.L. at $x = 0$,

$$f(0^-) = \lim_{x\to 0^-} (x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$$

$$= \lim_{h\to 0} (0-h+1)^{2-\left(\frac{1}{|0-h|}+\frac{1}{0-h}\right)}$$

$$= \lim_{h\to 0} (1-h)^{2-\left(\frac{1}{h}-\frac{1}{h}\right)}$$

$$= \lim_{h\to 0} (1-h)^2$$

$$= 1$$

R.H.L. at $x = 0$,

$$f(0^+) = \lim_{x\to 0^+} (x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$$

$$= \lim_{h\to 0} (0+h+1)^{2-\left(\frac{1}{|0+h|}+\frac{1}{0+h}\right)}$$

$$= \lim_{h\to 0} (1+h)^{2-\frac{2}{h}}$$

$$= \frac{\lim_{h\to 0} (1+h)^2}{\lim_{h\to 0} (1+h)^{\frac{2}{h}}}$$

$$= \frac{1}{e^2}$$

Thus, L.H.L. $\neq$ R.H.L., $f(x)$ has jump type of non-removable discontinuity at $x = 0$.

ILLUSTRATION 4.15

Discuss continuity of $f(x) = [\sin x] - [\cos x]$ at $x = \pi/2$, where $[\cdot]$ represent the greatest integer function.

Sol. $\lim_{x\to\frac{\pi}{2}^-} ([\sin x]-[\cos x])$

$$= \lim_{h\to 0}\left(\left[\sin\left(\frac{\pi}{2}-h\right)\right]-\left[\cos\left(\frac{\pi}{2}-h\right)\right]\right)$$

$$= [1^-] - [0^+]$$

$$= 0 - 0 = 0$$

$$\lim_{x\to\frac{\pi}{2}^+}([\sin x]-[\cos x])$$

$$= \lim_{h\to0}\left(\left[\sin\left(\frac{\pi}{2}+h\right)\right]-\left[\cos\left(\frac{\pi}{2}+h\right)\right]\right)$$

$$= [1^-]-[0^-]$$

$$= 0-(-1)=1$$

Thus, L.H.L. ≠ R.H.L., $f(x)$ has jump type of non-removable discontinuity at $x = \frac{\pi}{2}$

ILLUSTRATION 4.16

Let $f(x)=\begin{cases}\dfrac{\log_e \cos x}{\sqrt[4]{1+x^2}-1}, & x>0\\ \dfrac{e^{\sin 4x}-1}{\log_e(1+\tan 2x)}, & x<0\end{cases}$. Then find the value of $f(0)$ which makes $f(x)$ continuous at $x = 0$.

Sol. Since $f(x)$ is continuous at $x = 0$, we must have

$$f(0)=\lim_{x\to0^-}f(x)=\lim_{x\to0^+}f(x)$$

Now, $\text{L.H.L.} = \lim_{x\to0^-}\dfrac{e^{\sin 4x}-1}{\log_e(1+\tan 2x)}$

$$= \lim_{x\to0^-}\frac{\dfrac{e^{\sin 4x}-1}{\sin 4x}\sin 4x}{\dfrac{\log_e(1+\tan 2x)}{\tan 2x}\tan 2x}$$

$$= \lim_{x\to0^-}\frac{\sin 4x}{\tan 2x}$$

$$=2$$

$\text{R.H.L.} = \lim_{x\to0^+}\dfrac{\log_e \cos x}{\sqrt[4]{1+x^2}-1}$

$$= \lim_{x\to0^+}\frac{-\tan x}{\frac{1}{4}(1+x^2)^{-\frac{3}{4}}2x} \quad \text{(Using L'Hospital Rule)}$$

$$= -2$$

Here L.H.L. ≠ R.H.L.

So, $f(x)$ has non-removable discontinuity at $x = 0$.

Hence $f(0)$ does not exist.

ILLUSTRATION 4.17

If function $f(x)=\begin{cases}ax-b, & x\le1\\ 3x, & 1<x<2\\ bx^2-a, & x\ge2\end{cases}$ is continuous at $x = 1$ but discontinuous at $x = 2$ then prove that the locus of the point (a, b) is a straight line excluding the point where it cuts the line $y = 3$.

Sol. Given $f(x)$ is continuous at $x = 1$

$\therefore \quad \lim_{x\to1^-}f(x)=f(1)=\lim_{x\to1^+}f(x)$

$\Rightarrow \quad \lim_{x\to1^-}(ax-b)=a-b=\lim_{x\to1^+}3x$

$\Rightarrow \quad a-b=3$ ...(1)

$\therefore \quad$ Locus of (a, b) is $x-y=3$

Now $f(x)$ is discontinuous at $x = 2$.

$\therefore \quad \text{L.H.L.} \ne f(2)$

$\Rightarrow \quad \lim_{x\to2^-}f(x)\ne f(2)$

$\Rightarrow \quad \lim_{x\to2^-}3x\ne b(2)^2-a$

$\Rightarrow \quad 6\ne 4b-a$

Assume: $6 = 4b - a$ ...(2)

From (1) and (2), we get $b = 3$

So, $f(x)$ is discontinuous at $x = 2$ if line $x - y = 3$ is not satisfied by the point for which $y = 3$.

Therefore, locus of (a, b) is $x - y = 3$ excluding the point where it cuts the line $y = 3$.

Continuity and Discontinuity of Functions Obtained by Algebraic Operations on Two Given Functions

I. If $f(x)$ and $g(x)$ are continuous functions in the given interval, then their sum, difference, product and quotient functions are also continuous.

However, for quotient function denominator function should not be zero in the interval where continuity is checked.

II. If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$, then $f(x) \pm g(x)$ is a discontinuous at $x = a$.

Let $f(x)$ be continuous at $x = a$ and $g(x)$ is discontinuous at $x = a$, then $\lim_{x\to a}f(x)=f(a)$ and $\lim_{x\to a}g(x)\ne g(a)$.

$$\therefore \quad \lim_{x\to a}(f(x)+g(x))\ne f(a)+g(a)$$

Thus function $(f(x) + g(x))$ is discontinuous at $x = a$.

Similarly, we can prove that function $(f(x) - g(x))$ is discontinuous at $x = a$.

III. If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$ then the product function $h(x) = f(x) \times g(x)$ may or may not be continuous at $x = a$.

Consider, $f(x) = x^3$ and $g(x) = \text{sgn}(x)$

Here $f(x)$ is continuous at $x = 0$ and $g(x)$ is discontinuous at $x = 0$.

Now product function,

$$h(x)=f(x)\times g(x)=\begin{cases}x^3, & x>0\\ 0, & x=0\\ -x^3, & x<0\end{cases} \text{ is continuous at } x=0.$$

On the other hand, if $f(x) = x$, which is continuous for all real x and $g(x) = [x]$ (greatest integer function), which is discontinuous at all integral x, the product function $h(x) = f(x)\,g(x) = x[x]$ is discontinuous at $x = 1$ as

$\lim_{x\to1^-}x[x]=1\times0=0$ and $\lim_{x\to1^+}x[x]=1\times1=1$.

However, $x[x]$ is continuous at $x = 0$.

In a particular case if $f(x)$ is discontinuous at $x = a$ in its domain and $g(x)$ is continuous function with $g(a) = 0$, then product function $h(x) = f(x) \times g(x)$ is continuous at $x = a$.

IV. If both the functions $f(x)$ and $g(x)$ are discontinuous at $x = a$, then function obtained by algebraic operation of $f(x)$ and $g(x)$ may or may not be continuous at $x = a$.

Consider following examples

(i) $f(x) = [x]$ (Greatest integer function) and $g(x) = \{x\}$ (fractional part function) both are discontinuous at $x = 1$ but their sum $f(x) + g(x) = [x] + \{x\} = x$ is continuous at $x = 1$.

(ii) $f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases}$ and $g(x) = \begin{cases} 1, & x \le 0 \\ -1, & x > 0 \end{cases}$.

Here both the functions are discontinuous at $x = 0$, but their product $f(x) \times g(x) = -1, \forall x \in R$ is continuous at $x = 0$.

Also, $f(x) + g(x) = 0 \ \forall x \in R$ and $\dfrac{f(x)}{g(x)} = -1, \forall x \in R$ are also continuous at $x = 0$.

However, $f(x) - g(x) = \begin{cases} -2, & x \le 0 \\ 2, & x > 0 \end{cases}$ is discontinuous at $x = 0$.

ILLUSTRATION 4.18

If $f(x) = \begin{cases} |x+1|, & x \le 0 \\ x, & x > 0 \end{cases}$ and $g(x) = \begin{cases} |x|+1, & x \le 1 \\ -|x-2|, & x > 1 \end{cases}$, then discuss continuity of $f(x) + g(x)$.

Sol. Here each of the branch functions of $f(x)$ and $g(x)$ are continuous in the given interval.

But $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} |x+1| = 1$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x = 0$.

So, $f(x)$ is discontinuous at $x = 0$.

Also, $\lim\limits_{x\to 1^-} g(x) = \lim\limits_{x\to 1^-} (|x|+1) = 2$ and

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (-|x-2|) = -1$

So, $g(x)$ is discontinuous at $x = 1$.

Now, $f(x)$ is continuous at $x = 1$ and $g(x)$ is continuous at $x = 0$.

Hence $f(x) + g(x)$ is discontinuous at $x = 0$ and $x = 1$.

ILLUSTRATION 4.19

Prove that function $f(x) = [x]\cos\left(\dfrac{(2x-1)\pi}{2}\right)$, where [.] denotes the greatest integer function, is continuous for all real x.

Sol. Function $\cos\left(\dfrac{(2x-1)\pi}{2}\right)$ is continuous for all real x and function $[x]$ is discontinuous at all integral values of x.

Now, for any integer 'n', we have

$$\lim_{x\to n^-} f(x) = \lim_{h\to 0}[n-h]\cos\frac{(2(n-h)-1)\pi}{2}$$
$$= (n-1)\cos\frac{(2n-1)\pi}{2} = 0$$

and
$$\lim_{x\to n^+} f(x) = \lim_{h\to 0}[n+h]\cos\frac{(2(n+h)-1)\pi}{2}$$
$$= n\cos\frac{(2n-1)\pi}{2} = 0$$

Also, $f(n) = [n]\cos\left(\dfrac{(2n-1)\pi}{2}\right) = 0$.

Thus $f(x)$ is continuous at $x = n, n \in I$.

Hence $f(x)$ is continuous for all real x.

CONCEPT APPLICATION EXERCISE 4.1

1. A function $f(x)$ satisfies the following property: $f(x \cdot y) = f(x)\, f(y)$. Show that the function $f(x)$ is continuous for all values of x if it is continuous at $x = 1$.
2. Find the value of $f(0)$ so that the function $f(x) = \dfrac{\sqrt{1+x} - \sqrt[3]{1+x}}{x}$ becomes continuous at $x = 0$.
3. If the function $f: R\setminus\{0\} \to R$ given by $f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$ is continuous at $x = 0$, then find the value of $f(0)$.
4. Let $f(x) = \dfrac{1-\tan x}{4x-\pi}$, $x \ne \dfrac{\pi}{4}$, $x \in \left[0, \dfrac{\pi}{2}\right]$. If $f(x)$ is continuous in $\left[0, \dfrac{\pi}{4}\right]$, then find the value of $f\left(\dfrac{\pi}{4}\right)$.
5. If $f(x) = \left(\tan\left(\dfrac{\pi}{4} + \log_e x\right)\right)^{\log_x e}$ is to be made continuous at $x = 1$, then what is the value of $f(1)$?
6. If the function $f(x) = \dfrac{x^2-(A+2)x+A}{x-2}$, for $x \ne 2$ and $f(2) = 2$, is continuous at $x = 2$, then find the value of A.
7. Find the values of a and b so that the function $f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \pi/4 \\ 2x\cot x + b, & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x, & \pi/2 < x \le \pi \end{cases}$ is continuous for $0 \le x \le \pi$.
8. Let $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & x < 0 \\ a, & x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16-\sqrt{x}} - 4}, & x > 0 \end{cases}$. Then determine the value of a, if possible, so that the function is continuous at $x = 0$.
9. Let $f(x) = \begin{cases} (1+|\sin x|)^{\frac{a}{|\sin x|}}, & -\dfrac{\pi}{6} < x < 0 \\ b, & x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}}, & 0 < x < \dfrac{\pi}{6} \end{cases}$. Then determine a and b such that $f(x)$ is continuous at $x = 0$.

10. Let $f(x)$ be a continuous at $x = 0$ such that

$$f(x) = \begin{cases} \dfrac{\log_e(px^2+qx+r)}{x^2}, & -1 \le x < 0 \\ 1, & x = 0 \\ \dfrac{\sin(e^{x^2}-1)}{x^2}, & 0 < x \le 1 \end{cases}.$$

Then find p and q.

11. Match the type of discontinuity at $x = 1$ in List II for the function in List I:

List I	List II
a. $f(x) = \dfrac{1}{x-1}$	**p.** Removable discontinuity
b. $f(x) = \dfrac{x^3-x}{x^2-1}$	**q.** Non-removable discontinuity
c. $f(x) = \dfrac{\lvert x-1\rvert}{x-1}$	**r.** Jump discontinuity
d. $f(x) = \sin\left(\dfrac{1}{x-1}\right)$	**s.** Discontinuity due to vertical asymptote
	t. Missing point discontinuity
	u. Oscillating discontinuity

12. Which of the following functions is not continuous $\forall\, x \in R$?

(a) $\sqrt{2\sin x+3}$ **(c)** $\dfrac{e^x+1}{e^x+3}$

(c) $\left(\dfrac{2^{2x}+1}{2^{3x}+5}\right)^{5/7}$ **(d)** $\sqrt{\text{sgn}\,x+1}$

13. Discuss the continuity of $f(x) = \begin{cases} \dfrac{x^2}{\lvert x\rvert}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

14. Let $f(x) = \begin{cases} (1+3x)^{1/x}, & x \ne 0 \\ e^3, & x = 0 \end{cases}$. Discuss the continuity of $f(x)$ at **(a)** $x = 0$, **(b)** $x = 1$.

15. Discuss the continuity of $f(x) = \begin{cases} \dfrac{x-1}{e^{\frac{1}{x-1}}+1}, & x \ne 1 \\ 0, & x = 1 \end{cases}$ at $x = 1$.

16. Discuss the continuity of

$$f(x) = \begin{cases} \dfrac{x^4-5x^2+4}{\lvert (x-1)(x-2)\rvert}, & x \ne 1, 2 \\ 6, & x = 1 \\ 12, & x = 2 \end{cases}.$$

17. Discuss the continuity of the function $f(x) = [x] + \lvert 1 - x\rvert$, $-1 \le x \le 3$, where [.] represents the greatest integer function.

ANSWERS

2. $\dfrac{1}{6}$ **3.** 1 **4.** $-\dfrac{1}{2}$

5. e^2 **6.** $A = 0$ **7.** $a = \dfrac{\pi}{6}$, $b = \dfrac{-\pi}{12}$ **8.** $a = 8$

9. $a = \dfrac{2}{3}$, $b = e^{2/3}$ **10.** $p = 1, q = 0, r = 1$

11. a → s, q; b → t, p; c → r, q; d → u, q;

12. (d) **13.** Continuous everywhere

14. Continuous at $x = 0, 1$ **15.** Continuous at $x = 0$

16. Discontinuous at $x = 1, 2$

17. Continuous at $x = 0, 1, 2, 3$

CONTINUITY OF FEW SPECIFIC FUNCTIONS

CONTINUITY OF FUNCTION INVOLVING GREATEST INTEGER FUNCTION

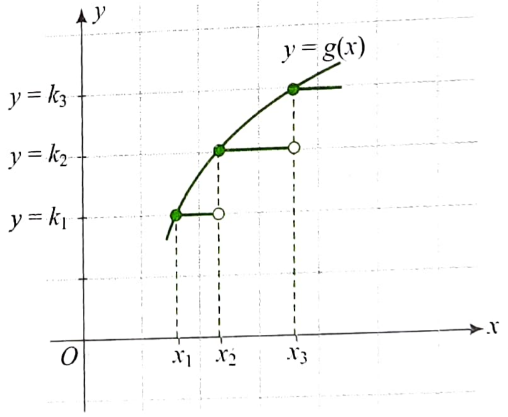
We know that $f(x) = [x]$ (greatest integer function) is discontinuous whenever x is an integer.

Graphically we can say that $f(x) = [x]$ is discontinuous for values of x, where it crosses the lines $y = k$, $(k \in Z)$.

Here function 'x' has no turning point. It keeps increasing with the increase in the value of x.

We can extend this idea to check continuity of $f(x) = [g(x)]$ if $g(x)$ has no turning point.

If the graph of continuous function $y = g(x)$ crosses the line $y = k$ (i.e. when $g(x)$ is an integer) for $x = x_0$, then certainly $[g(x)]$ is discontinuous at $x = x_0$.



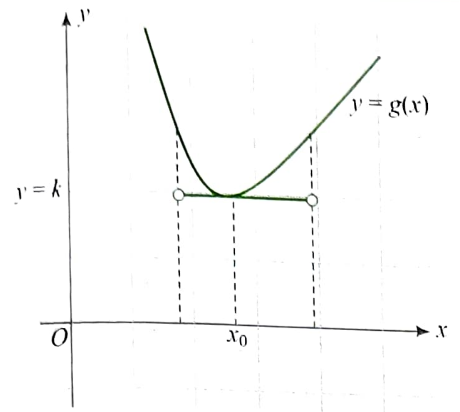
From the above figure $[g(x)]$ is discontinuous for the values of x at which graph of $y = g(x)$ crosses the lines $y = k_1$, $y = k_2$ and $y = k_3$; where $k_1, k_2, k_3 \in Z$. So, $[g(x)]$ is discontinuous at $x = x_1$, x_2, x_3.

We can say that $f(x) = [g(x)]$ is discontinuous at all values of x for which $g(x)$ is an integer.

For example, $f(x) = [\sqrt{x}]$ is discontinuous whenever $\sqrt{x}$ takes integral values as $\sqrt{x}$ increases and has no turning point.

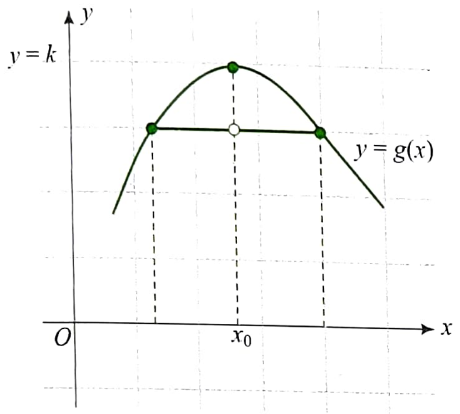
With the similar reasoning $f(x) = [x^2]$, $x \ge 0$ is discontinuous whenever x^2 is an integer.

However, if function $y = g(x)$ has turning point and it crosses the line $y = k$ for some k, exactly at the turning point, then continuity is to be checked cautiously.



In the above graph of $y = g(x)$ crosses line $y = k$ for which $x = x_0$, where $[g(x)]$ is continuous.

For example $[\sin x]$ is continuous at $x = \frac{3\pi}{2}$, where graph of $y = \sin x$ has above trend.



In the above graph of $y = g(x)$ crosses line $y = k$ for which $x = x_0$, where $[g(x)]$ is discontinuous.

For example $[\sin x]$ is discontinuous at $x = \frac{\pi}{2}$, where graph of $y = \sin x$ has above trend.

ILLUSTRATION 4.20

Find the points of discontinuity of following functions. (where, $[\cdot]$ represents greatest integer function)

(i) $f(x) = [x/3], x \in [0, 30]$

(ii) $f(x) = [\log_e x]$

(iii) $f(x) = [\sin^{-1} x]$

(iv) $f(x) = \left[\frac{2}{1+x^2}\right], x \geq 0$

Sol.

(i) $f(x) = [x/3], x \in [0, 30]$

$f(x)$ is discontinuous at all values of x for which $x/3 = k, k \in Z$.

$\therefore \quad x = 3k$ or $x = 3, 6, 9, 12, 15, 18, 21, 24, 27$

At $\quad x = 0$, $[x/3]$ is continuous as $\lim\limits_{x\to 0^+} [x/3] = 0 = f(0)$

And at $x = 30$, $[x/3]$ is discontinuous as

$$\lim_{x\to 30^-} [x/3] = 9 \neq f(30)$$

Hence, there are ten points of discontinuity,

$x = 3, 6, 9, \ldots, 27, 30$

(ii) $\log_e x$ function increases in its domain.

So, $f(x) = [\log_e x]$ is discontinuous at all values of x for which $\log_e x = k$ or $x = e^k, k \in Z$.

Hence the set of points of discontinuity is $\{\ldots e^{-2}, e^{-1}, e^0, e^1, e^2, e^3, \ldots\}$

(iii) $\sin^{-1} x$ function increases in its domain.

So, $f(x) = [\sin^{-1} x]$ is discontinuous at all values of x for which $\sin^{-1} x$ takes integral values.

Since, range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$, $\sin^{-1} x = -1, 0, 1$

or $\quad x = \sin(-1), \sin 0, \sin 1$

Hence the set of points of discontinuity is $\{-\sin 1, 0, \sin 1\}$

(iv) Function $\frac{2}{1+x^2}$ decreases for $x \geq 0$.

So, $f(x) = \left[\frac{2}{1+x^2}\right], x \geq 0$ is discontinuous at all values of x for which $\frac{2}{1+x^2}$ takes integral values.

For $x \geq 0$, $\frac{2}{1+x^2} \in (2, 0]$

$\therefore \quad \frac{2}{1+x^2} = 1, 2$

$\Rightarrow \quad x = 1, 0$

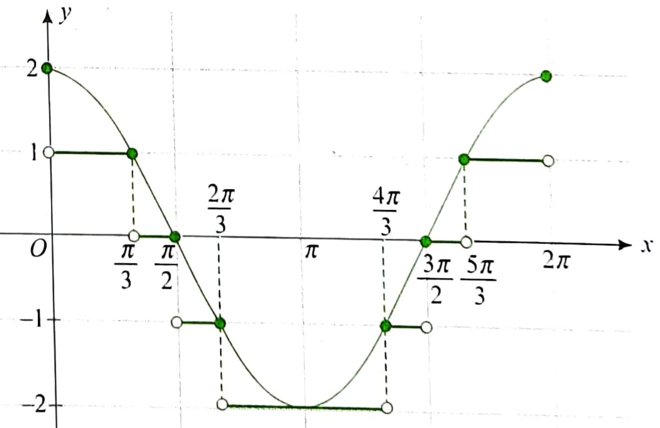
Hence the set of points of discontinuity is $\{0, 1\}$

ILLUSTRATION 4.21

Draw the graph and find the points of discontinuity of $f(x) = [2\cos x]$, $x \in [0, 2\pi]$ ($[\cdot]$ represents greatest integer function)

Sol. We have $f(x) = [2 \cos x]$

Graph of the function $2\cos x$ and $[2\cos x]$ are as shown in the following figure.



From the graph $[2\cos x]$ is discontinuous at $x = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, 2\pi$.

ILLUSTRATION 4.22

Draw the graph and discuss the continuity of $f(x) = [\sin x + \cos x]$, $x \in [0, 2\pi]$ ([·] represents greatest integer function)

Sol. We have

$$f(x) = [\sin x + \cos x] = \left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]$$

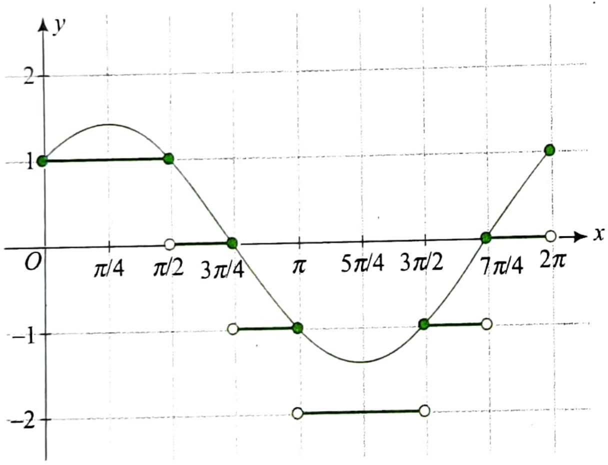
The graph of $g(x) = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ can be drawn by plotting the following points and then joining them by smooth curve.

$$g(0) = 1,\ g\left(\frac{\pi}{4}\right) = \sqrt{2},\ g\left(\frac{\pi}{2}\right) = 1$$

$$g\left(\frac{3\pi}{4}\right) = 0,\ g(\pi) = -1,\ g\left(\frac{5\pi}{4}\right) = -\sqrt{2}$$

$$g\left(\frac{3\pi}{2}\right) = -1,\ g\left(\frac{7\pi}{4}\right) = 0,\ g(2\pi) = 1$$

Graph of the function $\sin x + \cos x$ and $[\sin x + \cos x]$ are as shown in the following figure.



From the graph $[\sin x + \cos x]$ is discontinuous at $x = \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$.

ILLUSTRATION 4.23

If the function $f(x) = \left[\frac{(x-2)^3}{a}\right]\sin(x-2) + \cos(x-2)$, is continuous in [4, 6], then find the values of a. Here [.] denotes the greatest integer function.

Sol. For $x \in [4, 6]$, $(x-2)^3 \in [8, 64]$

Now suppose $a = 2$.

$\therefore \quad \frac{(x-2)^3}{2} \in [4, 32]$

$\therefore \quad \left[\frac{(x-2)^3}{2}\right]$ will be discontinuous when

$$\frac{(x-2)^3}{2} = 5, 6, \ldots, 32 \quad \ldots(1)$$

Since $\sin(x-2)$ and $\cos(x-2)$ are continuous for all x and $\sin(x-2)$ is non-zero where $\left[\frac{(x-2)^3}{2}\right]$ is discontinuous

$f(x)$ will be discontinuous at all the values of x obtained in (1).

Thus, $f(x)$ will be discontinuous if $\left[\frac{(x-2)^3}{a}\right]$ is allowed to take different values in [4, 6].

So, $f(x)$ will be continuous if $\left[\frac{(x-2)^3}{a}\right]$ takes only one integral value for all values of a.

Therefore, we must have $a > 64$, for which $\left[\frac{(x-2)^3}{a}\right] = 0$ and $f(x)$ is continuous.

CONTINUITY OF FUNCTION INVOLVING SIGNUM FUNCTION

We know that

$$f(x) = \text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$ is discontinuous at $x = 0$.

In general $f(x) = \text{sgn}(g(x)) = \begin{cases} -1, & g(x) < 0 \\ 0, & g(x) = 0 \\ 1, & g(x) > 0 \end{cases}$ is discontinuous at $x = a$ if $g(a) = 0$.

ILLUSTRATION 4.24

Discuss continuity of

(i) $f(x) = \text{sgn}(x^3 - x)$ **(ii)** $f(x) = \text{sgn}(2\cos x - 1)$

(iii) $f(x) = \text{sgn}(x^2 - 2x + 3)$

Sol.

(i) $f(x) = \text{sgn}(x^3 - x)$

$f(x)$ is discontinuous when, $x^3 - x = 0$

$\therefore \quad x = 0, -1, 1$

Hence $f(x)$ is discontinuous at $x = 0, -1, 1$

(ii) $f(x) = \text{sgn}(2\cos x - 1)$

$f(x)$ is discontinuous when, $2\cos x - 1 = 0$

$\therefore \quad \cos x = \frac{1}{2}$

$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{3}$, $n \in Z$, where $f(x)$ is discontinuous.

(iii) $f(x) = \text{sgn}(x^2 - 2x + 3)$

$= \text{sgn}((x-1)^2 + 2)$

$= 1$ (as $(x-1)^2 + 2 > 0\ \forall\ x \in R$)

$\therefore \quad f(x)$ is continuous for all real x.

ILLUSTRATION 4.25

If $f(x) = \text{sgn}(2\sin x + a)$ is continuous for all real x, then find possible values of a.

Sol. Given that $f(x) = \text{sgn}(2\sin x + a)$ is continuous for all x

$\therefore \quad 2\sin x + a \neq 0$ for any real x.

$\Rightarrow \quad \sin x \neq -a/2$

$\Rightarrow \quad |a/2| > 1$

$\Rightarrow \quad |a| > 2$

$\Rightarrow \quad a < -2$ or $a > 2$.

ILLUSTRATION 4.26

Discuss the continuity of $f(x) = |x|\ \text{sgn}\ (x^3 - x)$.

Sol. $\text{sgn}(x^3 - x)$ is discontinuous when $x^3 - x = 0$ or $x = 0, \pm 1$.

But $\lim\limits_{x\to 0^+} |x|\,\text{sgn}(x(x^2-1)) = 0 \times (-1) = 0 = f(0)$

and $\lim\limits_{x\to 0^-} |x|\,\text{sgn}(x(x^2-1)) = 0 \times (1) = 0 = f(0)$

So, $f(x)$ is continuous at $x = 0$.

Hence, $f(x)$ discontinuous at $x = \pm 1$ only

ILLUSTRATION 4.27

If $f(x) = \begin{cases} \text{sgn}(x-2)\times[\log_e x], & 1 \le x \le 3 \\ \{x^2\}, & 3 < x \le 3.5 \end{cases}$

where $[\cdot]$ denotes greatest integer function and $\{\cdot\}$ represents fractional part function. Find the points where the continuity of $f(x)$ should be checked. Hence find the points of discontinuity.

Sol. We have following points of interest to check the continuity.

- $\text{sgn}(x - 2)$ is discontinuous when $x - 2 = 0$ or $x = 2$
- $[\log_e x]$ is discontinuous when $\log_e x$ is integer as $\log_e x$ is monotonically increasing function.

  For $x \in [1, 3]$, $\log_e x \in [0, \log_e 3]$

  $\therefore$ Integral value of $\log_e x$ is 1, which occurs when $x = e$.
- Function is changing its definition at $x = 3$.
- $\{x^2\}$ is discontinuous when x^2 is integer as x^2 is monotonically increasing function in the given interval.

  For $x \in [3, 3.5]$, $x^2 \in [9, 12.25]$

  $\therefore$ Integral values of x^2 are 9, 10, 11, 12.

  $\therefore \quad x = 3, \sqrt{10}, \sqrt{11}, \sqrt{12}$
- End points of interval $x = 1$ and $x = 3.5$

So, continuity should be checked at $x = 1, 2, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5$

At $x = 1$

Now $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \text{sgn}(x-2)\times[\log_e x] = 0 = f(1)$

So, $f(x)$ is continuous at $x = 1$.

At $x = 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \text{sgn}(x-2)\times[\log_e x] = (-1)\times 0 = 0$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \text{sgn}(x-2)\times[\log_e x] = (1)\times 0 = 0$$

Also $f(2) = 0$

Hence $f(x)$ is continuous at $x = 2$

At $x = e$

$$\lim_{x\to e^-} f(x) = \lim_{x\to e^-} \text{sgn}(x-2)\times[\log_e x] = (1)\times 0 = 0$$

$$\lim_{x\to e^+} f(x) = \lim_{x\to e^+} \text{sgn}(x-2)\times[\log_e x] = (1)\times 1 = 1$$

Hence $f(x)$ is discontinuous at $x = e$

At $x = 3$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} \text{sgn}(x-2)\times[\log_e x] = 1\times 1 = 1$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \{x^2\} = \lim_{x\to 3^+} (x^2 - [x^2]) = 9 - 9 = 0$$

Hence $f(x)$ is discontinuous at $x = 3$.

At $x = \sqrt{10}, \sqrt{11}, \sqrt{12}$

Clearly $f(x)$ is discontinuous at these points.

At $x = 3.5$

$$\lim_{x\to 3.5^-} f(x) = \lim_{x\to 3.5^-} \{x^2\}$$

$$= \lim_{x\to 3.5^-} (x^2 - [x^2])$$

$$= 12.25 - [12.25]$$

$$= 0.25 = f(3.5)$$

Hence $f(x)$ continuous at $x = 3.5$

is discontinuous at $x = e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}$

CONTINUITY OF FUNCTION INVOLVING LIMIT AT INFINITY

We know that $\lim\limits_{n\to\infty} a^n = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \to \infty, & a > 1 \end{cases}$

Also, $\lim\limits_{n\to\infty} a^{2n} = \lim\limits_{n\to\infty} (a^2)^n$

$$= \begin{cases} 0, & 0 \le a^2 < 1 \\ 1, & a^2 = 1 \\ \to \infty, & a^2 > 1 \end{cases}$$

$$= \begin{cases} 0, & -1 < a < 1 \\ 1, & a = \pm 1 \\ \to \infty, & a < -1 \text{ or } a > 1 \end{cases}$$

There are functions which use these limits. Let us discuss continuity of such functions.

ILLUSTRATION 4.28

Discuss continuity of function $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n} - 1}{x^{2n} + 1}$.

Sol. We have $f(x) = \lim\limits_{n\to\infty} \dfrac{(x^2)^n - 1}{(x^2)^n + 1}$

$$= \lim_{n\to\infty} \frac{1 - \dfrac{1}{(x^2)^n}}{1 + \dfrac{1}{(x^2)^n}} = \begin{cases} \dfrac{0-1}{0+1}, & 0 \le x^2 < 1 \\ 0, & x^2 = 1 \\ \dfrac{1-0}{1+0}, & x^2 > 1 \end{cases}$$

$$= \begin{cases} 1, & x < -1 \\ 0, & x = -1 \\ -1, & -1 < x < 1 \\ 0, & x = 1 \\ 1, & x > 1 \end{cases}$$

Clearly $f(x)$ is discontinuous at $x = \pm 1$

ILLUSTRATION 4.29

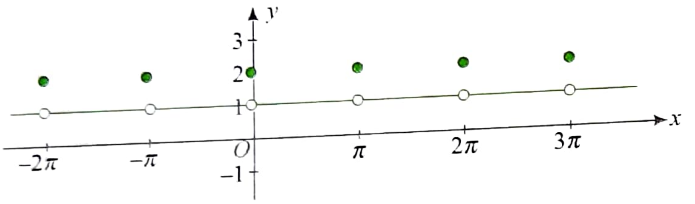
Discuss continuity of function $f(x) = 1 + \lim\limits_{n\to\infty} \cos^{2n} x$. Draw the graph of the function and find the period of the function.

Sol. We have $f(x) = 1 + \lim\limits_{n\to\infty} (\cos^2 x)^n$

$$= \begin{cases} 1+0, & 0 \le \cos^2 x < 1 \\ 1+1, & \cos^2 x = 1 \end{cases}$$

$$= \begin{cases} 1, & x \ne n\pi,\ n \in Z \\ 2, & x = n\pi,\ n \in Z \end{cases}$$

Hence, $f(x)$ is discontinuous when $x = n\pi$, $n \in Z$.
Graph of the function is as shown in the following figure.



Clearly, period of the function is π.

ILLUSTRATION 4.30

Find the values of a if $f(x) = \lim\limits_{n\to\infty} \dfrac{ax^{2n}+2}{x^{2n}+a+1}$ is continuous at $x = 1$.

Sol. $f(x)$ is continuous at $x = 1$.

So, $\lim\limits_{x\to 1^-}\left(\lim\limits_{n\to\infty} \dfrac{a(x^2)^n+2}{(x^2)^n+a+1}\right) = \lim\limits_{x\to 1^+}\left(\lim\limits_{n\to\infty} \dfrac{a+\dfrac{2}{(x^2)^n}}{1+\dfrac{a+1}{(x^2)^n}}\right) = \dfrac{a+2}{1+a+1}$

$\Rightarrow \quad \lim\limits_{n\to\infty} \dfrac{a(1^-)^n+2}{(1^-)^n+a+1} = \lim\limits_{n\to\infty} \dfrac{a+\dfrac{2}{(1^+)^n}}{1+\dfrac{a+1}{(1^+)^n}} = 1$

$\Rightarrow \quad \dfrac{0+2}{0+a+1} = \dfrac{a+0}{1+0} = 1$

$\Rightarrow \quad a = 1$

CONTINUITY OF FUNCTION DIFFERENTLY DEFINED FOR RATIONAL AND IRRATIONAL VALUES

Sometimes function is differently defined for rational and irrational real numbers. Let us discuss the continuity of such functions.

ILLUSTRATION 4.31

Discuss the continuity of function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Sol. We have $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

Let $x = a$, where a is rational number

$f(a) = 1$

L.H.L. at $x = a$ is

$f(a^-) = \lim\limits_{x\to a^-} f(x) = \lim\limits_{h\to 0} f(a-h)$

Now $\lim\limits_{h\to 0}(a-h)$ may be rational or irrational.

$\therefore \quad \lim\limits_{h\to 0} f(a-h)$ is either 0 or 1 and hence not fixed.

Also, we find that $\lim\limits_{h\to 0} f(a+h)$ may be 0 or 1.

So, $f(x)$ is discontinuous at $x = a$, where a is rational.
Similarly, we can prove that $f(x)$ is discontinuous at $x = a$, where a is irrational real number.
Thus, $f(x)$ is discontinuous for all real numbers.

Note:
This function is called Dirichlet function

ILLUSTRATION 4.32

Find the value of x where function,

$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$ is continuous.

Sol. Consider $x = a$, where a is rational.

$\therefore \quad f(a) = a$ (Using $f(x) = x$)

Now, $\lim\limits_{x\to a^-} f(x) = \lim\limits_{h\to 0} f(a-h)$

$$= \begin{cases} \lim\limits_{h\to 0}(a-h), & \text{if } (a-h) \text{ is rational} \\ \lim\limits_{h\to 0}(1-(a-h)), & \text{if } (a-h) \text{ is irrational} \end{cases}$$

$$= \begin{cases} a, & \text{if } (a-h) \text{ is rational} \\ 1-a, & \text{if } (a-h) \text{ is irrational} \end{cases}$$

For $f(x)$ to be continuous at $x = a$, we must have fixed value of $\lim\limits_{h\to 0} f(a-h)$.

So, $a = 1 - a$ or $a = 1/2$.

Similarly, for fixed value of $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{h\to 0} f(a+h)$, we must have $a = 1/2$.

Thus, $\lim\limits_{x\to \frac{1}{2}^-} f(x) = \lim\limits_{x\to \frac{1}{2}} x = \dfrac{1}{2}$

or $\lim\limits_{x\to \frac{1}{2}^-} f(x) = \lim\limits_{x\to \frac{1}{2}} (1-x) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

Similarly, $\lim\limits_{x\to \frac{1}{2}^+} f(x) = \dfrac{1}{2}$ in any case.

Hence $f(x)$ is continuous at $x = \dfrac{1}{2}$.

At all other values of x except $x = \dfrac{1}{2}$, $f(x)$ is discontinuous.

ILLUSTRATION 4.33

For $x > 0$, let $f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$, where p and $q > 0$ are relatively prime integers, then prove that $f(x)$ is continuous for all irrational values of x.

Sol. Let us check the continuity of $f(x)$ at some irrational number say, $x = \sqrt{2}$

$\therefore \quad f(\sqrt{2}) = 0$

Now $f(\sqrt{2}) \cong f(1.414043583)$

$$= f\left(\frac{1414023583}{10^9}\right)$$

$$= \frac{1}{10^9} \to 0 \qquad \left(\because f(x) = \frac{1}{q}, \text{ if } x = \frac{p}{q}\right)$$

Thus $f(x)$ is continuous at $x = \sqrt{2}$

With the similar reasoning, we can say that $f(x)$ is continuous for all irrational values of x.

CONTINUITY OF COMPOSITE FUNCTION

Consider function $f(x) = \dfrac{1}{1-x}$, which is discontinuous at $x = 1$.

$\therefore$ function $g(x) = f(f(x))$ is discontinuous when $f(x) = 1$,

i.e. $\dfrac{1}{1-x} = 1$ or $x = 0$.

Also, $g(x) = \dfrac{1}{1-f(x)} = \dfrac{1}{1-\frac{1}{1-x}} = \dfrac{x-1}{x}$.

Now $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} f(f(x))$

$$= \lim_{x\to 1^+} f\left(\frac{1}{1-x}\right)$$

$$= \lim_{y\to -\infty} f(y) = 0$$

Similarly, $\lim_{x\to 1^-} g(x) = 0$

But $g(1) = f(f(1))$ is not defined.

So, $f(f(x))$ has missing point discontinuity at $x = 1$.

Thus, $f(f(x))$ is discontinuous at $x = 0$ and $x = 1$.

Now, consider $h(x) = f(f(f(x)))$

$$= f\left(\frac{x-1}{x}\right)$$

$$= \frac{1}{1-\frac{x-1}{x}} = x$$

This function will be discontinuous at $x = 0$ (where $f(f(x))$ is discontinuous) and also at $x = 1$(where $f(x)$ is discontinuous).

At both these points $f(f(f(x)))$ has missing point discontinuity.

ILLUSTRATION 4.34

If $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{1}{x-2}$, then discuss the continuity of $f(x)$, $g(x)$ and fog (x).

Sol. $f(x)$ is not defined at $x = 1$

$\therefore$ f is discontinuous at $x = 1$

$g(x)$ is not defined at $x = 2$

$\therefore$ g is discontinuous at $x = 2$

Now $\quad \text{fog}(x) = \dfrac{g(x)+1}{g(x)-1}$

$$= \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}$$

$$= \frac{x-1}{3-x}$$

Therefore, $f(g(x))$ is discontinuous at $x = 3$ and at $x = 2$, where $g(x)$ is discontinuous.

At $x = 3$, $f(g(x))$ has nonremovable discontinuity.

At $x = 2$, $f(g(x))$ has missing point discontinuity, which is of removable type.

ILLUSTRATION 4.35

If $f(x) = \begin{cases} x-2, & x \le 0 \\ 4-x^2, & x > 0 \end{cases}$, then discuss the continuity of $f(f(x))$.

Sol. Since $f(x)$ is discontinuous at $x = 0$, $f(f(x))$ is also discontinuous at $x = 0$.

Also, $f(f(x))$ is discontinuous when $f(x) = 0$.

From $f(x) = x - 2 = 0$, $x = 2$.

This is not possible as $x \le 2$.

From $f(x) = 4 - x^2 = 0$, $x = 2$ (as $x > 0$).

So, $f(f(x))$ is discontinuous at $x = 0$ and $x = 2$.

CONCEPT APPLICATION EXERCISE 4.2

1. Find the values of x in $[1, 3]$ where the function $[x^2 + 1]$ ($[\cdot]$ represents the greatest integer function) is discontinuous.
2. Find the number of points of discontinuity for $f(x) = [6\sin x]$, $0 \le x \le \pi$ ($[\cdot]$ represents greatest integer function).
3. Discuss the continuity of $f(x) = [\tan^{-1}x]$ ($[\cdot]$ represents the greatest integer function).
4. Discuss the continuity of $f(x) = \{\cot^{-1}x\}$ ($\{\cdot\}$ represents the fractional part function).
5. Discuss the continuity of $f(x) = (\log|x|)\text{sgn}(x^2 - 1)$, $x \ne 0$.
6. Let $f(x) = \begin{cases} [\sin \pi x], & 0 \le x < 1 \\ \text{sgn}\left(x - \frac{5}{4}\right) \times \left\{x - \frac{2}{3}\right\}, & 1 \le x \le 2 \end{cases}$,
   where $[\cdot]$ denotes the greatest integer function and $\{.\}$ represents the fractional part function. At what points should the continuity be checked? Hence, find the points of discontinuity.
7. Let $f(x) = \lim_{n\to\infty} \dfrac{x^n - \sin x^n}{x^n + \sin x^n}$ for $x > 0$, $x \ne 1$, and $f(1) = 0$. Discuss the continuity at $x = 1$.

8. Discuss the continuity of $f(x)$ in $[0, 2]$, where
$f(x) = \lim_{n\to\infty}\left(\sin\frac{\pi x}{2}\right)^{2n}$.

9. Discuss the continuity of $f(x) = \begin{cases} x^2, & x \text{ is rational} \\ -x^2, & x \text{ is irrational} \end{cases}$.

10. Find the value of a for which $f(x) = \begin{cases} x^2, & x \in Q \\ x + a, & x \notin Q \end{cases}$ is not continuous at any x.

11. If $y = \frac{1}{t^2 + t - 2}$, where $t = \frac{1}{x-1}$, then find the number of points where $f(x)$ is discontinuous.

12. Find the points of discontinuity of $h(x) = \frac{1}{1 - e^{-\frac{x-1}{x-2}}}$.

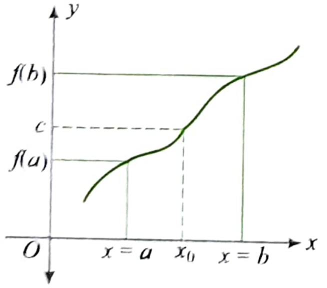
13. Let $f(x) = \begin{cases} 1 + x, & 0 \le x \le 2 \\ 3 - x, & 2 < x \le 3 \end{cases}$. Then discuss the continuity of $g(x) = f(f(x))$.

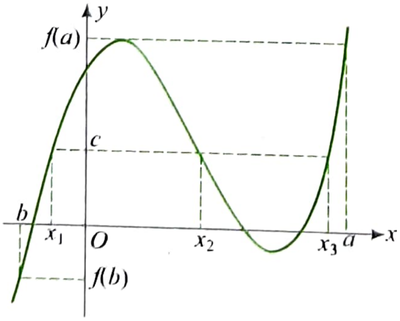
ANSWERS

1. $x = \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, 3$
2. 11 points
3. Discontinuous when $x = -\tan 1, 0, \tan 1$
4. Discontinuous when $x = \cot 1, \cot 2, \cot 3$
5. Continuous in entire domain
6. $x = \frac{1}{2}, 1, \frac{5}{4}, \frac{5}{3}$
7. Discontinuous
8. Continuous for all x except odd integers.
9. Continuous when $x = 0$
10. $a < -\frac{1}{4}$
11. Three
12. $x = 0, 2$
13. Discontinuous at $x = 1, 2$

INTERMEDIATE VALUE THEOREM (IVT)

The **intermediate value theorem** states that, if $f(x)$ is continuous on $[a, b]$ and $f(a) \neq f(b)$ then for any value $c \in (f(a), f(b))$, there is at least one number x_0 in (a, b) for which $f(x_0) = c$.





In other words, if a graph of a function can be traced continuously from $(a, f(a))$ to $(b, f(b))$, then it must take on every intermediate value from $f(a)$ to $f(b)$, possibly more than once along the way.

From the above two figures it is clear that continuity in the interval $[a, b]$ is essential for the validity of this theorem.

This theorem has two important corollaries:

I. If a continuous function has values of opposite sign inside an interval, then it has a root in that interval.
i.e. if function $f(x)$ is continuous in the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs then the graph of $y = f(x)$ must have crossed at least once in (a, b), i.e. $f(x) = 0$ has at least one root in (a, b).
This corollary is useful for the equation whose root cannot be found easily. We can find the smallest interval in which root of such equation occurs. Narrowing this interval using this corollary we reach at the root of equation.

II. The image of a continuous function over an interval is itself an interval.

ILLUSTRATION 4.36

Show that the function $f(x) = (x-a)^2 (x-b)^2 + x$ takes the value $\frac{a+b}{2}$ for some value of $x \in [a, b]$.

Sol. $f(a) = a$; $f(b) = b$. Also, f is continuous in $[a, b]$ and $\frac{a+b}{2} \in [a, b]$.

Hence, using intermediate value theorem, there exists at least one $c \in [a, b]$ such that $f(c) = \frac{a+b}{2}$.

ILLUSTRATION 4.37

Using intermediate value theorem, prove that there exists a number $x \in [0, 2]$ such that $x^{2005} + \frac{1}{1 + \sin^2 x} = 2005$.

Sol. Let $f(x) = x^{2005} + (1 + \sin^2 x)^{-2}$.
Thus, f is continuous and $f(0) = 1 < 2005$ and $f(2) > 2^{2005}$. Hence, from the intermediate value theorem, there exists a number c in $(0, 2)$ such that $f(c) = 2005$.

ILLUSTRATION 4.38

Let $f(x)$ be a continuous function defined for $1 \le x \le 3$. If $f(x)$ takes rational values for all x and $f(2) = 10$, then find the value of $f(1.5)$

Sol. As $f(x)$ is continuous in $[1, 3]$, $f(x)$ will attain all values between $f(1)$ and $f(3)$. If $f(1) \neq f(3)$, then $f(x)$ will attain innumerable irrational values between $f(1)$ and $f(3)$. But given that $f(x)$ attains rational values only, then we must have $f(1) = f(3)$, in fact $f(x) =$ constant for $x \in [1, 3]$. Since $f(2) = 10$, $f(x) = 2$, $\forall x \in [1, 3]$. So $f(1.5) = 10$

ILLUSTRATION 4.39

Let $f: [0, 1] \to [0, 1]$ be a continuous function. Then prove that for some $c \in [0, 1]$

(i) $f(c) = c$ (ii) $f(c) = 1 - c$

Sol.

(i) We have $f(c) = c$ for some $c \in [0, 1]$
i.e. $f(c) - c = 0$
$\therefore$ Consider function $g(x) = f(x) - x$

Now $g(0) = f(0) - 0 \geq 0$ (as $f(0) \geq 0$)

and $g(1) = f(1) - 1 \leq 0$ (as $f(1) \leq 0$)

If at least one $g(0)$ and $g(1)$ is zero then we are done.

Otherwise $g(0)$ and $g(1)$ have opposite signs.

Also, $g(x)$ is continuous function.

Hence there exists at least one $c \in (0, 1)$ such that $g(c) = 0$

$\therefore$ $f(c) - c = 0$ or $f(c) = c$.

(ii) We have $f(c) = 1 - c$ for some 'c' in $[0, 1]$.

i.e. $f(c) - 1 + c = 0$

$\therefore$ Consider function $g(x) = f(x) - 1 + x$

Now $g(0) = f(0) - 1 \leq 0$ (as $f(0) \leq 1$)

and $g(1) = f(1) \geq 0$ (as $f(1) \geq 0$)

If at least one $g(0)$ and $g(1)$ is zero then we are done.

Otherwise $g(0)$ and $g(1)$ have opposite signs.

Also, $g(x)$ is continuous function.

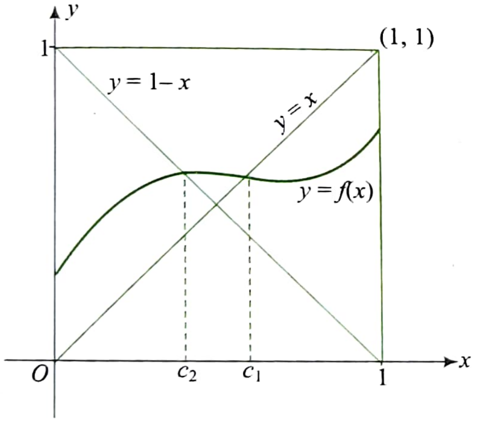
Hence there exists at least one $c \in (0, 1)$ such that $g(c) = 0$

$\therefore$ $f(c) - 1 + c = 0$ or $f(c) = 1 - c$.

Alternative method:

We can prove both (i) and (ii) graphically.

Since $f: [0, 1] \to [0, 1]$ be a continuous function the rough sketch of the graph of $y = f(x)$ along with $y = x$ and $y = 1 - x$ is as shown in the following figure.



We observe that lines $y = x$ and an $y = 1 - x$ both intersect the graph of $y = f(x)$.

That means for some $c_1, c_2 \in (0, 1)$ we have

$f(c_1) = c_1$ and $f(c_2) = 1 - c_2$.

CONCEPT APPLICATION EXERCISE 4.3

1. Prove that $f(x) = x^3/4 - \sin \pi x + 3$ takes the value 7/3 for $x \in [-2, 2]$.
2. Let f be continuous on the interval $[0, 1]$ to R such that $f(0) = f(1)$. Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$.
3. Suppose $f: R \to R$ is a continuous function and $f(f(a)) = a$ for some a. Show that there is some b such that $f(b) = b$.

DIFFERENTIABILITY: DEFINITION

In the previous chapter we defined derivative as slope of tangent and found the derivative of different functions assuming that in each function it is possible to draw a unique tangent at any point on the graph of it. In mathematical form the derivative of function $f(x)$ at point any point $x = a$ is defined by $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \left(= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right)$ provided this limit exists. Now the question is, what if it doesn't?

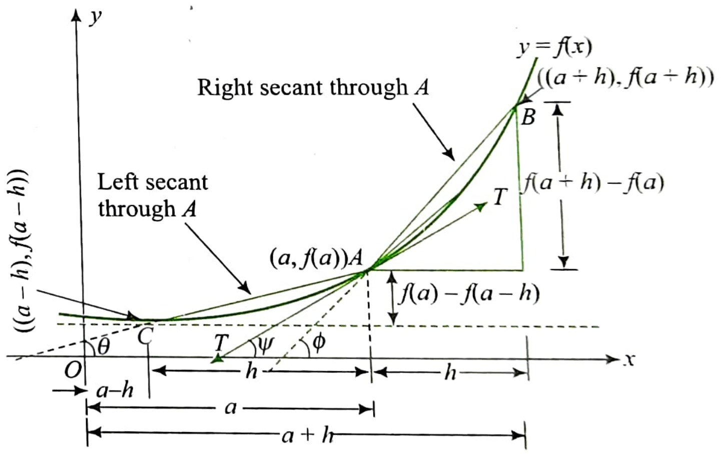
If $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ does not exist, we say that f is not differentiable at a. In other words, we say that a function f has derivative at point a or simply $f(x)$ is differentiable at a point 'a' in its domain if limit of the function $f'(x)$ exists at $x = a$,

i.e., both $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ $(= f'(a^+))$

and $\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$ $(= f'(a^-))$ are finite and equal.

$f'(a^+)$ is called the right-hand derivative of $f(x)$ at $x = a$ and $f'(a^-)$ is called left-hand derivative of $f(x)$ at $x = a$.

The graphical interpretation of this is as follows:



For the right-hand derivative of $f(x)$ at $x = a$, when $h \to 0$, the point B moves along the curve to approach point A. So, the chord AB approaches the tangent line AT at the point A and ϕ approaches ψ.

$$\therefore \quad f'(a^+) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \tan \phi = \tan \psi$$

For the left-hand derivative of $f(x)$ at $x = a$, when $h \to 0$, the point C moves along the curve to approach point A. So, the chord CA approaches the tangent line AT at the point A and θ approaches ψ.

$$\therefore \quad f'(a^-) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \tan \theta = \tan \psi$$

Thus, at point A, $f(x)$ is differentiable if both $f'(a^-)$ and $f'(a^+)$, exist finitely and are equal.

In other words, $f(x)$ is differentiable at $x = a$, if unique tangent can be drawn at this point.

Differentiability and Continuity

If $f(x)$ is differentiable at every point of its domain, then it must be continuous in that domain.

Proof:

To prove that f is continuous at a, given that $f'(a)$ exists finitely, we have to show that $\lim_{x\to a} f(x) = f(a)$.

or we have to show that $\lim_{x\to a}[f(x) - f(a)] = 0$

$$\text{Now} \quad \lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}(x-a)$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \times \lim_{x\to a}(x-a)$$

$$= f'(a) \times 0$$

$$= 0 \qquad (\because f'(a) \text{ is finite})$$

Therefore, f is continuous at a.

The converse of the above statement is not true, i.e. if $f(x)$ is continuous at $x = a$ then it need not to be differentiable at $x = a$.

For instance, the function $f(x) = |x|$ is continuous but not differentiable at $x = 0$ as

$$f'(0^+) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{h}{h} = 1 \text{ and}$$

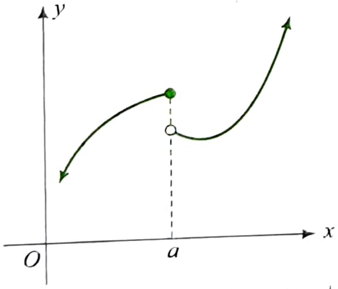
$$f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|-h|-0}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

Thus $f'(0^+) \neq f'(0^-)$.

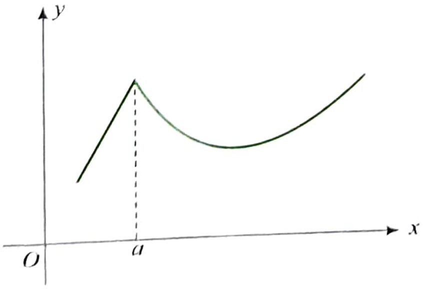
Reasons for Non-Differentiability

The function $f(x)$ is said to be non-differentiable at $x = a$ if

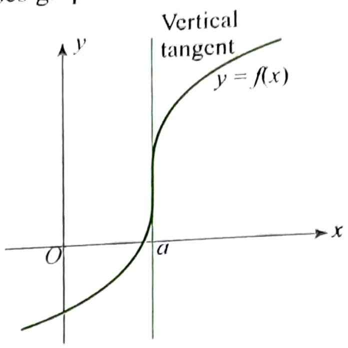
(i) $f(x)$ is discontinuous at $x = a$.
Obviously if $f(x)$ is discontinuous at $x = a$ (with any type of discontinuity), we cannot draw a unique tangent to the curve at this point. (there is no point in talking about tangent line at point of discontinuity.)



(ii) $f(x)$ is continuous at $x = a$ and both $f'(a^+)$ and $f'(a^-)$ exist but are not equal.
In such cases graph of the function has sharp turn at $x = a$.



(iii) $f(x)$ is continuous at $x = a$ and either or both $f'(a^+)$ and $f'(a^-)$ are not finite.
In such cases graph has vertical tangent at $x = a$.



Differentiability in an Interval

When function $f(x)$ is given with the interval $[a, b]$, we usually check continuity in closed interval $[a, b]$ and differentiability in an open interval (a, b). We say that function is differentiable in $[a, b]$ when it is differentiable at each intermediate points of the interval except end points. Also, differentiability at a point means "able to draw a unique tangent at a point". There is no point in drawing tangent at end points of interval. However, some sources define differentiability at end points of interval as a continuity of the function $f'(x)$ at these points. In that case $f(x)$ is differentiable at $x = a$ if $f'(a) = f'(a^+)$ and at $x = b$ if $f'(b) = f'(b^-)$.

However, if function is discontinuous at any of the end points then certainly it is not differentiable at this point. i.e. $f(x) = [x]$. for $x \in [1, 2]$ is discontinuous at $x = 2$ and so, non-differentiable at $x = 2$.

Note:
Polynomial, rational, trigonometric, exponential, logarithmic, inverse trigonometric functions are differentiable in their respective domains.

ILLUSTRATION 4.40

Let $f(x)$ be a continuous function satisfying the condition $f(-x) = f(x)$ for all real x. If $f'(0)$ exists, then find its value.

Sol. Given that $f(x)$ is a continuous function satisfying

$$f(-x) = f(x),\ \forall\, x \in R \qquad \ldots(1)$$

Also, $f'(0)$ exists

$\therefore \quad Rf'(0) = Lf'(0)$

Now, $\quad Rf'(0) = f'(0)$

$$\Rightarrow \quad \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{f(h)-f(0)}{h} = -\lim_{h\to 0} \frac{f(h)-f(0)}{h} \quad (\text{as } f(x) = f(-x))$$

$$\Rightarrow \quad f'(0) = -f'(0)$$

$$\Rightarrow \quad f'(0) = 0$$

ILLUSTRATION 4.41

Discuss the differentiability of $f(x) = (2x - 5)^{3/5}$ at $x = 5/2$.

Sol. We have $f(x) = (2x - 5)^{3/5}$

Clearly $f(x)$ is continuous for all real x.

Now, $f'(5/2) = \lim_{h \to 0} \dfrac{f(5/2+h) - f(5/2)}{h}$

$$= \lim_{h \to 0} \frac{(2(5/2+h)-5))^{3/5} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^{3/5}}{h} = \lim_{h \to 0} \frac{1}{h^{2/5}} \to \infty$$

Thus $f'(5/2)$ does not exists.

Here function has vertical tangent at $x = 5/2$.

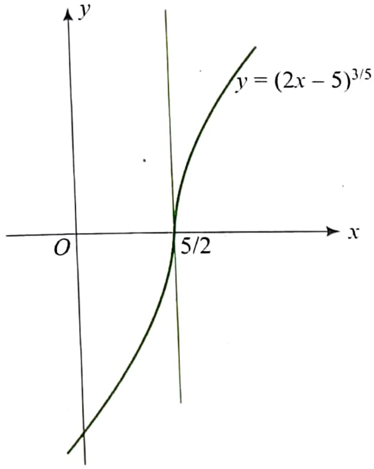


ILLUSTRATION 4.42

Discuss differentiability of $f(x) = \begin{cases} \dfrac{\sin x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

Sol. First check continuity at $x = 0$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x^2}{x} = \lim_{x \to 0} x \frac{\sin x^2}{x^2}$$

$$= \lim_{x \to 0} x \times \lim_{x \to 0} \frac{\sin x^2}{x^2} = 0 \times 1 = 0 = f(0)$$

So, $f(x)$ is continuous at $x = 0$.

For differentiability,

$$f'(0^+) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\sin h^2}{h^2} = 1$$

and $f'(0^-) = \lim_{h \to 0} \dfrac{f(-h) - f(0)}{-h}$

$$= \lim_{h \to 0} \frac{\sin h^2}{(-h)^2} = \lim_{h \to 0} \frac{\sin h^2}{h^2} = 1$$

Thus, L.H.D. = R.H.D. at $x = 0$.

So, $f(x)$ is differentiable at $x = 0$.

ILLUSTRATION 4.43

Discuss differentiability of $f(x) = \begin{cases} x \sin(\log_e x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$.

Sol. First check continuity

$$\lim_{x \to 0} x \sin(\log_e x^2) = \lim_{x \to 0} x \times \lim_{x \to 0} \sin(\log_e x^2)$$

$$= 0 \times \lim_{y \to -\infty} \sin y$$

$$= 0$$

So, $f(x)$ is continuous at $x = 0$.

For differentiability,

$$f'(0^+) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h \sin(\log_e h^2) - 0}{h}$$

$$= \lim_{h \to 0} \sin(\log_e h^2)$$

$$= \lim_{y \to -\infty} \sin y$$

= any value between −1 and 1

Thus, $f'(0^+)$ does not take any fixed value.

Hence $f(x)$ is not differentiable at $x = 0$.

ILLUSTRATION 4.44

Discuss the differentiability of $f(x) = \begin{cases} (x-e)2^{-2^{\frac{1}{e-x}}}, & x \neq e \\ 0, & x = e \end{cases}$ at $x = e$.

Sol. For continuity at $x = e$,

$$\text{R.H.L.} = f(e^+) = \lim_{h \to 0} (e + h - e) \,.\, 2^{-2^{\frac{1}{e-(e+h)}}}$$

$$= \lim_{h \to 0} h \times 2^{-2^{-\frac{1}{h}}}$$

$$= 0 \times 1$$

$$= 0 \text{ (as for } h \to 0, -\frac{1}{h} \to -\infty, \text{ then } 2^{-\frac{1}{h}} \to 0)$$

$$\text{L.H.L.} = f(e^-) = \lim_{h \to 0} (e - h - e) \,.\, 2^{-2^{\frac{1}{e-(e-h)}}}$$

$$= \lim_{h \to 0} (-h) . 2^{-2^{\frac{1}{h}}}$$

$$= 0 \times 0 = 0$$

Thus, $f(x)$ is continuous $x = e$.

For differentiability,

$$f'(e^+) = \lim_{h \to 0} \frac{f(e+h) - f(e)}{h}$$

$$= \lim_{h \to 0} \frac{h . 2^{-2^{-\frac{1}{h}}} - 0}{h}$$

$$= \lim_{h \to 0} 2^{-2^{-\frac{1}{h}}} = 1$$

$$f'(e^-) = \lim_{h\to 0}\frac{f(e-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{(-h).2^{-2^{\frac{1}{h}}}-0}{-h}$$

$$= \lim_{h\to 0} 2^{-2^{\frac{1}{h}}} = 0$$

Hence $f(x)$ is non-differentiable at $x = e$.

ILLUSTRATION 4.45

Discuss the differentiability of $f(x) = \sqrt{1-\sqrt{1-x^2}}$ at $x = 0$.

Sol. We have, $f(x) = \sqrt{1-\sqrt{1-x^2}}$.

Clearly $f(x)$ is continuous at $x = 0$ as

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\sqrt{1-\sqrt{1-x^2}} = 0 = f(0)$$

For differentiability,

$$f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-\sqrt{1-h^2}}-0}{-h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-(1-h^2)}}{-h}\cdot\frac{1}{\sqrt{1+\sqrt{1-h^2}}}$$

$$= \lim_{h\to 0}\frac{h}{-h}\cdot\frac{1}{\sqrt{1+\sqrt{1-h^2}}} = -\frac{1}{\sqrt{2}}$$

Similarly, we find that $f'(0^+) = \frac{1}{\sqrt{2}}$.

Hence, $f(x)$ is not differentiable at $x = 0$

CONCEPT APPLICATION EXERCISE 4.4

1. If f is an even function such that $\lim_{h\to 0}\frac{f(h)-f(0)}{h}$ has some finite non-zero value, then prove that $f(x)$ is not differentiable at $x = 0$.
2. Let $f(x)$ be a function satisfying $f(x + y) = f(x) + f(y)$ and $f(x) = xg(x)$ for all $x, y \in R$, where $g(x)$ is continuous. Then prove that $f'(x) = g(0)$.
3. If $f(x) = [2x] \sin 3\pi x$ then prove that $f'(k^+) = 6k\pi(-1)^k$, (where $[\cdot]$ denotes the greatest integer function and $k \in N$).
4. Let $f(x) = \begin{cases} g(x)\,.\cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0\end{cases}$, where $g(x)$ is an even function differentiable at $x = 0$, passing through the origin. Then find the value of $f'(0)$.
5. If $f(x) = \begin{cases} (\sin^{-1}x)^2.\cos\frac{1}{x}, & x\neq 0 \\ 0, & x = 0\end{cases}$ then discuss differentiability of $f(x)$ at $x = 0$.
6. Let $f(x) = \begin{cases}\sqrt{x}(1+x\sin(1/x)), & x>0 \\ -\sqrt{(-x)}(1+x\sin(1/x)), & x<0 \\ 0, & x=0\end{cases}$.

   Discuss differentiability at $x = 0$.

ANSWERS

4. 0 5. Differentiable at $x = 0$
6. Non-differentiable at $x = 0$

EXAMINING DIFFERENTIABILITY USING DIFFERENTIATION AND GRAPH OF THE FUNCTION

In previous section we discussed differentiability mainly using comparison of left-hand and right-hand derivative. Now there are functions defined piecewise in which it is easier to differentiate the branch function. In such case we differentiate each branch function and compare left hand and right-hand derivative at each split points. i.e. if $f(x) = \begin{cases} g_1(x), & x<a \\ g_2(x), & x\geq a\end{cases}$ and $f(x)$ is continuous at $x=a$, then we find $f'(x) = \begin{cases}(g_1(x))', & x<a \\ (g_2(x))', & x>a\end{cases}$. Differentiability can be checked at $x = a$ by comparing $\lim_{x\to a^-}(g_1(x))'$ and $\lim_{x\to a^+}(g_2(x))'$.

Also, sometimes it is easier to draw the graph of the function using graphical transformations. From the graph we can identify the points where function has sharp turn. These are points of non-differentiability. Sometimes we use combination of differentiation and graph plotting to find the points of non-differentiability.

ILLUSTRATION 4.46

Prove that each of the following function is differentiable at $x = 0$.

(i) $f(x) = \cos|x|$ **(ii)** $f(x) = x|x|$

(iii) $f(x) = |x^3|$ **(iv)** $f(x) = \frac{x}{1+|x|}$

Sol.

(i) We know that $\cos(-x) = \cos x$.

$\therefore$ $\cos|x| = \cos x$, which is differentiable for all real x.

(ii) $f(x) = x|x| = \begin{cases} x\cdot x, & x\geq 0 \\ x\cdot(-x), & x<0\end{cases}$

$$= \begin{cases} x^2, & x\geq 0 \\ -x^2, & x<0\end{cases}$$

$$\therefore \quad f'(x) = \begin{cases} 2x, & x>0 \\ -2x, & x<0\end{cases}$$

$\therefore$ $f'(0^-) = -2(0) = 0$ and $f(0^+) = 2(0) = 0$

Hence $f(x)$ is differentiable at $x = 0$.

(iii) $f(x) = |x^3| = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$

$$\therefore \quad f'(x) = \begin{cases} 3x^2, & x > 0 \\ -3x^2, & x < 0 \end{cases}$$

$\therefore \quad f'(0^-) = -3(0)^2 = 0$ and $f'(0^+) = 3(0)^2 = 0$

Hence $f(x)$ is differentiable at $x = 0$.

(iv) $f(x) = \dfrac{x}{1+|x|} = \begin{cases} \dfrac{x}{1-x}, & x < 0 \\ \dfrac{x}{1+x}, & x \geq 0 \end{cases}$

$$\therefore \quad f'(x) = \begin{cases} \dfrac{1}{(1-x)^2}, & x < 0 \\ \dfrac{1}{(1+x)^2}, & x > 0 \end{cases}$$

$\therefore \quad f'(0^-) = \dfrac{1}{(1-0)^2} = 1$ and $f'(0^+) = \dfrac{1}{(1+0)^2} = 1$

Hence $f(x)$ is differentiable at $x = 0$.

Note:
From this illustration we learned that mere presence of modulus function does not make function non-differentiable.

ILLUSTRATION 4.47

If $f(x) = \begin{cases} x, & x \leq 1 \\ x^2 + bx + c, & x > 1 \end{cases}$, then find the values of b and c if $f(x)$ is differentiable at $x = 1$.

Sol. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2 + bx + c, & x > 1 \end{cases}$

$$\therefore \quad f'(x) = \begin{cases} 1, & x < 1 \\ 2x + b, & x > 1 \end{cases}$$

$f(x)$ is differentiable at $x = 1$.

Then, it must be continuous at $x = 1$ for which

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$$

or $\quad 1 + b + c = 1$

or $\quad b + c = 0 \qquad (1)$

Also, $f'(1^+) = f'(1^-)$

or $\quad \lim_{x \to 1^+} f'(x) = \lim_{x \to 1^-} f'(x)$

or $\quad 2 + b = 1$ or $b = -1$

$\therefore \quad c = 1 \qquad$ [From (1)]

ILLUSTRATION 4.48

Find the values of a and b if $f(x) = \begin{cases} a + \sin^{-1}(x+b), & x \geq 1 \\ x, & x < 1 \end{cases}$ is differentiable at $x = 1$.

Sol. $f(x) = \begin{cases} a + \sin^{-1}(x+b), & x \geq 1 \\ x, & x < 1 \end{cases}$

$$\therefore \quad f'(x) = \begin{cases} \dfrac{1}{\sqrt{1-(x+b)^2}}, & x > 1 \\ 1, & x < 1 \end{cases}$$

For $f(x)$ to be continuous at $x = 1$,

$f(1^+) = f(1^-)$ or $a + \sin^{-1}(1 + b) = 1 \qquad (1)$

Also, $f'(1^+) = f'(1^-)$ or $\dfrac{1}{\sqrt{1-(1+b)^2}} = 1$ or $b = -1$.

Therefore, from (1), $a = 1$.

ILLUSTRATION 4.49

$$f(x) = \begin{cases} ax(x-1) + b, & x < 1 \\ x - 1, & 1 \leq x \leq 3. \\ px^2 + qx + 2, & x > 3 \end{cases}$$

Find the values of the constants a, b, p, and q so that all the following conditions are satisfied.

(i) $f(x)$ is continuous for all x.
(ii) $f'(1)$ does not exist.
(iii) $f'(x)$ is continuous at $x = 3$.

Sol. $f(x)$ is continuous $\forall\, x \in R$.

Hence, it must be continuous at $x = 1, 3$.

$$f(1^-) = \lim_{x \to 1} ax(x-1) + b = b$$

$$f(1^+) = \lim_{x \to 1} (x-1) = 0$$

Now, $\quad f(1^-) = f(1^+) \qquad$ (For continuity at $x = 1$)

or $\quad b = 0$

$$f(3^-) = \lim_{x \to 3} (x-1) = 2$$

$$f(3^+) = \lim_{x \to 3} (px^2 + qx + 2) = 9p + 3q + 2$$

Now, $\quad f(3^-) = f(3^+) \qquad$ (For continuity at $x = 3$)

or $\quad 9p + 3q = 0 \qquad (1)$

$$f'(x) = \begin{cases} 2ax - a, & x < 1 \\ 1, & 1 < x < 3 \\ 2px + q, & x > 3 \end{cases}$$

Now, given that $f'(1)$ does not exist. Therefore,

$$f'(1^+) \neq f'(1^-)$$

or $\quad 1 \neq 2a - a$

or $\quad a \neq 1$

Also, given that $f'(3)$ exists. Therefore,

$$f'(3^-) = f'(3^+)$$

or $\quad 1 = 6p + q \qquad (2)$

Solving (1) and (2) for p and q, we get $p = 1/3$, $q = -1$.

ILLUSTRATION 4.50

Let $f(x) = \begin{cases} a\sqrt{x+1}, & 0 < x < 3 \\ bx+2, & 3 \le x < 5 \end{cases}$. If $f(x)$ is differentiable at $x = 3$ then find the values of a and b.

Sol. $f(x)$ must be continuous at $x = 3$.

$\therefore \quad \lim_{x\to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

$\Rightarrow \quad a\sqrt{3+1} = 3b+2$

$\Rightarrow \quad 2a = 3b + 2 \qquad \ldots(1)$

Now $f'(x) = \begin{cases} \dfrac{a}{2\sqrt{x+1}}, & 0<x<3 \\ b, & 3<x<5 \end{cases}$

$f(x)$ is differentiable at $x = 3$,

$\therefore \quad f'(3^-) = f'(3^+)$

$\Rightarrow \quad \dfrac{a}{2\sqrt{3+1}} = b$

$\Rightarrow \quad a = 4b \qquad \ldots(2)$

Solving (1) and (2), we get $b = \dfrac{2}{5}$ and $a = \dfrac{8}{5}$

ILLUSTRATION 4.51

Discuss the differentiability of $f(x) = \sin^{-1}\dfrac{2x}{1+x^2}$

Sol. We know that

$$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & -1 \le x \le 1 \\ \pi - 2\tan^{-1}x, & x > 1 \\ -\pi - 2\tan^{-1}x, & x < -1 \end{cases} \qquad \ldots(1)$$

$$\Rightarrow f'(x) = \begin{cases} \dfrac{2}{1+x^2}, & -1 < x < 1 \\ -\dfrac{2}{1+x^2}, & x > 1 \\ -\dfrac{2}{1+x^2}, & x < -1 \end{cases} \qquad \ldots(2)$$

For differentiability at $x = -1$

$$\lim_{x\to -1^-} f'(x) = \lim_{x\to -1^-}\left(-\frac{2}{1+x^2}\right) = -1$$

$$\lim_{x\to -1^+} f'(x) = \lim_{x\to -1^+}\frac{2}{1+x^2} = 1$$

Thus L.H.D. $\neq$ R.H.D. at $x = 1$.

$\therefore \quad f(x)$ is non-differentiable at $x = 1$.

Similarly, we get $f'(1^-) = 1$ and $f'(1^+) = -1$

$\therefore \quad f(x)$ is non-differentiable at $x = 1$ also.

Alternative method:

Often it is difficult to remember the branch functions for different interval of (1).

In that case we can differentiate $f(x) = \sin^{-1}\dfrac{2x}{1+x^2}$ without considering its differentiability at $x = \pm 1$.

$\therefore \quad$ We have $f'(x) = \dfrac{1}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}} \cdot \dfrac{d}{dx}\left(\dfrac{2x}{1+x^2}\right)$

$$= \frac{(1+x^2)}{\sqrt{(1+x^2)^2 - 4x^2}} \cdot \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)|1-x^2|}$$

$\therefore \quad f'(1^+) = \lim_{x\to 1^+}\dfrac{2(1-x^2)}{(1+x^2)(x^2-1)} = -1$

and $\quad f'(1^-) = \lim_{x\to 1^+}\dfrac{2(1-x^2)}{(1+x^2)(1-x^2)} = 1$

$\therefore \quad f(x)$ is non-differentiable at $x = 1$.

Similarly, it can be proved that $f(x)$ is non-differentiable at $x = -1$.

This method is useful for those inverse trigonometric function which splits into branches.

ILLUSTRATION 4.52

Draw the graph of the following functions and discuss differentiability.

(i) $f(x) = \sin|x|$ **(ii)** $f(x) = |\log_e|x||, x \neq 0.$

(iii) $y = \sin^{-1}(\sin x)$, **(iv)** $y = \sin^{-1}|\sin x|$

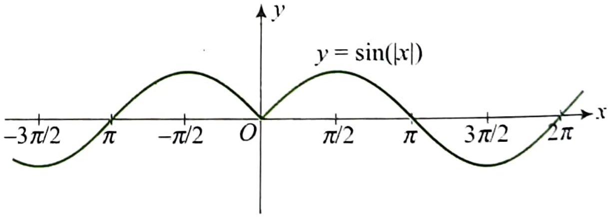
(v) $f(x) = e^{-|x|}$

Sol.

(i) $f(x) = \sin|x|$

In function chapter, we learned to draw the graph of such functions using graphical transformations.

The graph of the function is as shown in the following figure.



Also, $f(x) = \begin{cases} -\sin x, & x < 0 \\ \sin x, & x \ge 0 \end{cases}$

$\therefore \quad f'(x) = \begin{cases} -\cos x, & x < 0 \\ \cos x, & x > 0 \end{cases}$

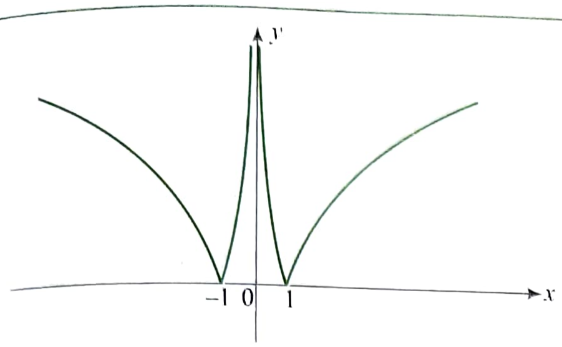
$\therefore \quad f'(0^-) = -1$ and $f'(0^+) = 1$

So, $f(x)$ is non-differentiable at $x = 0$

Also, it is evident from the graph as graph has sharp turn at $x = 0$.

(ii) $f(x) = |\log_e|x||, x \neq 0.$

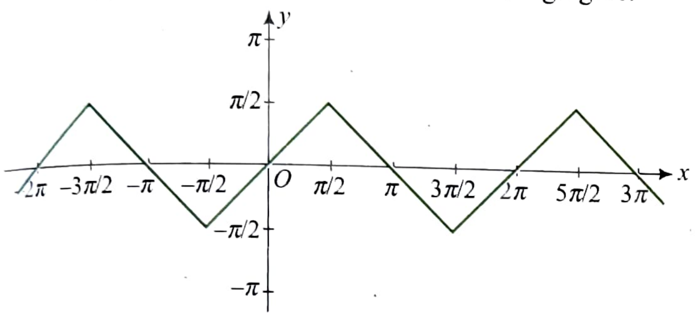
Using graphical transformation on $y = \log_e x$, we can get the graph of $y = |\log_e|x||$ as follows.



From the graph of the function, $f(x)$ is non-differentiable at $x = \pm 1$ as graph has sharp turn at these points.

(iii) $f(x) = \sin^{-1}(\sin x)$

The graph of $f(x)$ is as shown in the following figure.

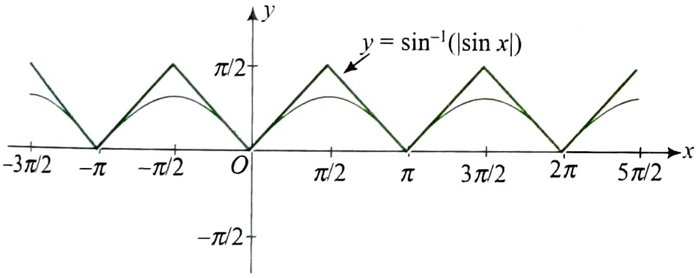


From the graph of the function is non-differentiable at $x = (2n+1)\frac{\pi}{2}$, $n \in Z$ as graph has sharp turn at these points.

(iv) $f(x) = \sin^{-1}|\sin x|$

The graph of $f(x)$ can be easily drawn from the graph of $y = |\sin x|$ function has period π.

Graphs of $|\sin x|$ and $\sin^{-1}|\sin x|$ are as shown in the following figure.



From the graph of the function is non-differentiable at $x = \frac{n\pi}{2}$, $n \in Z$ as graph has sharp turn at these points.

(v) We have, $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0 \end{cases}$

Clearly $f(x)$ is continuous for all real x.

Now, $f'(x) = \begin{cases} -e^{-x}, & x > 0 \\ e^{x}, & x < 0 \end{cases}$

$\therefore \quad f'(0^+) = -e^0 = -1$ and $f'(0^-) = e^0 = 1$

Thus $f'(0^+) \neq f'(0^-)$.

Hence, $f(x)$ is not differentiable at $x = 0$.

Graph of $y = f(x)$ is as shown in the following figure.

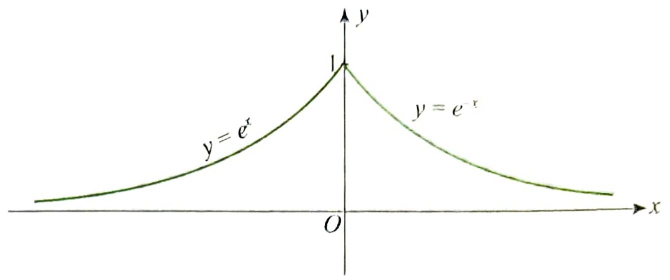


ILLUSTRATION 4.53

Draw the graph of the following function and find the points of non-differentiability.

(i) $f(x) = \max.\{\sec^{-1}x, \operatorname{cosec}^{-1}x\}$

(ii) $f(x) = \max.\{2\sin x, 1 - \cos x\}, x \in (0, \pi)$

(iii) $f(x) = \min.\{x, \sin x\}$

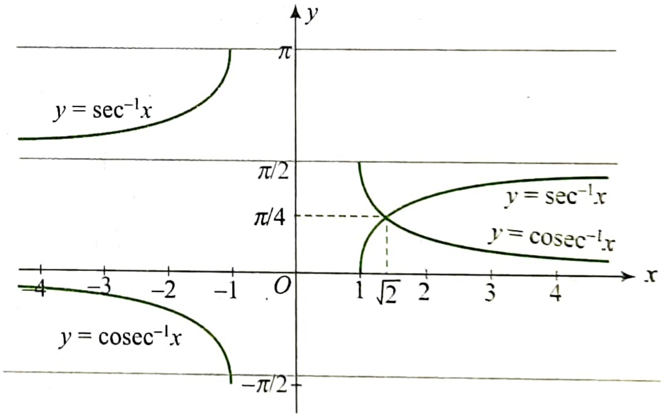
Sol.

(i) $f(x) = \max.\ \{\sec^{-1}x, \operatorname{cosec}^{-1}x\}$

The element functions $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$ are continuous and differentiable in their common domain $R - (-1, 1)$.

So, point of non-differentiability may lie at a point where graphs of these element functions cross each other.

Graphs of $y = \sec^{-1}x$ and $y = \operatorname{cosec}^{-1}x$ are as shown in the following figure.



From the graph $f(x) = \begin{cases} \sec^{-1}x, & x \leq -1 \\ \operatorname{cosec}^{-1}x, & 1 \leq x \leq \sqrt{2} \\ \sec^{-1}x, & x > \sqrt{2} \end{cases}$

Clearly, $f(x)$ is continuous in its domain.

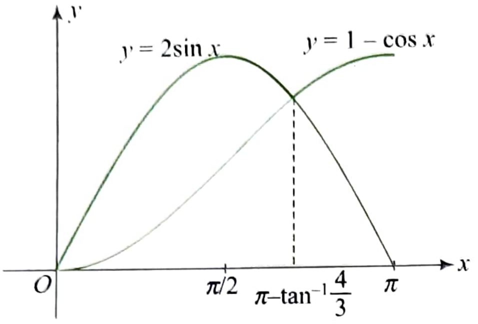
From the graph, $f(x)$ is non-differentiable at $x = \sqrt{2}$, where graphs of $y = \sec^{-1}x$ and $y = \operatorname{cosec}^{-1}x$ cross each other.

Also, $f'(x) = \begin{cases} -\dfrac{1}{|x|\sqrt{x^2-1}}, & 1 < x < \sqrt{2} \\ \dfrac{1}{|x|\sqrt{x^2-1}}, & x > \sqrt{2} \end{cases}$

$\therefore \quad f'(\sqrt{2}^-) = -\frac{1}{\sqrt{2}}$ and $f'(\sqrt{2}^+) = \frac{1}{\sqrt{2}}$

(ii) $f(x) = \max.\{2\sin x, 1 - \cos x\}, x \in (0, \pi)$

First, we draw the graphs of $y = 2\sin x$ and $y = 1 - \cos x$



Now to find the point of intersection of these graphs, we have $2\sin x = 1 - \cos x$

$$\therefore \quad 4\sin\frac{x}{2}\cos\frac{x}{2} = 2\sin^2\frac{x}{2}$$

$$\Rightarrow \quad \sin\frac{x}{2} = 0 \text{ or } \tan\frac{x}{2} = 2$$

But $x \in (0, \pi)$.

$$\therefore \quad \tan\frac{x}{2} = 2 \text{ is possible for which } x = 2\tan^{-1} 2.$$

$$\text{or} \quad x = \pi + \tan^{-1}\frac{4}{-3} = \pi - \tan^{-1}\frac{4}{3}$$

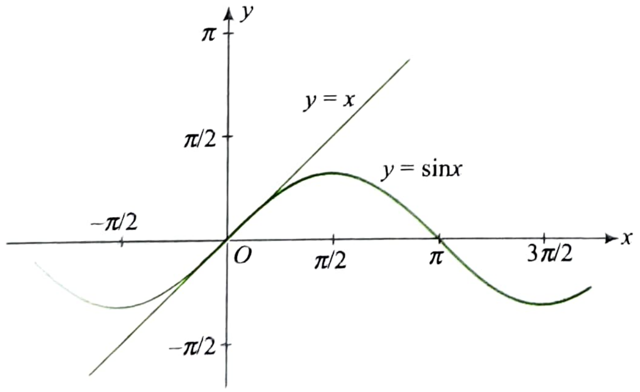
$$\therefore \quad f(x) = \begin{cases} 2\sin x, & 0 < x < \pi - \tan^{-1}\frac{4}{3} \\ 1 - \cos x, & \pi - \tan^{-1}\frac{4}{3} \le x < \pi \end{cases}$$

Clearly $f(x)$ is continuous at $x = \pi - \tan^{-1}\frac{4}{3}$

From the figure we observe graphs of $y = 2\sin x$ and $y = 1 - \cos x$ are crossing each other at $\pi - \tan^{-1}\frac{4}{3}$ creating sharp turn, where $f(x)$ is non-differentiable.

(iii) $f(x) = \min.\{x, \sin x\}$

First, we draw the graphs of $y = x$ and $y = \sin x$



We observe that graphs intersect at $x = 0$.

$$\therefore \quad f(x) = \begin{cases} x, & x < 0 \\ \sin x, & x \ge 0 \end{cases}$$

Clearly $f(x)$ is continuous at $x = 0$.

$$f'(x) = \begin{cases} 1, & x < 0 \\ \cos x, & x > 0 \end{cases}$$

$$\therefore \quad f'(0^-) = 1 \text{ and } f'(0^+) = \cos 0 = 1$$

So, $f(x)$ is differentiable at $x = 0$.

Here $y = x$ crosses $y = \sin x$ tangentially at $x = 0$.

Thus, such functions are not necessarily non-differentiable when graphs of element functions cross each other.

ILLUSTRATION 4.54

If $f(x) = \max\{x^2 + 2ax + 1, b\}$ has two points of non-differentiability, then prove that $a^2 > 1 - b$.

Sol. $f(x) = \max\{x^2 + 2ax + 1, b\}$ has two points of non-differentiability if

$y = x^2 + 2ax + 1$ and $y = b$ intersect at two points

or $\quad x^2 + 2ax + 1 = b$ has real and distinct roots

or $\quad x^2 + 2ax + 1 - b = 0$ has real and distinct roots

or $\quad 4a^2 - 4(1 - b) > 0 \quad$ or $\quad a^2 > 1 - b$

ILLUSTRATION 4.55

Test the continuity and differentiability of the function $f(x) = \left|\left(x + \frac{1}{2}\right)[x]\right|$, by drawing the graph of the function when $-2 \le x \le 2$.

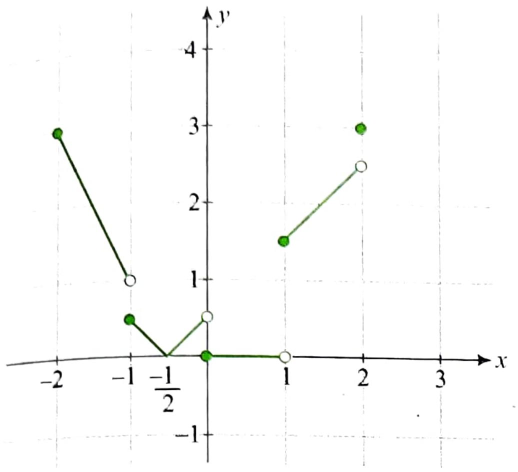
Sol. We have,

$$f(x) = \left|\left(x + \frac{1}{2}\right)[x]\right|, -2 \le x \le 2$$

$$\Rightarrow \quad f(x) = \begin{cases} \left|\left(x + \frac{1}{2}\right)(-2)\right| = |2x + 1|, & -2 \le x < -1 \\ \left|\left(x + \frac{1}{2}\right)(-1)\right| = \left|x + \frac{1}{2}\right|, & -1 \le x < 0 \\ \left|\left(x + \frac{1}{2}\right)(0)\right| = 0, & 0 \le x < 1 \\ \left|\left(x + \frac{1}{2}\right)(1)\right| = \left|x + \frac{1}{2}\right|, & 1 \le x < 2 \\ \left|\frac{3}{2} \cdot 2\right| = 3, & x = 2 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -(2x + 1), & -2 \le x < -1 \\ -x - \frac{1}{2}, & -1 \le x < -\frac{1}{2} \\ x + \frac{1}{2}, & -\frac{1}{2} \le x < 0 \\ 0, & 0 \le x < 1 \\ x + \frac{1}{2}, & 1 \le x < 2 \\ 3, & x = 2 \end{cases}$$

The graph of the function is as shown in the following figure.



From the figure, $f(x)$ is not continuous at $x = -1, 0, 1, 2$ as at these points $f(x)$ has jump.

Graph has sharp turn at $x = 1/2$.

So, $f(x)$ is non-differentiable at $x = -1, -1/2, 0, 1, 2$.

Differentiability of Functions Obtained by Algebraic Operations on Two Given Functions

I. If $f(x)$ and $g(x)$ are differentiable functions in the given interval, then their sum, difference, product and quotient functions are also differentiable.

i.e. function $f(x) = x^2 + \sin x$, $g(x) = \log_e x + \tan^{-1}x$,

$h(x) = 2^x + \dfrac{x}{1+x^2}$ are differentiable.

However, for quotient function denominator function should not be zero in the interval where differentiability is checked.

II. If $f(x)$ is differentiable and $g(x)$ is not differentiable at $x = a$, then $f(x) \pm g(x)$ is a not differentiable at $x = a$.

i.e., function $f(x) = |x| + \cos x$ is not differentiable at $x = 0$, $g(x) = x^3 + \text{sgn}(x)$ is not differentiable at $x = 0$.

III. If both the functions $f(x)$ and $g(x)$ are non-differentiable at $x = a$, then function obtained by algebraic operation of $f(x)$ and $g(x)$ may or may not be differentiable at $x = a$.

Consider following examples

(i) $f(x) = |x| - \sin|x|$.

Here both $|x|$ and $\sin|x|$ are non-differentiable at $x = 0$.

But $f(x) = \begin{cases} -x + \sin x, & x < 0 \\ x - \sin x, & x \ge 0 \end{cases}$

$\therefore \quad f'(x) = \begin{cases} -1 + \cos x, & x < 0 \\ 1 - \cos x, & x > 0 \end{cases}$

$\therefore \quad f'(0^-) = -1 + \cos 0 = -1 + 1 = 0$

And $f'(0^+) = 1 - \cos 0 = 1 - 1 = 0$

So, $f(x)$ is differentiable at $x = 0$.

(ii) $f(x) = |x| + \sin|x|$

Here both $|x|$ and $\sin|x|$ are non-differentiable at $x = 0$.

Also, it can be verified that $f(x)$ is non-differentiable at $x = 0$.

(iii) $f(x) = |x|(\text{sgn}(x))$

Here both $|x|$ and $\text{sgn}(x)$ are non-differentiable at $x = 0$.

But $f(x) = \begin{cases} (-x)(-1), & x < 0 \\ 0, & x = 0 \\ (x)(1), & x > 0 \end{cases} = x \ \forall \in R$

Thus $f(x)$ is differentiable at $x = 0$.

IV. If $f(x)$ is differentiable and $g(x)$ is non-differentiable at $x = a$ then the product function $h(x) = f(x) \times g(x)$ may or may not be differentiable at $x = a$.

Consider following examples

(i) $f(x) = x^3\,\text{sgn}(x)$

Here x^3 is differentiable at $x = 0$ and $\text{sgn}(x)$ is not differentiable at $x = 0$.

Now product function,

$h(x) = f(x) \times g(x)$

$= \begin{cases} x^3, & x > 0 \\ 0, & x = 0 \\ -x^3, & x < 0 \end{cases}$ is continuous at $x = 0$.

Also, $h'(x) = \begin{cases} 3x^2, & x > 0 \\ -3x^2, & x < 0 \end{cases}$

$\therefore \quad h'(0^-) = h'(0^+) = 0.$

So, $h(x)$ is differentiable at $x = 0$.

(ii) $f(x) = x|x - 1|$

Here x is differentiable and $|x - 1|$ is not differentiable at $x = 1$.

And $f(x) = \begin{cases} x(1-x), & x < 1 \\ x(x-1), & x \ge 1 \end{cases}$

$\therefore \quad f'(x) = \begin{cases} 1 - 2x, & x < 1 \\ 2x - 1, & x > 1 \end{cases}$

$\therefore \quad f'(1^-) = 1 - 2 = -1$

And $f'(1^+) = 2 - 1 = 1$

Thus $f(x)$ is non-differentiable at $x = 1$.

In a special case if $f(x)$ and $g(x)$ are continuous at $x = a$ with $f(a) = 0$ and $g(x)$ being non-differentiable at $x = a$ then $f(x) \times g(x)$ is differentiable at $x = a$.

i.e., $(x - 1)|x - 1|$, $|x - 1| \log_e x$, $|x - 1| \sin \pi x$ are differentiable at $x = 1$ as $(x - 1)$, $\log_e x$ and $\sin \pi x$ vanish at $x = 1$, where adjoining fucntion is non-differentiable.

ILLUSTRATION 4.56

Discuss differentiability of $f(x) = |x| + |x - 1|$

Sol. $f(x)$ is continuous everywhere as $|x|$ and $|x - 1|$ are continuous for all x.

$|x|$ is not differentiable and $|x - 1|$ is differentiable at $x = 0$.

So, $f(x)$ is not differentiable at $x = 0$.

$|x|$ is differentiable and $|x - 1|$ is not differentiable at $x = 1$.

So, $f(x)$ is not differentiable at $x = 1$.

ILLUSTRATION 4.57

Discuss the differentiability of $f(x) = [x] + |1 - x|, x \in (-1, 3)$

Sol. $[x]$ is discontinuous and hence, not differentiable at $x = 0, 1, 2$.

$|1 - x|$ is continuous everywhere but not differentiable at $x = 1$.

Thus, definitely $f(x)$ is discontinuous at $x = 0, 1, 2$ and hence, not differentiable at $x = 0, 1, 2$.

ILLUSTRATION 4.58

Let $f(x) = \begin{cases} (x-1)^2 \sin \dfrac{1}{(x-1)} - |x|, & \text{if } x \neq 1 \\ -1, & \text{if } x = 1 \end{cases}$ be a real-valued function. Then find the number of points where $f(x)$ is not differentiable.

Sol. Clearly $(x-1)^2 \sin \dfrac{1}{(x-1)}$ is differentiable at $x = 0$, but $|x|$ is not.

So, $f(x)$ is not differentiable at $x = 0$.

Now for differentiability at $x = 1$,

$$\lim_{x\to 1}\left[(x-1)^2 \sin \frac{1}{(x-1)} - |x|\right]$$

$$= 0 - 1 = -1 = f(1)$$

$\therefore$ $f(x)$ is continuous at $x = 1$.

Now, $f'(1^+) = \lim\limits_{h\to 0} \dfrac{f(1+h) - f(1)}{h}$

$$= \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h} - |1+h| - (-1)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h} - h}{h}$$

$$= 0 - 1 = -1$$

Similarly, we get $f'(1^-) = -1$.

So, $f(x)$ is differentiable at $x = 1$

ILLUSTRATION 4.59

Discuss the differentiability of $f(x) = (x^2 - 1)|x^2 - x - 2| + \sin(|x|)$.

Sol. $f(x) = (x^2 - 1)|x^2 - x - 2| + \sin(|x|)$

$$= (x - 1)(x + 1)|x + 1|\,|x - 2| + \sin(|x|)$$

$(x + 1)|x + 1|$ is differentiable at $x = -1$.

$|x - 2|$ is non-differentiable at $x = 2$.

$\sin(|x|)$ is non-differentiable at $x = 0$.

Hence, $f(x)$ is differentiable at $x = -1$ but not at $x = 0$ and $x = 2$.

ILLUSTRATION 4.60

Discuss the differentiability of $f(x) = |x| \sin x + |x - 2| \operatorname{sgn}(x - 2) + |x - 3|$.

Sol. $|x| \sin x$ is differentiable at $x = 0$, though $|x|$ is not differentiable at $x = 0$, as $\sin 0 = 0$.

$$|x - 2|\operatorname{sgn}(x - 2) = \begin{cases} (2 - x)(-1), & x < 2 \\ 0, & x = 2 \\ x - 2, & x > 2 \end{cases}$$

$$= x - 2, \forall \in R$$

$|x - 3|$ is not differentiable at $x = 3$.

Thus $f(x)$ is not differentiable at $x = 3$ only.

ILLUSTRATION 4.61

Discuss differentiability of $f(x) = (e^{|x|-1} - 1)|x^4 - x^2|$.

Sol. We have $f(x) = (e^{|x|-1} - 1)x^2|x^2 - 1|$

Clearly, $|x|$ and $(e^{|x|-1} - 1)$ is not differentiable at $x = 0$.

But $f(x)$ has factor x^2, which is zero when $x = 0$.

So, $f(x)$ is differentiable at $x = 0$.

Now $|x^2 - 1|$ is not differentiable at $x = \pm 1$.

But $(e^{|x|-1} - 1) = 0$ at $x = \pm 1$.

So, $f(x)$ is differentiable at $x = \pm 1$.

Thus $f(x)$ is differentiable for all real x.

ILLUSTRATION 4.62

Let $f(x) = e^{x-1} - ax^2 + b$ and $g(x) = \begin{cases} e^{x-1} & x \le 1 \\ x^2 + 1 & x > 1 \end{cases}$, then find the values of a and b such that $f(x) \times g(x)$ is differentiable at $x = 1$.

Sol. If $h(x) = f(x)\, g(x)$

Then $h'(x) = f'(x)g(x) + f(x)\, g'(x)$

Now $g(x)$ is discontinuous at $x = 1$

$\therefore$ $h'(x)$ exists if $f'(1) = f(1) = 0$ otherwise $h'(0^-)$ and $h'(0^+)$ will be different.

$\therefore$ $1 - a + b = 0$ and $1 - 2a = 0$

So, $a = \dfrac{1}{2}$ and $b = -\dfrac{1}{2}$

CONTINUITY OF DERIVATIVE

The derivative of function (if exists) is just another function. Saying that a function is differentiable just mean that the derivative exists, while saying that a function has a continuous derivative means that it is differentiable and its derivative is continuous function.

If $y = f(x)$ is differentiable at $x = a$, then it is not necessary that derivative is continuous at $x = a$.

For example, consider function $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

For $x \neq 0$

$$f'(x) = 2x \sin \frac{1}{x} + x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x}$$

$$= 2x \sin \frac{1}{x} - \cos\frac{1}{x}$$

For $x = 0$,

$$f'(x) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h}$$

$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0$$

Thus $f'(x) = \begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Now, $f'(x)$ is continuous at $x = 0$ if $\lim_{x\to 0} f'(x) = f'(0)$

But $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)$, does not exists as $\lim_{x\to 0}\cos\frac{1}{x}$ does not exists.

Hence, $f'(x)$ is not continuous at $x = 0$.

ILLUSTRATION 4.63

Prove that function $f(x) = (\sin\pi x)(x-1)^{1/5}$ is differentiable at $x = 1$. Also prove that its derivative is continuous at $x = 1$.

Sol. We have $f(x) = (\sin\pi x)(x-1)^{1/5}$

Clearly $f(x)$ is continuous as both $\sin\pi x$ and $(x-1)^{1/5}$ are continuous.

So, $f(x)$ is continuous at $x = 1$.

Now, $(x-1)^{1/5}$ is not differentiable (vertical tangent) at $x = 1$, but $\sin\pi x = 0$ at $x = 1$.

So, $f(x)$ is differentiable at $x = 1$.

Also, $f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$$= \lim_{h\to 0}\frac{\sin[\pi(1-h)](1-h-1)^{1/5}-0}{-h}$$

$$= \lim_{h\to 0}\frac{\sin(\pi h)(-h)^{1/5}}{-h} = 0$$

Similarly, we get $f'(1^+) = 0$

Further, $f'(x) = \begin{cases} \pi(\cos\pi x)(x-1)^{\frac{1}{5}} + \dfrac{\sin\pi x}{5(x-1)^{\frac{4}{5}}}, & x \neq 1 \\ 0, & x = 1 \end{cases}$

Now $\lim_{x\to 1} f'(x) = \lim_{x\to 1}\left[\pi(\cos\pi x)(x-1)^{\frac{1}{5}} + \dfrac{\sin\pi x}{5(x-1)^{\frac{4}{5}}}\right]$

$$= 0 + \lim_{x\to 1}\frac{\sin\pi(1-x)}{\pi(1-x)}\cdot\frac{\pi(1-x)}{5(x-1)^{\frac{4}{5}}}$$

$$= 0 + \lim_{x\to 1}\frac{\pi}{5}(1-x)^{\frac{1}{5}}$$

$$= 0 = f'(0)$$

Thus $f'(x)$ is continuous at $x = 1$.

ILLUSTRATION 4.64

Find the set of points where $f(x) = x^2|x|$ is thrice differentiable.

Sol. $f(x) = \begin{cases} -x^3, & x < 0 \\ x^3, & x \geq 0 \end{cases}$

$\therefore\quad f'(x) = \begin{cases} -3x^2, & x < 0 \\ 3x^2, & x > 0 \end{cases}$

Clearly $f'(0^-) = f'(0^+) = 0$

So, $f(x)$ is differentiable at $x = 0$.

Also, derivative is continuous at $x = 0$.

So, we have $f''(x) = \begin{cases} -6x, & x < 0 \\ 6x, & x > 0 \end{cases}$

Clearly $f''(0^-) = f''(0^+) = 0$ and $f''(x)$ is continuous at $x = 0$.

$\therefore\quad f'''(x) = \begin{cases} -6, & x < 0 \\ 6, & x > 0 \end{cases}$

Clearly $f'''(0)$ does not exists, as $f'''(0^-) = -6$ and $f'''(0^+) = 6$.

Thus, the set of points where $f(x)$ is thrice differentiable at $R - \{0\}$.

CONCEPT APPLICATION EXERCISE 4.5

1. Discuss the continuity and differentiability of $f(x) = |x+1| + |x| + |x-1|\ \forall\ x \in R$; also draw the graph of $f(x)$.
2. Find x where $f(x) = \text{maximum}\{\sqrt{x(2-x)}, 2-x\}$ is non-differentiable.
3. Discuss the differentiability of $f(x) = \min\{|x|, |x-2|, 2-|x-1|\}$.
4. Discuss the differentiability of function $f(x) = x - |x - x^2|$.
5. Discuss the differentiability of $f(x) = |[x]x|$ in $-1 < x \leq 2$, where [.] represents the greatest integer function.
6. Discuss the differentiability of $f(x) = \cos^{-1}(\cos x)$.
7. Discuss the differentiability of $f(x) = \max\{\tan^{-1}x, \cot^{-1}x\}$.
8. Find the values of a and b if $f(x) = \begin{cases} ax^2+1, & x \leq 1 \\ x^2+ax+b, & x > 1 \end{cases}$ is differentiable at $x = 1$.
9. Discuss the differentiability of $f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.
10. Discuss the differentiability of $f(x) = ||x^2-4| - 12|$.
11. Discuss differentiability of $f(x) = x[x]\{x\}$ in interval $[-1, 2]$, where $[\cdot]$ and $\{\cdot\}$ denotes the greatest integer function and fractional part function, respectively.
12. Discuss the continuity and differentiability of $f(x) = \begin{cases} \sin\left(\dfrac{\pi x}{2}\right), & 0 \leq x < 1 \\ |2x-3|[x], & 1 \leq x \leq 2 \end{cases}$, where $[\cdot]$ denotes the greatest integer function.
13. For which values of x function $f(x) = (\sin\pi x)|x-1||x-2||x-3|$ is non-differentiable?

14. Find the number of points where $f(x) = |x \text{ sgn } (1 - x^2)|$ is non-differentiable.

15. Let $f(x) = \begin{cases} \left(\cos\frac{1}{x}\right)(\log(1+x))^2, & x > 0 \\ 0, & x \le 0 \end{cases}$. Prove that $f(x)$ is differentiable but derivative is not continuous at $x = 0$.

Answers

1. Continuous for all x; Non-differentiable at $x = -1, 0, 1$
2. $x = 1$
3. Non-differentiable at $x = \frac{-1}{2}, 0, 1, 2, \frac{5}{2}$
4. Non-differentiable at $x = 0, 1$
5. Non-differentiable at $x = 0, 1, 2$
6. Non-differentiable at $x = n\pi, n \in Z$
7. Non-differentiable at $x = 1$
8. $a = 2, b = 0$
9. Non-differentiable at $x = 0$
10. Non-differentiable at $x = \pm 2, \pm 4$
11. Non-differentiable at $x = 0, 1, 2$
12. Dicontinuous at $x = 2$; Non-differentiable at $x = 1, \frac{3}{2}, 2$
13. Differentiable for all $x \in R$
14. Three

Solved Examples

Example 4.1

If $f(x) = \frac{x}{(1+x)} + \frac{x}{(1+x)(1+2x)} + \frac{x}{(1+2x)(1+3x)} + \ldots$ infinite terms. Discuss continuity at $x = 0$

Sol. Sum of n terms of the series is

$$\left(1 - \frac{1}{1+x}\right) + \left(\frac{1}{1+x} - \frac{1}{1+2x}\right) + \ldots + \left(\frac{1}{1+(n-1)x} - \frac{1}{1+nx}\right)$$

$$= 1 - \frac{1}{1+nx}$$

$\therefore \quad f(x) = \lim_{n \to \infty}\left(1 - \frac{1}{1+nx}\right) = 1 - 0 = 1$

For $x = 0$ sum of the series is $f(0) = 0$

Thus, $f(x) = \begin{cases} 1, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Clearly $f(x)$ is discontinuous at $x = 0$.

Example 4.2

A function $f(x)$ is defined as

$$f(x) = \begin{cases} x^2 + ax + 1, & x \text{ is rational} \\ ax^2 + 2x + b, & x \text{ is irrational} \end{cases}$$

is continuous at $x = 1$ and 2, then find the values of a and b.

Sol. $f(x)$ is continuous at $x = 1$ and $x = 2$

$\therefore$ $x = 1$ and $x = 2$ are roots of the equation

$x^2 + ax + 1 = ax^2 + 2x + b$

or $(a - 1)x^2 + (2 - a)x + b - 1 = 0$

Putting $x = 1$ and $x = 2$ in above equation

$a - 1 + 2 - a + b - 1 = 0 \Rightarrow b = 0$

and $4(a - 1) + 2(2 - a) + 0 - 1 = 0 \Rightarrow a = 1/2$

Example 4.3

If $g(x) = \begin{cases} [f(x)], & x \in (0, \pi/2) \cup (\pi/2, \pi) \\ 3, & x = \pi/2 \end{cases}$

and $f(x) = \frac{2(\sin x - \sin^n x) + |\sin x - \sin^n x|}{2(\sin x - \sin^n x) - |\sin x - \sin^n x|}$, $n \in N$

where [.] denotes the greatest integer function. Prove that $g(x)$ is continuous at $x = \pi/2$ when $n > 1$.

Sol. When $x \to \frac{\pi}{2}^+$, $\sin x$ is very close to 1 but less than 1.

$\therefore \quad \sin^n x < \sin x$

$\therefore \quad |\sin x - \sin^n x| = \sin x - \sin^n x$

So, $\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} \frac{2(\sin x - \sin^n x) + (\sin x - \sin^n x)}{2(\sin x - \sin^n x) - (\sin x - \sin^n x)}$

$= 3$ (exactly 3)

Similarly, we get $\lim_{x \to \frac{\pi}{2}^-} f(x) = 3$ (exactly 3)

$\therefore \quad \lim_{x \to \frac{\pi}{2}} g(x) = \lim_{x \to \frac{\pi}{2}} [f(x)] = \lim_{x \to \frac{\pi}{2}} [3] = 3 = g(3)$

So, $g(x)$ is continuous at $x = \pi/2$.

Example 4.4

Discuss the differentiability of $f(x) = [x] + \sqrt{\{x\}}$, where [.] and {.} denote the greatest integer function and fractional part respectively.

Sol. We have $f(x) = [x] + \sqrt{x - [x]}$

$[x]$ is discontinuous at all integral values of x.

So, we must check continuity and differentiability at integral values of x.

For continuity at some integer $x = k$

$f(k) = [k] + \sqrt{k - [k]} = k + \sqrt{k - k} = k$

L.H.L. $= f(k^-) = [k^-] + \sqrt{k^- - [k^-]}$

$= k - 1 + \sqrt{k - (k - 1)}$

$= k$

R.H.L. $= f(k^+) = [k^+] + \sqrt{k^+ - [k^+]}$

$= k + \sqrt{k - k}$

$= k$

Thus $f(x)$ is continuous at $x = k$.

For differentiability at $x = k$

$$\text{L.H.D. } f'(k^-) = \lim_{h\to 0} \frac{f(k-h)-f(k)}{-h}$$

$$= \lim_{h\to 0} \frac{[k-h]+\sqrt{(k-h)-[k-h]}-k}{-h}$$

$$= \lim_{h\to 0} \frac{k-1+\sqrt{k-h-(k-1)}-k}{-h}$$

$$= \lim_{h\to 0} \frac{-1+\sqrt{1-h}}{-h}$$

$$= \lim_{h\to 0} \frac{-1+(1-h)}{-h(1+\sqrt{1-h})}$$

$$= \frac{1}{2}$$

$$\text{R.H.D. } f'(k^+) = \lim_{h\to 0} \frac{f(k+h)-f(k)}{h}$$

$$= \lim_{h\to 0} \frac{[k+h]+\sqrt{(k+h)-[k+h]}-k}{h}$$

$$= \lim_{h\to 0} \frac{k+\sqrt{k+h-k}-k}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{h}}{h} \to \infty$$

So, $f(x)$ is not differentiable at $x = k$, $k \in Z$.

Example 4.5

Let $y = f(x)$ be defined parametrically as $y = t^2 + t|t|$, $x = 2t - |t|$, $t \in R$. Find $f(x)$ and discuss its differentiability.

Sol. We have $y = t^2 + t|t|$ and $x = 2t - |t|$, $t \in R$.

So, we have following two cases,

Case I: $t \geq 0$

$\therefore \quad x = 2t - t = t$ and $y = t^2 + t^2 = 2t^2$

Eliminating t, we get $y = 2x^2$

Also $x = t \geq 0$

Case II: $t < 0$

$\therefore \quad x = 2t - (-t) = 3t$ and $y = t^2 + t(-t) = 0$

So, we have $y = 0$.

Also, $x = 3t < 0$

$$\text{Thus, } y = f(x) = \begin{cases} 2x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Clearly, $f(x)$ is continuous at $x = 0$.

$$f'(x) = \begin{cases} 4x, & x > 0 \\ 0, & x < 0 \end{cases}$$

$\therefore \quad f'(0^+) = 0$ and $f'(0^-) = 0$

Therefore, $f(x)$ is differentiable at $x = 0$.

Example 4.6

If $f(x) = \begin{cases} x-3, & x < 0 \\ x^2-3x+2, & x \geq 0 \end{cases}$ and let $g(x) = f(|x|) + |f(x)|$. Discuss the differentiability of $g(x)$.

Sol.
$$f(|x|) = \begin{cases} |x|-3, & |x| < 0 \to \text{(Not possible)} \\ |x|^2 - 3|x| + 2, & |x| \geq 0 \end{cases}$$

$$= \begin{cases} x^2+3x+2, & x < 0 \\ x^2-3x+2, & x \geq 0 \end{cases} \quad \ldots(1)$$

$$|f(x)| = \begin{cases} |x-3|, & x < 0 \\ |x^2-3x+2|, & x \geq 0 \end{cases}$$

$$= \begin{cases} 3-x, & x < 0 \\ (x^2-3x+2), & 0 \leq x < 1 \\ -(x^2-3x+2), & 1 \leq x < 2 \\ (x^2-3x+2), & x \geq 2 \end{cases} \to \text{(Using sign scheme of } (x-1)(x-2)) \quad \ldots(2)$$

Adding (1) and (2), we get

$$g(x) = f(|x|) + |f(x)| = \begin{cases} x^2+2x+5, & x < 0 \\ 2x^2-6x+4, & 0 \leq x < 1 \\ 0, & 1 \leq x < 2 \\ 2x^2-6x+4, & x \geq 2 \end{cases}$$

Clearly $g(x)$ is discontinuous at $x = 0$ and continuous at $x = 1$ and 2.

$$g'(x) = \begin{cases} 4x-6, & 0 < x < 1 \\ 0, & 1 < x < 2 \\ 4x-6, & x > 2 \end{cases}$$

Clearly, $g(x)$ is not differentiable at $x = 1$ and 2.

Thus, $g(x)$ is discontinuous at $x = 0$ and not differentiable at $x = 0, 1, 2$.

Example 4.7

Let $f: R \to R$ satisfies $|f(x)| \leq x^2 \ \forall \ x \in R$. Then show that $f(x)$ is differentiable at $x = 0$.

Sol. Since $|f(x)| \leq x^2 \ \forall \ x \in R$, we have at $x = 0$, $|f(0)| \leq 0$

$\therefore \quad f(0) = 0 \quad (1)$

$$\therefore \quad f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} \quad (2)$$

$$\text{Now, } \left|\frac{f(h)}{h}\right| \leq |h| \quad (\because |f(x)| \leq x^2)$$

$$\text{or } -|h| \leq \frac{f(h)}{h} \leq |h|$$

$$\text{or } \lim_{h\to 0} \frac{f(h)}{h} = 0 \quad \text{(Using sandwich theorem)} \quad (3)$$

Therefore, from equations (2) and (3), we get $f'(0) = 0$, i.e., $f(x)$ is differentiable at $x = 0$.

Example 4.8

If a function $f: [-2a, 2a] \to R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left-hand derivative at $x = a$ is 0, then find the left-hand derivative at $x = -a$.

Sol. Given $f(2a - x) = f(x) \ \forall \ x \in (a, 2a) \quad (1)$

and $f'(a^-) = 0$ or $\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = 0$

Now, $f'(-a^-) = \lim_{h\to 0} \frac{f(-a-h)-f(-a)}{-h}$

$$= \lim_{h\to 0} \frac{-f(a+h)+f(a)}{-h}$$ [$\because f(x)$ is an odd function]

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Now, in equation (1), replacing x by $a-h$, we get

$$f(a-h) = f(2a-(a-h)) = f(a+h)$$

or $f'(-a^-) = -\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = -f'(a^-) = 0$

EXAMPLE 4.9

Given a real-valued function $f(x)$ as follows:

$$f(x) = \begin{cases} \dfrac{x^2+2\cos x-2}{x^4}, & \text{for } x<0 \\ 1/12, & \text{for } x=0 \\ \dfrac{\sin x-\log(e^x\cos x)}{6x^2}, & \text{for } x>0 \end{cases}$$

Test the continuity and differentiability of $f(x)$ at $x = 0$.

Sol. $f(0^+) = \lim_{h\to 0} \frac{\sin h-\log(e^h\cos h)}{6h^2}$

$$= \lim_{h\to 0} \frac{\cos h-\dfrac{e^h(\cos h-\sin h)}{e^h\cos h}}{12h}$$ (Using L′ Hospital's rule)

$$= \lim_{h\to 0} \frac{\cos h-(1-\tan h)}{12h}$$

$$= \lim_{h\to 0} \frac{-\sin h+\sec^2 h}{12}$$

$$= \frac{1}{12}$$ (Using L′ Hospital's rule)

$$f(0^-) = \lim_{h\to 0} \frac{h^2+2\cos h-2}{h^4}$$

$$= \lim_{h\to 0} \frac{h^2+2\left[1-\dfrac{h^2}{2!}+\dfrac{h^4}{4!}\right]-2}{h^4}$$

$$= \frac{1}{12}$$ (Using expansion formula of cos h)

Therefore, $f(x)$ is continuous at $x = 0$.

$$f'(0^+) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{\sin h-\log(e^h\cos h)}{6h^2}-\dfrac{1}{12}}{h}$$

$$= \lim_{h\to 0} \frac{2\sin h-2\log(e^h\cos h)-h^2}{12h^3}$$

$$\lim_{h\to 0} \frac{2\cos h-2(1-\tan h)-2h}{36h^2}$$ (Using L′ Hospital's rule)

$$= \lim_{h\to 0} \frac{\cos h-(1-\tan h)-h}{18h^2}$$

$$= \lim_{h\to 0} \frac{-\sin h+\sec^2 h-1}{36h}$$ (Using L′ Hospital's rule)

$$= \lim_{h\to 0} \frac{-\cos h+2\sec^2 h\tan h}{36}$$

$$= -\frac{1}{36}$$ (Using L′ Hospital's rule)

$$f'(0^-) = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{\dfrac{h^2+2\cos h-2}{h^4}-\dfrac{1}{12}}{-h}$$

$$= \lim_{h\to 0} \frac{12h^2+24\cos h-24-h^4}{-12h^5}$$

$$= \lim_{h\to 0} \frac{12h^2+24\left[1-\dfrac{h^2}{2!}+\dfrac{h^4}{4!}\right]-24-h^4}{-12h^5} = 0$$

Hence, $f(x)$ is continuous but non-differentiable at $x = 0$.

EXAMPLE 4.10

If $f(x)$ is continuous and differentiable function and $f\left(\frac{1}{n}\right)=0$ $\forall\, n \geq 1$ and $n \in Z$, then prove that $f(0) = 0$ and $f'(0) = 0$

Sol. We have R.H.L. at $x = 0$,

$$f(0^+) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0 \text{ (exact zero)}$$

$\therefore$ $f(0) = 0$ as $f(x)$ is continuous at $x = 0$.

Now, $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h-0}$

$$= \lim_{h\to 0} \frac{f(h)}{h}$$

$$= \frac{\text{Exact zero}}{\to 0} = 0$$

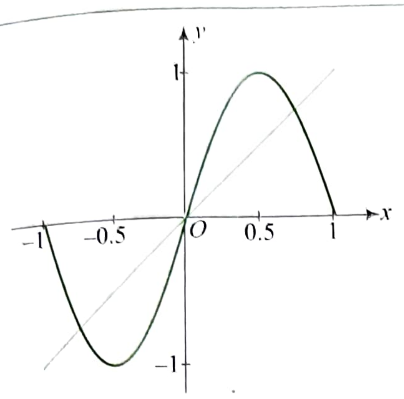
EXAMPLE 4.11

Discuss continuity and differentiability of $f(x) = [x \sin \pi x]$ in $(-1, 1)$. Here [.] denotes the greatest integer function.

Sol. We have $f(x) = [x \sin \pi x]$, $x \in (-1, 1)$.

Let us first check the nature of the function $g(x) = x \sin \pi x$

The graph of $y = x$ and $y = \sin \pi x$ are as shown in the following figure.



For $x \in (-1, 0)$, $\sin \pi x \in (-1, 0)$

$\therefore$ $x \sin \pi x \in (0, 1)$

At $x = 0$, $x \sin \pi x = 0$

For $x \in (0, 1)$, $\sin \pi x \in (0, 1)$

$\therefore$ $x \sin \pi x \in (0, 1)$

Thus, for $x \in (-1, 1)$

$x \sin \pi x \in [0, 1)$

$\therefore$ $[x \sin \pi x] = 0, \forall x \in (-1, 1)$

Thus $f(x)$ is continuous and differentiable in $(-1, 1)$.

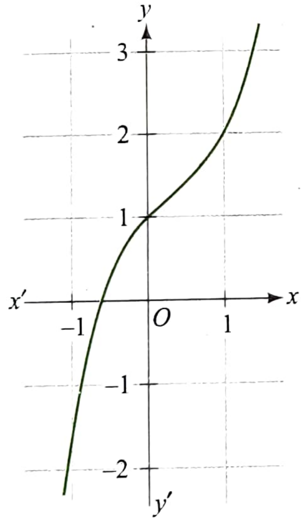
EXAMPLE 4.12

Let $f(x) = x^3 - x^2 + x + 1$ and

$$g(x) = \begin{cases} \max. f(t), & 0 \le t \le x \quad \text{for } 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

Discuss the continuity and differentiability of $g(x)$ in $(0, 2)$.

Sol. Here, $f(x) = x^3 - x^2 + x + 1$



Therefore, $f'(x) = 3x^2 - 2x + 1$, which is strictly increasing in $(0, 2)$. Thus,

$$g(x) = \begin{cases} f(x); & 0 \le x \le 1 \\ 3 - x; & 1 < x \le 2 \end{cases}$$

[as $f(x)$ is increasing, $f(x)$ is maximum when $0 \le t \le x$]

So, $$g(x) = \begin{cases} x^3 - x^2 + x + 1, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

Also, $$g'(x) = \begin{cases} 3x^2 - 2x + 1; & 0 \le x \le 1 \\ -1; & 1 < x \le 2 \end{cases},$$

which clearly shows $g(x)$ is continuous for all $x \in [0, 2]$, but $g(x)$ is not differentiable at $x = 1$.

EXAMPLE 4.13

Let $\alpha \in R$. Prove that a function $f: R \to R$ is differentiable at $x = \alpha$ if and only if there is a function $g: R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $\alpha \in R$.

Sol. Let $f: R \to R$ be differentiable at $x = \alpha \in R$.

Then, $\lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{(x - \alpha)} = f'(\alpha)$ exists and is finite, i.e.,

$$Lf'(\alpha) = Rf'(\alpha) = f'(\alpha)$$

or $$\lim_{x \to \alpha^-} \frac{f(x) - f(\alpha)}{(x - \alpha)} = \lim_{x \to \alpha^+} \frac{f(x) - f(\alpha)}{(x - \alpha)} = f'(\alpha)$$

$$\lim_{x \to \alpha^-} g(x) = \lim_{x \to \alpha^+} g(x) = f'(\alpha) \quad (1)$$

$[\because f(x) - f(\alpha) = g(x)(x - \alpha)]$

Again, $$f'(\alpha) = \lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{(x - \alpha)}$$

$$= \lim_{x \to \alpha} g(x) = g(\alpha) \quad (2)$$

From equations (1) and (2), we get

$$\lim_{x \to \alpha^-} g(x) = \lim_{x \to \alpha^+} g(x) = g(\alpha)$$

L.H.L. = R.H.L. = $g(\alpha)$

Thus, $g(x)$ is continuous function at $x = \alpha \in R$.

Conversely,

Assume $g(x)$ is continuous at $x = \alpha$ on R. Thus,

$$\lim_{x \to \alpha} g(x) = g(\alpha) = \text{a finite quantity} \quad (3)$$

and given $f(x) - f(\alpha) = g(x)(x - \alpha)$

For $x \ne \alpha$, $$g(x) = \frac{f(x) - f(\alpha)}{(x - \alpha)} \quad (4)$$

From equations (3) and (4), we get

$$\lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{(x - \alpha)} = g(\alpha)$$

$\therefore$ $f'(\alpha) = g(\alpha) = $ a finite quantity

Thus, $f(x)$ is differentiable at $x = \alpha \in R$.

EXAMPLE 4.14

Consider $f'(0) = \lim_{n \to \infty} nf\left(\frac{1}{n}\right)$ and $f(0) = 0$. Using this, find $\lim_{n \to \infty} \left((n+1)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right) - n\right), \left|\cos^{-1}\frac{1}{n}\right| < \frac{\pi}{2}$.

Sol. To find

$$\lim_{n \to \infty} \left[(n+1)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right) - n\right]$$

$$= \lim_{n \to \infty} n \left[\left(1 + \frac{1}{n}\right)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right) - 1\right]$$

$$= \lim_{n \to \infty} nf\left(\frac{1}{n}\right),$$

where $f(x) = \left[(1+x)\frac{2}{\pi}\cos^{-1}x - 1\right]$

such that $f(0) = \left[(1+0)\frac{2}{\pi}\cos^{-1}0 - 1\right] = \frac{2}{\pi}\cdot\frac{\pi}{2} - 1 = 0.$

$$\text{Now, } f'(0) = \lim_{n\to\infty}\frac{f\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0}\frac{f(x)-f(0)}{x}$$

$$f'(0) = \frac{d}{dx}\left[(1+x)\frac{2}{\pi}\cos^{-1}x - 1\right]\Bigg|_{x=0}$$

$$= \frac{2}{\pi}\left[\cos^{-1}x - \frac{1+x}{\sqrt{1-x^2}}\right]\Bigg|_{x=0}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - 1\right] = 1 - \frac{2}{\pi} = \frac{\pi-2}{\pi}$$

EXAMPLE 4.15

$f(x-y) = f(x)\,g(y) - f(y)\,g(x)$ and $g(x-y) = g(x)\,g(y) - f(x)\,f(y)$ for all $x, y \in R$. If the right-hand derivative at $x = 0$ exists for $f(x)$, find the derivative of $g(x)$ at $x = 0$.

Sol. Given that

$$f(x-y) = f(x)\,g(y) - f(y)\,g(x) \quad (1)$$

$$g(x-y) = g(x)\,g(y) + f(x)\,f(y) \quad (2)$$

In equation (1), putting $x = y$, we get

$$f(0) = f(x)\,g(x) - f(x)\,g(x) \text{ or } f(0) = 0$$

Putting $y = 0$ in equation (1), we get

$$f(x) = f(x)\,g(0) - f(0)\,g(x)$$

$$= f(x)\,g(0) \quad [\text{using } f(0) = 0]$$

$$\therefore \quad g(0) = 1$$

Putting $x = y$ in equation (2), we get

$$g(0) = g(x)\,g(x) + f(x)\,f(x)$$

or $\quad 1 = [g(x)]^2 + [f(x)]^2 \quad [\text{using } f(0) = 0]$

or $\quad [g(x)]^2 = 1 - [f(x)]^2 \quad (3)$

Clearly, $g(x)$ will be differentiable only if $f(x)$ is differentiable.

Therefore, first, let us check the differentiability of $f(x)$.

Given that $Rf'(0)$ exists,

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \text{ exists}$$

or $\quad \lim_{h\to 0}\frac{f(0)\,g(-h) - f(-h)\,g(0)}{h}$ exists

or $\quad \lim_{h\to 0}\frac{-f(-h)}{h}$ exists $\quad [\text{using } f(0) = 0 \text{ and } g(0) = 1]$

which can be written as

$$\lim_{h\to 0}\frac{f(-h)-f(0)}{-h} = Lf'(0)$$

or $\quad Lf'(0) = Rf'(0)$

Hence, f is differentiable at $x = 0$.

By differentiating equation (3), we get

$$2g(x)\,g'(x) = -2f(x)\,f'(x)$$

For $x = 0$,

$$g(0)\,g'(0) = -f(0)\,f'(0) \quad \text{or} \quad g'(0) = 0$$

Exercises

Single Correct Answer Type

1. Which of the following functions has finite number of points of discontinuity in R ([·] represents the greatest integer function)?
   (1) $\tan x$ (2) $x[x]$ (3) $\frac{|x|}{x}$ (4) $\sin[\pi x]$

2. The function $f(x) = \frac{4-x^2}{4x-x^3}$ is
   (1) discontinuous at only one point
   (2) discontinuous exactly at two points
   (3) discontinuous exactly at three points
   (4) none of these

3. If $f(x) = \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}$, $(x \neq \pi/4)$, is continuous at $x = \pi/4$, then the value of $f\left(\frac{\pi}{4}\right)$ is
   (1) 1 (2) 1/2 (3) 1/3 (4) −1

4. The value of $f(0)$ so that the function $f(x) = \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$ is continuous at each point in its domain, is equal to
   (1) 2 (2) 1/3 (3) 2/3 (4) −1/3

5. The function $f(x) = \frac{(3^x - 1)^2}{\sin x \cdot \ln(1+x)}$, $x \neq 0$, is continuous at $x = 0$. Then the value of $f(0)$ is
   (1) $2\log_e 3$ (2) $(\log_e 3)^2$
   (3) $\log_e 6$ (4) none of these

6. If $f(x) = \frac{x - e^x + \cos 2x}{x^2}$, $x \neq 0$, is continuous at $x = 0$, then
   (1) $f(0) = 5/2$ (2) $[f(0)] = -2$
   (3) $\{f(0)\} = 0$ (4) $[f(0)]\{f(0)\} = -1.5$
   where $[x]$ and $\{x\}$ denote the greatest integer and fractional part functions, respectively.

7. If $f(x) = \begin{cases} \frac{8^x - 4^x - 2^x + 1}{x^2}, & x > 0 \\ e^x \sin x + \pi x + \lambda \ln 4, & x \leq 0 \end{cases}$ is continuous at $x = 0$, then the value of λ is
   (1) $4\log_e 2$ (2) $2\log_e 2$
   (3) $\log_e 2$ (4) none of these

8. If $f(x) = \frac{a\cos x - \cos bx}{x^2}$, $x \neq 0$ and $f(0) = 4$, is continuous at $x = 0$, then the ordered pair (a, b) is
   (1) $(\pm 1, 3)$ (2) $(1, \pm 3)$
   (3) $(-1, -3)$ (4) $(1, 3)$

9. Let f be a continuous function on R such that $f(1/4n) = (\sin e^n)e^{-n^2} + \frac{n^2}{n^2+1}$. Then the value of $f(0)$ is
   (1) 1 (2) 1/2
   (3) 0 (4) none of these

10. If $f(x) = \frac{x^2 - bx + 25}{x^2 - 7x + 10}$ for $x \neq 5$ is continuous at $x = 5$, then the value of $f(5)$ is
   (1) 0 (2) 5 (3) 10 (4) 25

11. For a real number y, let $[y]$ denotes the greatest integer less than or equal to y. Let $f(x) = \frac{\tan(\pi[x-\pi])}{1+[x]^2}$. Then
   (1) $f(x)$ is discontinuous at some x
   (2) $f(x)$ is continuous at all x, but the derivative $f'(x)$ does not exist for some x
   (3) $f'(x)$ exists for all x, but $f''(x)$ does not exist for some x
   (4) $f'(x)$ exists for all x

12. If $f(x) = \begin{cases} \frac{1-|x|}{1+x}, & x \neq -1 \\ 1, & x = -1 \end{cases}$, then $f([2x])$, where $[\cdot]$ represents the greatest integer function, is
   (1) discontinuous at $x = -1$
   (2) continuous at $x = 0$
   (3) continuous at $x = 1/2$
   (4) continuous at $x = 1$

13. Let $f(x) = \begin{cases} \frac{x-4}{|x-4|} + a, & x < 4 \\ a + b, & x = 4 \\ \frac{x-4}{|x-4|} + b, & x > 4 \end{cases}$
   Then $f(x)$ is continuous at $x = 4$ when
   (1) $a = 0, b = 0$ (2) $a = 1, b = 1$
   (3) $a = -1, b = 1$ (4) $a = 1, b = -1$

14. Which of the following is true about
   $f(x) = \begin{cases} \frac{(x-2)}{|x-2|}\left(\frac{x^2-1}{x^2+1}\right), & x \neq 2 \\ \frac{3}{5}, & x = 2 \end{cases}$?
   (1) $f(x)$ is continuous at $x = 2$.
   (2) $f(x)$ has removable discontinuity at $x = 2$.
   (3) $f(x)$ has non-removable discontinuity at $x = 2$.
   (4) Discontinuity at $x = 2$ can be removed by redefining the function at $x = 2$.

15. If $f(x) = \begin{cases} x+2, & x<0 \\ -x^2-2, & 0 \le x < 1, \\ x, & x \ge 1 \end{cases}$

then the number of points of discontinuity of $|f(x)|$ is

(1) 1 (2) 2
(3) 3 (4) none of these

16. Which of the following statements is always true? (Here, [·] represents the greatest integer function)

(1) If $f(x)$ is discontinuous, then $|f(x)|$ is discontinuous.
(2) If $f(x)$ is discontinuous, then $f(|x|)$ is discontinuous.
(3) $f(x) = [g(x)]$ is discontinuous when $g(x)$ is an integer.
(4) None of these.

17. The number of points $f(x) = \begin{cases} [\cos \pi x], & 0 \le x \le 1 \\ |2x-3|[x-2], & 1 < x \le 2 \end{cases}$

is discontinuous at is ([.] denotes the greatest integral function)

(1) two (2) three (3) four (4) zero

18. A point where function $f(x) = [\sin [x]]$ is not continuous in $(0, 2\pi)$, [.] denotes the greatest integer $\le x$, is

(1) (3, 0) (2) (2, 0)
(3) (1, 0) (4) none of these

19. The function $f(x) = \{x\}\sin(\pi[x])$, where [.] denotes the greatest integer function and {.} is the fractional part function, is discontinuous at

(1) all x (2) all integer points
(3) no x (4) x which is not an integer

20. The function $f(x)$ defined by

$$f(x) = \begin{cases} \log_{(4x-3)}(x^2-2x+5), & \frac{3}{4} < x < 1 \text{ and } x > 1 \\ 4, & x = 1 \end{cases}$$

(1) is continuous at $x = 1$
(2) is discontinuous at $x = 1$ since $f(1^+)$ does not exist though $f(1^-)$ exists
(3) is discontinuous at $x = 1$ since $f(1^-)$ does not exist though $f(1^+)$ exists
(4) is discontinuous at $x = 1$ since neither $f(1^+)$ nor $f(1^-)$ exists.

21. $f(x) = [x^2] - \{x\}^2$, where [.] and {.} denote the greatest integer function and the fractional part function, respectively, is

(1) continuous at $x = 1, -1$
(2) continuous at $x = -1$ but not at $x = 1$
(3) continuous at $x = 1$ but not at $x = -1$
(4) discontinuous at $x = 1$ and $x = -1$

22. If $f(x) = \{x^2\} - (\{x\})^2$, where $\{x\}$ denotes the fractional part of x, then

(1) $f(x)$ is continuous at $x = -2$ but not at $x = 2$
(2) $f(x)$ is continuous at $x = 2$ but not at $x = -2$
(3) $f(x)$ is continuous at $x = 2$ and at $x = -2$
(4) $f(x)$ is discontinuous at $x = -2$ and at $x = 2$

23. $f(x) = \begin{cases} 3 - \left[\cot^{-1}\dfrac{2x^3-3}{x^2}\right], & \text{if } x > 0 \\ \{x^2\}\cos(e^{1/x}), & \text{if } x < 0 \end{cases}$ is continuous at $x = 0$.

Then the value of $f(0)$, where $[x]$ and $\{x\}$ denote the greatest integer and fractional part functions, respectively, is

(1) 0 (2) 1
(3) −1 (4) none of these

24. Let $f(x)$ be defined in the interval [0, 4] such that

$$f(x) = \begin{cases} 1-x, & 0 \le x \le 1 \\ x+2, & 1 < x < 2 \\ 4-x, & 2 \le x \le 4 \end{cases}$$

Then the number of points where $f(f(x))$ is discontinuous is

(1) 1 (2) 2
(3) 3 (4) none of these

25. The function defined by $f(x) = (-1)^{[x^3]}$ ([·] denotes the greatest integer function) satisfies

(1) discontinuous for $x = n^{1/3}$, where n is any integer
(2) $f(3/2) = 1$
(3) $f'(x) = 1$ for $-1 < x < 1$
(4) none of these

26. $f(x) = \lim\limits_{n\to\infty} \dfrac{(x-1)^{2n}-1}{(x-1)^{2n}+1}$ is discontinuous at

(1) $x = 0$ only (2) $x = 2$ only
(3) $x = 0$ and 2 (4) none of these

27. Let $f: R \to R$ be given by $f(x) = 5x$. If $x \in Q$ and $f(x) = x^2 + 6$ if $x \in R \sim Q$, then

(1) f is continuous at $x = 2$ and $x = 3$
(2) f is not continuous at $x = 2$ and $x = 3$
(3) f is continuous at $x = 2$ but not at $x = 3$
(4) f is continuous at $x = 3$ but not at $x = 2$

28. Let $f(x) = \lim\limits_{n\to\infty} (\sin x)^{2n}$. Then which of the following is not true?

(1) Discontinuous at infinite number of points
(2) Discontinuous at $x = \dfrac{\pi}{2}$
(3) Discontinuous at $x = -\dfrac{\pi}{2}$
(4) None of these

29. A function $f(x)$ defined as

$f(x) = \begin{cases} \sin x, & x \text{ is rational} \\ \cos x, & x \text{ is irrational} \end{cases}$ is continuous at

(1) $x = n\pi + \pi/4, n \in I$ (2) $x = n\pi + \pi/8, n \in I$
(3) $x = n\pi + \pi/6, n \in I$ (4) $x = n\pi + \pi/3, n \in I$

30. Let $f(x) = \lim\limits_{n\to\infty} \dfrac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$. Then

(1) f is continuous at $x = 1$ (2) $\lim\limits_{x\to 1^+} f(x) = \log 3$
(3) $\lim\limits_{x\to 1^+} f(x) = -\sin 1$ (4) $\lim\limits_{x\to 1^-} f(x)$ does not exist

31. $f(x) = \lim_{n\to\infty} \sin^{2n}(\pi x) + \left[x + \frac{1}{2}\right]$, where [.] denotes the greatest integer function, is
(1) continuous at $x = 1$ but discontinuous at $x = 3/2$
(2) continuous at $x = 1$ and $x = 3/2$
(3) discontinuous at $x = 1$ and $x = 3/2$
(4) discontinuous at $x = 1$ but continuous at $x = 3/2$

32. If $f(x) = \text{sgn}(\sin^2 x - \sin x - 1)$ has exactly four points of discontinuity for $x \in (0, n\pi)$, $n \in N$, then
(1) minimum value of n is 5
(2) maximum value of n is 6
(3) there are exactly two possible values of n
(4) none of these

33. If $f(x) = \begin{cases} x^2 - ax + 3, & x \text{ is rational} \\ 2 - x, & x \text{ is irrational} \end{cases}$ is continuous at exactly two points, then the possible values of a are
(1) $(2, \infty)$
(2) $(-\infty, 3)$
(3) $(-\infty, -1) \cup (3, \infty)$
(4) none of these

34. Let $f(x) = [x]\sin\left(\frac{\pi}{[x+1]}\right)$, where [.] denotes the greatest integer function. Then $f(x)$ is discontinuous at
(1) all $x \in I - \{-1\}$
(2) all $x \in I - \{0\}$
(3) all $x \in I$
(4) all $x \in I - \{-1, 0\}$

35. Given that $\prod_{n=1}^{n} \cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)}$

and $f(x) = \begin{cases} \lim_{n\to\infty} \sum_{n=1}^{n} \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right), & x \in (0, \pi) - \left\{\frac{\pi}{2}\right\} \\ \frac{2}{\pi}, & x = \frac{\pi}{2} \end{cases}$

Then which one of the following is true?
(1) $f(x)$ has non-removable discontinuity of finite type at $x = \frac{\pi}{2}$.
(2) $f(x)$ has removable discontinuity at $x = \frac{\pi}{2}$.
(3) $f(x)$ is continuous at $x = \frac{\pi}{2}$.
(4) $f(x)$ has non-removable discontinuity of infinite type at $x = \frac{\pi}{2}$.

36. $f(x) = \begin{cases} \frac{x}{2x^2 + |x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Then $f(x)$ is
(1) continuous but non-differentiable at $x = 0$
(2) differentiable at $x = 0$
(3) discontinuous at $x = 0$
(4) none of these

37. Let a function $f(x)$ be defined by $f(x) = \frac{x - |x - 1|}{x}$. Then which of the following is not true?
(1) Discontinuous at $x = 0$
(2) Discontinuous at $x = 1$
(3) Not differentiable at $x = 0$
(4) Not differentiable at $x = 1$

38. Let $f(x) = \begin{cases} \min\{x, x^2\}, x \geq 0 \\ \max\{2x, x^2 - 1\}, x < 0 \end{cases}$. Then which of the following is not true?
(1) $f(x)$ is continuous at $x = 0$.
(2) $f(x)$ is not differentiable at $x = 1$.
(3) $f(x)$ is not differentiable at exactly three points
(4) None of these.

39. The function $f(x) = \sin^{-1}(\cos x)$ is
(1) not differentiable at $x = \frac{\pi}{2}$
(2) differentiable at $\frac{3\pi}{2}$
(3) differentiable at $x = 0$
(4) differentiable at $x = 2\pi$

40. Let [.] denotes the greatest integer function and $f(x) = [\tan^2 x]$. Then
(1) $\lim_{x\to 0} f(x)$ does not exist
(2) $f(x)$ is continuous at $x = 0$
(3) $f(x)$ is not differentiable at $x = 0$
(4) $f'(0) = 1$

41. If $f(x) = \begin{cases} 2x - [x] + x\sin(x - [x]); & x \neq 0 \\ 0; & x = 0 \end{cases}$ where [.] denotes the greatest integer function, then
(1) $f(x)$ is differentiable at $x = 0$
(2) $f(x)$ is differentiable at $x = 2$
(3) $f(x)$ is continuous but not differentiable at $x = 0$
(4) none of these

42. Which of the following functions is non-differentiable?
(1) $f(x) = (e^x - 1)|e^{2x} - 1|$ in R
(2) $f(x) = \frac{x - 1}{x^2 + 1}$ in R
(3) $f(x) = \begin{cases} ||x - 3| - 1|, & x < 3 \\ \frac{x}{3}[x] - 2, & x \geq 3 \end{cases}$ at $x = 3$

where [.] represents the greatest integer function
(4) $f(x) = 3(x - 2)^{\frac{3}{4}} + 3$ in R

43. The number of values of $x \in [0, 2]$ at which $f(x) = \left|x - \frac{1}{2}\right| + |x - 1| + \tan x$ is not differentiable is
(1) 0
(2) 1
(3) 3
(4) none of these

44. Which of the following function is not differentiable at $x = 1$?

(1) $f(x) = (x^2 - 1)\,|(x - 1)(x - 2)|$

(2) $f(x) = \sin(|x - 1|) - |x - 1|$

(3) $f(x) = \tan(|x - 1|) + |x - 1|$

(4) None of these

45. $f(x) = \begin{cases} xe^{-\left(\frac{1}{x}+\frac{1}{|x|}\right)}, & x \neq 0 \\ a, & x = 0 \end{cases}$. The value of a, such that $f(x)$ is differentiable at $x = 0$, is equal to

(1) 1 (2) –1

(3) 0 (4) none of these

46. If $f(x) = a\,|\sin x| + be^{|x|} + c\,|x|^3$ is differentiable at $x = 0$, then

(1) $a = b = c = 0$ (2) $a = 0, b = 0, c \in R$

(3) $b = c = 0, a \in R$ (4) $c = 0, a = 0, b \in R$

47. Let $f(x) = \begin{cases} \sin 2x, & 0, x \leq \pi/6 \\ ax + b, & \pi/6 < x < 1 \end{cases}$.

If $f(x)$ and $f'(x)$ are continuous, then

(1) $a = 1, b = \frac{1}{\sqrt{2}} + \frac{\pi}{6}$ (2) $a = \frac{1}{\sqrt{2}}, b = \frac{1}{\sqrt{2}}$

(3) $a = 1, b = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$ (4) none of these

48. If $f(x) = \begin{cases} x^3, & x^2 < 1 \\ x, & x^2 \geq 1 \end{cases}$, then the set of values of x where $f(x)$ is differentiable is

(1) $(-\infty, \infty) - \{1\}$ (2) $(-\infty, \infty) - \{1 - 1\}$

(3) $(-\infty, \infty) - \{1 -1, 0\}$ (4) $(-\infty, \infty) - \{-1\}$

49. If $f(x) = (x^2 - 4)\,|x^3 - 6x^2 + 11x - 6| + \frac{x}{1 + |x|}$, then the set of points at which the function $f(x)$ is not differentiable is

(1) $\{-2, 2, 1, 3\}$ (2) $\{-2, 0, 3\}$

(3) $\{-2, 2, 0\}$ (4) $\{1, 3\}$

50. If $f(x) = \cos \pi(|x| + [x])$, where [.] denotes the greatest integral, function then which is not true?

(1) Continuous at $x = 1/2$

(2) Continuous at $x = 0$

(3) Differentiable in $(-1, 0)$

(4) Differentiable in $(0, 1)$

51. If $f(x) = \begin{cases} e^{x^2 + x}, & x > 0 \\ ax + b, & x \leq 0 \end{cases}$ is differentiable at $x = 0$, then

(1) $a = 1, b = -1$ (2) $a = -1, b = 1$

(3) $a = 1, b = 1$ (4) $a = -1, b = -1$

52. If $f(x) = \begin{cases} x - 1, & x < 0 \\ x^2 - 2x, & x \geq 0 \end{cases}$, then

(1) $f(|x|)$ is discontinuous at $x = 0$

(2) $f(|x|)$ is differentiable at $x = 0$

(3) $|f(x)|$ is non-differentiable at $x = 0, 2$

(4) $|f(x)|$ is continuous at $x = 0$

53. If $f(x) = \begin{cases} |1 - 4x^2|, & 0 \leq x < 1 \\ [x^2 - 2x], & 1 \leq x < 2 \end{cases}$, where [.] denotes the greatest integer function, then $f(x)$ is

(1) differentiable for all x

(2) continuous at $x = 1$

(3) non-differentiable at $x = 1$

(4) none of these

54. If $f(x) = \begin{cases} x^a \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

is continuous but non-differentiable at $x = 0$, then

(1) $a \in (-1, 0)$ (2) $a \in (0, 2]$

(3) $a \in (0, 1]$ (4) $a \in [1, 2)$

55. If $x + 4\,|y| = 6y$, then y as a function of x is

(1) continuous at $x = 0$ (2) derivable at $x = 0$

(3) $\frac{dy}{dx} = \frac{1}{2}$ for all x (4) none of these

56. Let $g(x)$ be a polynomial of degree one and $f(x)$ be defined by $f(x) = \begin{cases} g(x), & x \leq 0 \\ |x|^{\sin x}, & x > 0 \end{cases}$.

If $f(x)$ is continuous satisfying $f'(1) = f(-1)$, then $g(x)$ is

(1) $(1 + \sin 1)\,x + 1$ (2) $(1 - \sin 1)\,x + 1$

(3) $(1 - \sin 1)\,x - 1$ (4) $(1 + \sin 1)\,x - 1$

57. If $f(x) = |1 - x|$, then the points where $\sin^{-1}(f|x|)$ is non-differentiable are

(1) $\{0, 1\}$ (2) $\{0, -1\}$

(3) $\{0, 1, -1\}$ (4) none of these

58. Given that $f(x) = xg(x) / |x|$, $g(0) = g'(0) = 0$ and $f(x)$ is continuous at $x = 0$. Then the value of $f'(0)$

(1) does not exist (2) is –1

(3) is 1 (4) is 0

59. If $f(x) = \begin{cases} \sin x, & x < 0 \\ \cos x - |x - 1|, & x \geq 0 \end{cases}$,

then $g(x) = f(|x|)$ is non-differentiable for

(1) no value of x

(2) exactly one value of x

(3) exactly three values of x

(4) none of these

60. $f(x) = \max\{x/n, |\sin \pi x|\}$, $n \in N$, has maximum points of non-differentiability for $x \in (0, 4)$. Then n cannot be

(1) 4 (2) 2 (3) 5 (4) 6

61. If $f(x) = [\log_e x] + \sqrt{\{\log_e x\}}$, $x > 1$, where [·] and {·} denote the greatest integer function and the fractional part function, respectively, then

(1) $f(x)$ is continuous but non-differentiable at $x = e$

(2) $f(x)$ is differentiable at $x = e$

(3) $f(x)$ is discontinuous at $x = e$

(4) none of these

62. $f(x)=\begin{cases}1-\sqrt{1-x^2}, & \text{if } -1\le x\le 1\\ 1+\log\dfrac{1}{x}, & \text{if } x>1\end{cases}$ is

(1) continuous and differentiable at $x = 1$
(2) continuous but not differentiable at $x = 1$
(3) neither continuous nor differentiable at $x = 1$
(4) none of these

63. The set of all points where $f(x) = \sqrt[3]{x^2\,|x|} - |x| - 1$ is not differentiable is

(1) $\{0\}$ (2) $\{-1, 0, 1\}$
(3) $\{0, 1\}$ (4) none of these

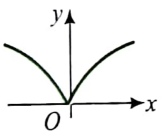
64. Let $f(x)$ be a continuous function for all $x \in R$ and $f'(0) = 1$. Then $g(x) = f(|x|) - \sqrt{\dfrac{1-\cos 2x}{2}}$, at $x = 0$,

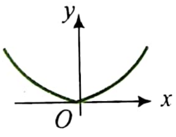
(1) is differentiable at $x = 0$ and its value is 1
(2) is differentiable at $x = 0$ and its value is 0
(3) is non-differentiable at $x = 0$ as its graph has sharp turn at $x = 0$
(4) is non-differentiable at $x = 0$ as its graph has vertical tangent at $x = 0$

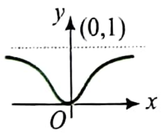
65. $f(x)=\begin{cases}x^2\left(\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right), & x\ne 0\\ 0, & x=0\end{cases}$. Then

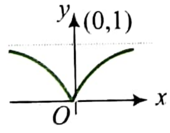
(1) $f(x)$ is discontinuous at $x = 0$
(2) $f(x)$ is continuous but non-differentiable at $x = 0$
(3) $f(x)$ is differentiable at $x = 0$
(4) $f'(0) = 2$

66. Let $y = f(x) = \begin{cases}e^{-\frac{1}{x^2}}, & \text{if } x\ne 0\\ 0, & \text{if } x=0\end{cases}$. Then which of the following can best represent the graph of $y = f(x)$?

(1) 

(2) 

(3) 

(4) 

67. If $f(2 + x) = f(-x)$ for all $x \in R$, then differentiability at $x = 4$ implies differentiability at

(1) $x = 1$ (2) $x = -1$
(3) $x = -2$ (4) cannot say anything

68. If number of points of discontinuity of the function $f(x) = [2 + 10 \sin x]$, in $x \in \left[0, \dfrac{\pi}{2}\right]$ is same as number of points of non-differentiability of the function $g(x) = (x - 1) \times (x - 2)|(x - 1)(x - 2) \cdots (x - 2m)|$, $(m \in N)$ in $x \in (-\infty, \infty)$, then the value of m is

(1) 4 (2) 5 (3) 6 (4) 7

69. The number of points where the function

$$f(x)=\begin{cases}1+\left[\cos\dfrac{\pi x}{2}\right], & 1<x\le 2\\ 1-\{x\}, & 0\le x<1\\ |\sin \pi x|, & -1\le x<0\end{cases}$$

and $f(1) = 0$ is continuous but non-differentiable is/are (where $[\cdot]$ and $\{\cdot\}$ represent greatest integer and fractional part functions, respectively)

(1) 0 (2) 1
(3) 2 (4) none of these

70. Let $f(x) = \lim\limits_{n\to\infty}\dfrac{(x^2+2x+3+\sin \pi x)^n-1}{(x^2+2x+3+\sin \pi x)^n+1}$. Then

(1) $f(x)$ is continuous and differentiable for all $x \in R$
(2) $f(x)$ is continuous but not differentiable for all $x \in R$
(3) $f(x)$ is discontinuous at infinite number of points
(4) $f(x)$ is discontinuous at finite number of points

Multiple Correct Answers Type

1. Which of the statement(s) is/are incorrect?

(1) If $f + g$ is continuous at $x = a$, then f and g are continuous at $x = a$.
(2) If $\lim\limits_{x\to a}(fg)$ exists, then both $\lim\limits_{x\to a} f$ and $\lim\limits_{x\to a} g$ exist.
(3) Discontinuity at $x = a \Rightarrow$ non-existence of limit.
(4) All functions defined on a closed interval attain a maximum or a minimum value in that interval.

2. A function f is defined on an interval $[a, b]$. Which of the following statement(s) is/are incorrect?

(1) If $f(a)$ and $f(b)$ have opposite signs, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.
(2) If f is continuous on $[a, b]$, $f(a) < 0$, and $f(b) > 0$, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.
(3) If f is continuous on $[a, b]$, and there is a point c in (a, b) such that $f(c) = 0$. Then $f(a)$ and $f(b)$ have opposite signs.
(4) If f has no zeros on $[a, b]$, then $f(a)$ and $f(b)$ have the same sign.

3. Which of the following functions has/have a removable discontinuity at the indicated point?

(1) $f(x) = \dfrac{x^2 - 2x - 8}{x + 2}$ at $x = -2$

(2) $f(x) = \dfrac{x - 7}{|x - 7|}$ at $x = 7$

(3) $f(x) = \dfrac{x^3 + 64}{x + 4}$ at $x = -4$

(4) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x}$ at $x = 9$

4. Which of the following function(s) has/have removable discontinuity at $x = 1$?

(1) $f(x) = \frac{1}{\ln|x|}$ (2) $f(x) = \frac{x^2-1}{x^3-1}$

(3) $f(x) = 2^{-2^{\frac{1}{1-x}}}$ (4) $f(x) = \frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

5. If $f(x) = [|x|]$, where $[\cdot]$ denotes the greatest integer function, then which of the following is not true?

(1) $f(x)$ is continuous $\forall x \in R$

(2) $f(x)$ is continuous from right and discontinuous from left $\forall x \in N$

(3) $f(x)$ is continuous from left and discontinuous from right $\forall x \in I$

(4) $f(x)$ is continuous at $x = 0$

6. If $f(x) = \lim\limits_{t\to\infty} \frac{|a+\sin\pi x|^t - 1}{|a+\sin\pi x|^t + 1}$, $x \in (0, 6)$, then

(1) if $a = 1$, then $f(x)$ has 5 points of discontinuity

(2) if $a = 3$, then $f(x)$ has no point of discontinuity

(3) if $a = 0.5$, then $f(x)$ has 6 points of discontinuity

(4) if $a = 0$, then $f(x)$ has 6 points of discontinuity

7. If $f(x) = \text{sgn}(x^2 - ax + 1)$ has maximum number of points of discontinuity, then

(1) $a \in (2, \infty)$ (2) $a \in (-\infty, -2)$

(3) $a \in (-2, 2)$ (4) none of these

8. A function is defined as

$$f(x) = \lim_{n\to\infty} \begin{cases} \cos^{2n} x, & \text{if } x < 0 \\ \sqrt[n]{1+x^n}, & \text{if } 0 \le x \le 1 \\ \frac{1}{1+x^n}, & \text{if } x > 1 \end{cases}$$

which of the following does not hold good?

(1) Continuous at $x = 0$ but discontinuous at $x = 1$

(2) Continuous at $x = 1$ but discontinuous at $x = 0$

(3) Continuous both at $x = 1$ and $x = 0$

(4) Discontinuous both at $x = 1$ and $x = 0$

9. The function $f(x) = \begin{cases} 1, & |x| \ge 1 \\ \frac{1}{n^2}, & \frac{1}{n} \le |x| < \frac{1}{n-1}, \; n = 2, 3, \ldots \\ 0, & x = 0 \end{cases}$

(1) is discontinuous at infinite points

(2) is continuous everywhere

(3) is discontinuous only at $x = \frac{1}{n}$, $n \in Z - \{0\}$

(4) none of these

10. Let $f(x) = [x]$ and $g(x) = \begin{cases} 0, & x \in Z \\ x^2, & x \in R - Z \end{cases}$

($[\cdot]$ represents the greatest integer function.) Then

(1) $\lim\limits_{x\to 1} g(x)$ exists but $g(x)$ is not continuous at $x = 1$

(2) $f(x)$ is not continuous at $x = 1$

(3) gof is continuous for all x

(4) fog is continuous for all x

11. If $f(x) = \sum\limits_{r=1}^{n} a_r |x|^r$, where a_i' sare real constants, then $f(x)$ is

(1) Continuous at $x = 0$ for all a_i

(2) Differentiable at $x = 0$ for all $a_{2k+1} = 0$

(3) Differentiable at $x = 0$ for all $a_i \in R$

(4) None of these

12. If $f(x) = \begin{cases} |x| - 3, & x < 1 \\ |x-2| + a, & x \ge 1 \end{cases}$ and $g(x) = \begin{cases} 2 - |x|, & x < 2 \\ \text{sgn}(x) - b, & x \ge 2 \end{cases}$

and $h(x) = f(x) + g(x)$ is discontinuous at exactly one point, then which of the following values of a and b are possible?

(1) $a = -3, b = 0$ (2) $a = 2, b = 1$

(3) $a = 2, b = 0$ (4) $a = -3, b = 1$

13. $f(x) = \begin{cases} \left(\frac{3}{2}\right)^{(\cot 3x)/(\cot 2x)}, & 0 < x < \frac{\pi}{2} \\ b + 3, & x = \frac{\pi}{2} \\ (1 + |\cot x|)^{(a|\tan x|)/b}, & \frac{\pi}{2} < x < \pi \end{cases}$

is continuous at $x = \pi/2$. Then

(1) $a = 0$ (2) $a = 2$

(3) $b = -2$ (4) $b = 2$

14. Let $f(x) = \text{sgn}(\cos 2x - 2\sin x + 3)$, where sgn $(\cdot)$ is the signum function. Then $f(x)$

(1) is continuous over its domain

(2) has a missing point discontinuity

(3) has isolated point discontinuity

(4) removable discontinuity

15. Let $f(x) = \begin{cases} \frac{e^x - 1 + ax}{x^2}, & x > 0 \\ b, & x = 0 \\ \frac{\sin\frac{x}{2}}{x}, & x < 0 \end{cases}$. Then

(1) $f(x)$ is continuous at $x = 0$ if $a = -1$, $b = \frac{1}{2}$

(2) $f(x)$ is discontinuous at $x = 0$ if $b \ne \frac{1}{2}$

(3) $f(x)$ has irremovable discontinuity at $x = 0$ if $a \ne -1$

(4) $f(x)$ has removable discontinuity at $x = 0$ if $a = -1$, $b \ne \frac{1}{2}$

16. If function $f(x) = \begin{cases} \left(\dfrac{e^{-x}+x^2-a}{-x}\right)^{-1/x}, & -1 \le x < 0 \\ \dfrac{e^{1/x}+e^{2/x}+e^{3/|x|}}{ae^{2/x}+be^{3/|x|}}, & 0 < x < 1 \end{cases}$

continuous at $x = 0$ then

(1) $a = 1$ (2) $b = e^{-3/2}$

(3) $f(0) = e^{3/2}$ (4) $f(0) = e^{-1/2}$

17. If function $f(x) = \begin{cases} \dfrac{ae^{1/(|x+2|)}-1}{2-e^{1/(|x+2|)}}; & -3 < x < -2 \\ b; & x = -2 \\ \sin\left(\dfrac{x^4-16}{x^5+32}\right); & -2 < x < 0 \end{cases}$

is continuous at $x = -2$ then

(1) $a = \sin\frac{2}{5}$ (2) $b = -\sin\frac{2}{5}$

(3) $a = -\sin\frac{1}{5}$ (4) $b = \sin\frac{1}{5}$

18. For the function $f : (0, 1) \to R$, $f(x) = [2^n x] + \{2^m x\}$, $(n, m \in N; n > m)$, the number of points of discontinuity of the function can be (where $[\cdot]$, $\{\}$ represent greatest integer function and fractional part of x respectively)

(1) 24 (2) 28 (3) 26 (4) 496

19. The function $f(x) = \begin{cases} 5x - 4, & \text{for } 0 < x \le 1 \\ 4x^2 - 3x, & \text{for } 1 < x < 2 \\ 3x + 4, & \text{for } x \ge 2 \end{cases}$ is

(1) continuous at $x = 1$ and $x = 2$
(2) continuous at $x = 1$ but not derivable at $x = 2$
(3) continuous at $x = 2$ but not derivable at $x = 1$
(4) continuous at $x = 1$ and 2 but not derivable at $x = 1$ and $x = 2$

20. Which of the following is/are true for $f(x) = \text{sgn}(x) \times \sin x$?
(1) Discontinuous no where
(2) An even function
(3) $f(x)$ is periodic
(4) $f(x)$ is differentiable for all x

21. $f(x)$ is differentiable function and $(f(x)\cdot g(x))$ is differentiable at $x = a$. Then
(1) $g(x)$ must be differentiable at $x = a$
(2) if $g(x)$ is discontinuous, then $f(a) = 0$
(3) if $f(a) \neq 0$, then $g(x)$ must be differentiable
(4) none of these

22. Let $f(x) = \begin{cases} xe^x, & x \le 0 \\ x + x^2 - x^3, & x > 0 \end{cases}$. Then the correct statement is
(1) f is continuous and differentiable for all x.
(2) f is continuous but not differentiable at $x = 0$.
(3) f' is continuous and differentiable for all x.
(4) f' is continuous but not differentiable at $x = 0$.

23. $f(x) = \dfrac{[x]+1}{\{x\}+1}$ for $f:\left[0, \frac{5}{2}\right) \to \left(\frac{1}{2}, 3\right]$, where $[.]$ represents the greatest integer function and $\{.\}$ represents the fractional part of x. Then which of the following is true?
(1) $f(x)$ is injective discontinuous function.
(2) $f(x)$ is surjective non-differentiable function.
(3) $\min\left(\lim\limits_{x\to 1^-} f(x), \lim\limits_{x\to 1^+} f(x)\right) = f(1)$.
(4) max(x values of point of discontinuity) $= f(1)$.

24. If $f(x) = \begin{cases} \dfrac{x\log\cos x}{\log(1+x^2)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then
(1) $f(x)$ is not continuous at $x = 0$
(2) $f(x)$ is continuous at $x = 0$
(3) $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$
(4) $f(x)$ is differentiable at $x = 0$

25. If $f(x) = x + |x| + \cos([\pi^2]x)$ and $g(x) = \sin x$, where $[.]$ denotes the greatest integer function, then
(1) $f(x) + g(x)$ is continuous everywhere
(2) $f(x) + g(x)$ is differentiable everywhere
(3) $f(x) \times g(x)$ is differentiable everywhere
(4) $f(x) \times g(x)$ is continuous but not differentiable at $x = 0$

26. Let $f: R \to R$ be any function and $g(x) = \dfrac{1}{f(x)}$. Then which of the following is/are not true?
(1) g is onto if f is onto.
(2) g is one-one if f is one-to-one.
(3) g is continuous if f is continuous.
(4) g is differentiable if f is differentiable.

27. If $f(x) = \begin{cases} x^2(\text{sgn}[x]) + \{x\}, & 0 \le x < 2 \\ \sin x + |x-3|, & 2 \le x < 4 \end{cases}$,

where [ ] and { } represent the greatest integer and the fractional part functions, respectively, then
(1) $f(x)$ is differentiable at $x = 1$
(2) $f(x)$ is continuous but non-differentiable at $x = 1$
(3) $f(x)$ is non-differentiable at $x = 2$
(4) $f(x)$ is discontinuous at $x = 2$

28. Which of the following function is thrice differentiable at $x = 0$?
(1) $f(x) = |x^3|$ (2) $f(x) = x^3|x|$
(3) $f(x) = |x|\sin^3 x$ (4) $f(x) = x|\tan^3 x|$

29. Let $f(x) = [\sin^4 x]$. Then (where $[\cdot]$ represents the greatest integer function)
(1) $f(x)$ is continuous at $x = 0$
(2) $f(x)$ is differentiable at $x = 0$
(3) $f(x)$ is non-differentiable at $x = 0$
(4) $f'(0) = 1$

30. $f: [0, 1] \to R$ is defined as

$$f(x) = \begin{cases} x^3(1-x)\sin\left(\dfrac{1}{x^2}\right), & 0 < x \le 1 \\ 0, & x = 0 \end{cases}, \text{ then}$$

(1) f is continuous but not derivable in [0, 1]
(2) f is differentiable in [0, 1]
(3) f is bounded in [0, 1]
(4) f' is bounded in (0, 1]

31. Let $f(x) = \begin{cases} x^2 + a, & 0 \le x < 1 \\ 2x + b, & 1 \le x \le 2 \end{cases}$ and

$$g(x) = \begin{cases} 3x + b, & 0 \le x < 1 \\ x^3, & 1 \le x \le 2 \end{cases}.$$

If $\dfrac{df}{dg}$ exists at $x = 1$, then

(1) $a = -1$ (2) $b = -2$
(3) $\left(\dfrac{df}{dg}\right)_{x=1} = 2/3$ (4) $a = \pm 1, b = \pm 2$

32. If $f(x) = \begin{cases} \dfrac{a\cos x + bx\sin x + ce^x - 2x}{x^2}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ is differentiable at $x = 0$, then

(1) $a + b + c = 2$ (2) $a + b = -4$
(3) $f'(0) = 1/3$ (4) $a - c = 4$

Linked Comprehension Type

For Problems 1–3

$$\text{Let } f(x) = \begin{cases} \dfrac{a(1 - x\sin x) + b\cos x + 5}{x^2}, & x < 0 \\ 3, & x = 0, \\ \left\{1 + \left(\dfrac{P(x)}{x^2}\right)\right\}^{1/x}, & x > 0 \end{cases}$$

where $P(x)$ is a cubic function and f is continuous at $x = 0$.

1. The range of function $g(x) = 3a\sin x - b\cos x$ is
(1) [–10, 10] (2) [–5, 5]
(3) [–12, 12] (4) none of these

2. The value of $P''(0)$ is
(1) $\log_e 9$ (2) $\log_e 2$
(3) 2 (4) 1

3. If the leading coefficient of $P(x)$ is positive, then the equation $P(x) = b$ has
(1) only one real, positive root
(2) only one real, negative root
(3) three real roots
(4) none of these

For Problems 4–6

Consider $f(x) = x^2 + ax + 3$ and $g(x) = x + b$ and

$$F(x) = \lim_{n\to\infty} \frac{f(x) + x^{2n}g(x)}{1 + x^{2n}}.$$

4. If $F(x)$ is continuous at $x = 1$, then
(1) $b = a + 3$ (2) $b = a - 1$
(3) $a = b - 2$ (4) none of these

5. If $F(x)$ is continuous at $x = -1$, then
(1) $a + b = -2$ (2) $a - b = 3$
(3) $a + b = 5$ (4) none of these

6. If $F(x)$ is continuous at $x = \pm 1$, then $f(x) = 2g(x)$ has
(1) imaginary roots (2) both the roots positive
(3) both the roots negative (4) roots of opposite signs

For Problems 7–9

$$\text{Let } f(x) = \begin{cases} x + 2, & 0 \le x < 2 \\ 6 - x, & x \ge 2 \end{cases}, \quad g(x) = \begin{cases} 1 + \tan x, & 0 \le x < \dfrac{\pi}{4} \\ 3 - \cot x, & \dfrac{\pi}{4} \le x < \pi \end{cases}$$

7. $f(g(x))$ is
(1) discontinuous at $x = \pi/4$
(2) differentiable at $x = \pi/4$
(3) continuous but non-differentiable at $x = \pi/4$
(4) differentiable at $x = \pi/4$, but derivative is not continuous

8. The number of points of non-differentiability of $h(x) = |f(g(x))|$ is
(1) 1 (2) 2 (3) 3 (4) 4

9. The range of $h(x) = f(g(x))$ is
(1) $(-\infty, \infty)$ (2) $(4, \infty)$
(3) $(-\infty, 4]$ (4) none of these

For Problems 10–12

$$\text{Let } f(x) = \begin{cases} [x], & -2 \le x \le -\dfrac{1}{2} \\ 2x^2 - 1, & -\dfrac{1}{2} < x \le 2 \end{cases} \text{ and } g(x) = f(|x|) + |f(x)|, \text{ where}$$

[·] represents the greatest integer function.

10. The number of points where $|f(x)|$ is non-differentiable is
(1) 3 (2) 4 (3) 2 (4) 5

11. The number of points where $g(x)$ is non-differentiable is
(1) 4 (2) 5 (3) 2 (4) 3

12. The number of points where $g(x)$ is discontinuous is
(1) 1 (2) 2
(3) 3 (4) none of these

For Problems 13–15

Given the continuous function

$$y = f(x) = \begin{cases} x^2 + 10x + 8, & x \le -2 \\ ax^2 + bx + c, & -2 < x < 0, \ a \ne 0 \\ x^2 + 2x, & x \ge 0 \end{cases}$$

If a line L touches the graph of $y = f(x)$ at three points, then

13. The slope of the line L is equal to
(1) 1 (2) 2 (3) 4 (4) 6

14. The value of $(a + b + c)$ is equal to
(1) $5\sqrt{2}$ (2) 5 (3) 6 (4) 7

15. If $y = f(x)$ is differentiable at $x = 0$, then the value of b
(1) is -1 (2) is 2
(3) is 4 (4) cannot be determined

For Problems 16 and 17

If $|c| \le \frac{1}{2}$ and $f(x)$ is a differentiable function at $x = 0$ given by

$$f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{c+x}{2}\right), & -\dfrac{1}{2} < x < 0 \\ \dfrac{1}{2}, & x = 0 \\ \dfrac{e^{ax/2}-1}{x}, & 0 < x < \dfrac{1}{2} \end{cases}, \text{ then}$$

16. The value of a is
(1) 1/2 (2) 1 (3) 3/2 (4) 2

17. Which of the following is true ?
(1) $16b^2 = 4 - c^2$ (2) $16b^2 = 1 - 4c^2$
(3) $64b^2 = 4 - c^2$ (4) none of these

For Problems 18-20

Consider, $f(x) = \dfrac{(x^2-1)(x^2-4)}{g(x)}$, $x \in R - \{a, b\}$, where $g(x)$ is a polynomial of degree ≤ 4.

$f(x)$ has a removable type of discontinuity at $x = a$ and $x = b$.

Also it is known that

(i) $\lim_{x\to\pm\infty} f(x) = +\infty$

(ii) $\lim_{x\to -1} f(x) = 6$

(iii) $\lim_{x\to -2} f(x) =$ a non-zero finite number

Let L and M denote the number of points of discontinuity and the number of points of non-derivatively respectively of any function in R. (R is the set of real numbers)

18. Number of points where $|f(x)|$ is non-differentiable in its domain is
(1) 2 (2) 3
(3) 4 (4) 5

19. Number of points where $f(|x|)$ is non-differentiable in its domain is
(1) 2 (2) 3
(3) 4 (4) 5

20. Number of points where $f(-|x|)$ is non-differentiable in its domain is
(1) 0 (2) 1
(3) 2 (4) 3

Matrix Match Type

1. Match the following lists:

List I	List II
a. $f(x) = \lvert x^3\rvert$ is	p. continuous in $(-1, 1)$
b. $f(x) = \sqrt{\lvert x\rvert}$ is	q. differentiable in $(-1, 1)$
c. $f(x) = \lvert\sin^{-1} x\rvert$ is	r. differentiable in $(0, 1)$
d. $f(x) = \cos^{-1}\lvert x\rvert$ is	s. not differentiable at least at one point in $(-1, 1)$

2. In the following, $[x]$ denotes the greatest integer less than or equal to x. Now, match the following lists:

List I	List II
a. $f(x) = x\lvert x\rvert$ is	p. Continuous in $(-1, 1)$
b. $f(x) = \sqrt{\lvert x\rvert}$ is	q. Differentiable in $(-1, 1)$
c. $f(x) = x + [x]$ is	r. Strictly increasing in $(-1, 1)$
d. $f(x) = \lvert x - 1\rvert$ is	s. Not differentiable at least at one point in $(-1, 1)$

3. Let $f(x) = \begin{cases} \dfrac{5e^{1/x}+2}{3-e^{1/x}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$. Then match the following lists:

List I	List II
a. $y = f(x)$ is	p. continuous at $x = 0$
b. $y = x f(x)$ is	q. discontinuous at $x = 0$
c. $y = x^2 f(x)$ is	r. differentiable at $x = 0$
d. $y = x^{-1} f(x)$ is	s. non-differentiable at $x = 0$

4. Match the following lists:

List I	List II
a. $f(x) = \begin{cases} \dfrac{1}{\lvert x\rvert}, & \text{for } \lvert x\rvert \ge 1 \\ ax^2 + b, & \text{for } \lvert x\rvert < 1 \end{cases}$ is differentiable everywhere and $\lvert k\rvert = a + b$. Then the value of k is	p. 2
b. If $f(x) = \text{sgn}(x^2 - ax + 1)$ has exactly one point of discontinuity, then the value of a can be	q. -2
c. $f(x) = [2 + 3\lvert n\rvert \sin x]$, $n \in N$, $x \in (0, \pi)$, has exactly 11 points of discontinuity. Then the value of n is	r. 1
d. $f(x) = \lvert\lvert\lvert x\rvert - 2\rvert + a\rvert$ has exactly three points of non-differentiability. Then the value of a can be	s. -1

5. Consider the function $f(x) = x^2 + bx + c$, where $D = b^2 - 4c > 0$. Now, match the following lists:

List I	List II
Condition on b and c	Number of points of non-differentiability of $g(x) = \lvert f(\lvert x\rvert)\rvert$
a. $b < 0, c > 0$	p. 1
b. $c = 0, b < 0$	q. 2
c. $c = 0, b > 0$	r. 3
d. $b = 0, c < 0$	s. 5

6. Match the following lists:

List I	List II
a. $f(x) = \lim_{n\to\infty} \cos^{2n}(2\pi x) + \left\{x + \frac{1}{2}\right\}$, where {·} denotes the fractional part function at $x = \frac{1}{2}$	p. continuous
b. $f(x) = (\log_e x)(x-1)^{1/5}$ at $x = 1$	q. discontinuous
c. $f(x) = [\cos 2\pi x] + \sqrt{\left\{\sin \pi \frac{x}{2}\right\}}$, where [.] and {.} denote the greatest integer and the fractional part function, respectively, at $x = 1$	r. differentiable
d. $f(x) = \begin{cases} \cos 2x, & x \in Q \\ \sin x, & x \notin Q \end{cases}$ at $x = \frac{\pi}{6}$	s. non-differentiable

7. Consider function $f(x) = \begin{cases} 2x+3, & x \le 1 \\ -x^2+6, & x > 1 \end{cases}$. Now, match the following lists and then choose the correct code.

List I	List II
a. Number of points of non-differentiability of $f(x)$	p. 1
b. Number of points of non-differentiability of $f(\|x\|)$	q. 3
c. Number of points of non-differentiability of $\|f(x)\|$	r. 4
d. Number of points of non-differentiability of $\|f(\|x\|)\|$	s. 5

Codes:

	a	b	c	d
(1)	p	q	q	r
(2)	p	s	q	r
(3)	p	q	q	s
(4)	p	r	q	r

Numerical Value Type

1. The number of points of discontinuity for $f(x) = \text{sgn}(\sin x)$, $x \in [0, 4\pi]$ is _______.

2. The number of points where $f(x) = \text{sgn}(x^2 - 3x + 2) + [x - 3]$, $x \in [0, 4]$, is discontinuous is (where [·] denotes the greatest integer function) _______.

3. Number of points where $f(x) = \sqrt{x^2 + [x]^2}$, $x \in [-2, 2]$ is discontinuous is (where [·] represents the greatest interger function)_______.

4. Let $f(x) = g(x)\,|(x-1)(x-2)(x-3)^2(x-4)^3|$, where $g(x) = x^3 + bx^2 + cx + d$. If $f(x)$ is differentiable for all $x \in R$ and $f'(3) + f'''(4) = 0$ then the value of $g(5)$ is _______.

5. Let $f(x) = \lim_{n\to\infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$. If $f(x)$ is continuous for all $x \in R$, then the value of $a + 8b$ is _______.

6. Let $f(x) = \begin{cases} \frac{x}{2} - 1, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x \le 2 \end{cases}$ and $g(x) = (2x+1)(x-k) + 3$, $x \ge 0$. Then $g(f(x))$ is continuous at $x = 1$ if $12k$ is equal to_______.

7. A differentiable function f is satisfying the relation $f(x+y) = f(x) + f(y) + 2xy(x+y) - \frac{1}{3}$ $\forall\ x, y \in R$ and $\lim_{h\to 0} \frac{3f(h) - 1}{6h} = \frac{2}{3}$. Then the value of $[f(2)]$ is (where $[x]$ represents the greatest integer function)_______.

8. The least integral value of p for which $f''(x)$ is everywhere continuous where $f(x) = \begin{cases} x^p \sin\left(\frac{1}{x}\right) + x|x|, & x \ne 0 \\ 0, & x = 0 \end{cases}$ is_______.

9. The number of points where $f(x) = [x] + [x + 1/3] + [x + 2/3]$, [·] denotes the greatest integer function, is discontinuous for $x \in (0, 3)$ is _______.

10. Let $f(x)$ and $g(x)$ be two continuous functions and $h(x) = \lim_{n\to\infty} \frac{x^{2n} \cdot f(x) + x^{2m} \cdot g(x)}{(x^{2n} + 1)}$. If the limit of $h(x)$ exists at $x = 1$, then one root of $f(x) - g(x) = 0$ is_______.

11. Number of points of discontinuity of $f(x) = [x^3 + 3x^2 + 7x + 2]$, where [.] represents the greatest integer function in [0, 1] is_______.

12. $f(x) = \frac{x}{1 + (\ln x)(\ln x)\cdots\infty}$ $\forall\ x \in [1, 3]$ is non-differentiable at $x = k$. Then the value of $[k^2]$ is (where [·] represents the greatest integer function)_______.

13. If the function $f(x) = \frac{\tan(\tan x) - \sin(\sin x)}{\tan x - \sin x}$, $(x \ne 0)$, is continuous at $x = 0$, then the value of $f(0)$ is_______.

14. The number of points of non-differentiability of function $f(x) = \max\{\sin^{-1}|\sin x|, \cos^{-1}|\sin x|\}$, $0 < x < 2\pi$, is_______.

15. The function $f(x)$ is discontinuous only at $x = 0$ such that $f^2(x) = 1$ $\forall\ x \in R$. Then, the total number of such functions is_______.

Archives

JEE MAIN

Single Correct Answer Type

1. Let $f(x) = x|x|$ and $g(x) = \sin x$

Statement 1: gof is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement 2: gof is twice differentiable at $x = 0$.

(1) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.
(2) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1.
(3) Statement 1 is true, statement 2 is false.
(4) Statement 1 is false, statement 2 is true.

(AIEEE 2009)

2. The value of p and q for which the function

$$f(x) = \begin{cases} \dfrac{\sin(p+1)x + \sin x}{x} & x < 0 \\ q, & x = 0 \\ \dfrac{\sqrt{x+x^2} - \sqrt{x}}{x^{3/2}} & x > 0 \end{cases}$$

is continuous for all x in R, are

(1) $p = \frac{1}{2}, q = \frac{3}{2}$
(2) $p = \frac{1}{2}, q = -\frac{3}{2}$
(3) $p = \frac{5}{2}, q = \frac{1}{2}$
(4) $p = -\frac{3}{2}, q = \frac{1}{2}$

(AIEEE 2011)

3. If $f: R \to R$ is a function defined by $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then f is

(1) continuous for every real x.
(2) discontinuous only at $x = 0$.
(3) discontinuous only at non-zero integral values of x.
(4) continuous only at $x = 0$.

(AIEEE 2012)

4. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx+2, & 3 < x \le 5 \end{cases}$ is differentiable, then the value of $k + m$ is

(1) 2 (2) $\frac{16}{5}$ (3) $\frac{10}{3}$ (4) 4

(JEE Main 2015)

5. For $x \in R$, $f(x) = |\log_e 2 - \sin x|$ and $g(x) = f(f(x))$, then

(1) $g'(0) = \cos(\log_e 2)$
(2) $g'(0) = -\cos(\log_e 2)$
(3) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log_e 2)$
(4) g is not differentiable at $x = 0$

(JEE Main 2016)

6. Let $S = \{t \in R : f(x) = |x - \pi| \times (e^{|x|} - 1) \sin|x|$ is not differentiable at $t\}$. Then the set S is equal to

(1) $\{0, \pi\}$
(2) ϕ (an empty set)
(3) $\{0\}$
(4) $\{\pi\}$

(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. Let $f(x) = \begin{cases} x^2 \left|\cos\dfrac{\pi}{x}\right|, & x \ne 0 \\ 0, & x = 0 \end{cases}$, $x \in R$. Then f is

(1) differentiable both at $x = 0$ and at $x = 2$
(2) differentiable at $x = 0$ but not differentiable at $x = 2$
(3) not differentiable at $x = 0$ but differentiable at $x = 2$
(4) differentiable neither at $x = 0$ nor at $x = 2$

(IIT-JEE, 2012)

Multiple Correct Answers Type

1. If $f(x) = \begin{cases} -x - \dfrac{\pi}{2}, & x \le -\dfrac{\pi}{2} \\ -\cos x, & -\dfrac{\pi}{2} < x \le 0 \\ x - 1, & 0 < x \le 1 \\ \ln x, & x > 1 \end{cases}$, then

(1) $f(x)$ is continuous at $x = -\pi/2$
(2) $f(x)$ is not differentiable at $x = 0$
(3) $f(x)$ is differentiable at $x = 1$
(4) $f(x)$ is differentiable at $x = -3/2$

(IIT-JEE, 2011)

2. Let $f: R \to R$ be a function such that $f(x + y) = f(x) + f(y)$ $\forall x, y \in R$. If $f(x)$ is differentiable at $x = 0$, then

(1) $f(x)$ is differentiable only in a finite interval containing zero
(2) $f(x)$ is continuous $\forall x \in R$
(3) $f'(x)$ is constant $\forall x \in R$
(4) $f(x)$ is differentiable except at finitely many points

(IIT-JEE, 2011)

3. Let $g: R \to R$ be a differentiable function with $g(0) = 0$, $g'(0) = 0$ and $g'(1) \ne 0$. Let $f(x) = \begin{cases} \dfrac{x}{|x|} g(x), & x \ne 0 \\ 0, & x = 0 \end{cases}$ and $h(x) = e^{|x|}$ for all $x \in R$. Let $(foh)(x)$ denote $f(h(x))$ and $(hof)(x)$ denote $h(f(x))$. Then which of the following is (are) true?

(1) f is differentiable at $x = 0$
(2) h is differentiable at $x = 0$
(3) foh is differentiable at $x = 0$
(4) hof is differentiable at $x = 0$

(JEE Advanced 2015)

4. Let $a, b \in R$ and $f: R \to R$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then f is
(1) differentiable at $x = 0$ if $a = 0$ and $b = 1$
(2) differentiable at $x = 1$ if $a = 1$ and $b = 0$
(3) NOT differentiable at $x = 0$ if $a = 1$ and $b = 0$
(4) NOT differentiable at $x = 1$ if $a = 1$ and $b = 1$

(JEE Advanced 2016)

5. Let $f:\left[-\frac{1}{2}, 2\right] \to R$ and $g: \left[-\frac{1}{2}, 2\right] \to R$ be functions defined by $f(x) = [x^2 - 3]$ and $g(x) = |x| f(x) + |4x - 7| f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y \in R$. Then
(1) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$
(2) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$
(3) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$
(4) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$

(JEE Advanced 2016)

6. Let $[x]$ be the greatest integer less than or equal to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous?
(1) $x = -1$ (2) $x = 0$
(3) $x = 2$ (4) $x = 1$

(JEE Advanced 2017)

Matrix Match Type

1. Let $f_1: R \to R, f_2: [0, \infty) \to R, f_3: R \to R$ and $f_4: R \to [0, \infty)$ be defined by

$$f_1(x) = \begin{cases} |x| & \text{if } x < 0 \\ e^x & \text{if } x \ge 0 \end{cases}; f_2(x) = x^2;$$

$$f_3(x) = \begin{cases} \sin x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases} \text{ and}$$

$$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0 \\ f_2(f_1(x)) - 1 & \text{if } x \ge 0 \end{cases}$$

Match the statements/expressions given in List I with the values given in List II.

List I	List II
a. f_4 is	**p.** onto but not one–one
b. f_3 is	**q.** neither continuous nor one–one
c. $f_2 \circ f_1$ is	**r.** differentiable but not one–one
d. f_2 is	**s.** continuous and one–one

Codes:

	a	b	c	d
(1)	r	p	s	q
(2)	p	r	s	q
(3)	r	p	q	s
(4)	p	r	q	s

(JEE Advanced 2014)

2. Match the statements/expressions given in List I with the values given in List II.

List I	List II
a. In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $\|\alpha\|$ is (are)	**p.** 1
b. Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \ge 1 \end{cases}$ is differentiable for all $x \in R$. Then possible value(s) of a is(are)	**q.** 2
c. Let $\omega \ne 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$, then possible value(s) of n is (are)	**r.** 3
d. Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of $\|q - a\|$ is (are)	**s.** 4
	t. 5

(JEE Advanced 2015)

3. Let $f_1: R \to R$, $f_2: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$, $f_3: \left(-1, e^{\frac{\pi}{2}} - 2\right) \to R$ and $f_4: R \to R$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$,
where the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iii) $f_3(x) = [\sin(\log_e(x + 2))]$, where for $t \in R$, $[t]$ denotes the greatest integer less than or equal to t.

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

List I	List II
P. The function f_1 is	**I.** NOT continuous at $x = 0$
Q. The function f_2 is	**II.** continuous at $x = 0$ and NOT differentiable at $x = 0$
R. The function f_3 is	**III.** differentiable at $x = 0$ and its derivative is NOT continuous at $x = 0$
S. The function f_4 is	**IV.** differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is:

(1) P → II; Q → III; R → I; S → IV

(2) P → IV; Q → I; R → II; S → III

(3) P → IV; Q → II; R → I; S → III

(4) P → II; Q → I; R → IV; S → III

(JEE Advanced 2018)

Numerical Value Type

1. Let $f: R \to R$ and $g: R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h: R \to R$ by

$$h(x) = \begin{cases} \max\{f(x), g(x)\}, & \text{if } x \le 0 \\ \min\{f(x), g(x)\}, & \text{if } x > 0 \end{cases}$$

Then number of points at which $h(x)$ is not differentiable is **(JEE Advanced 2014)**

Answers Key

EXERCISES

Single Correct Answer Type

1. (3)	2. (3)	3. (2)	4. (2)	5. (2)
6. (4)	7. (3)	8. (2)	9. (1)	10. (1)
11. (4)	12. (2)	13. (4)	14. (3)	15. (1)
16. (4)	17. (2)	18. (4)	19. (3)	20. (4)
21. (4)	22. (2)	23. (1)	24. (2)	25. (1)
26. (3)	27. (1)	28. (4)	29. (1)	30. (3)
31. (1)	32. (3)	33. (3)	34. (1)	35. (3)
36. (3)	37. (2)	38. (4)	39. (2)	40. (2)
41. (4)	42. (4)	43. (3)	44. (3)	45. (4)
46. (2)	47. (3)	48. (2)	49. (4)	50. (2)
51. (3)	52. (3)	53. (3)	54. (3)	55. (1)
56. (2)	57. (3)	58. (4)	59. (3)	60. (2)
61. (1)	62. (2)	63. (4)	64. (2)	65. (3)
66. (3)	67. (3)	68. (3)	69. (2)	70. (1)

Multiple Correct Answers Type

1. (1), (2), (3), (4)	2. (1), (3), (4)
3. (1), (3), (4)	4. (2), (4)
5. (2), (4)	6. (1), (2), (3), (4)
7. (1), (2)	8. (1), (2), (3)
9. (1), (3)	10. (1), (2), (3)
11. (1), (2)	12. (1), (2)
13. (1), (3)	14. (3), (4)
15. (1), (2), (3), (4)	16. (1), (2), (3)
17. (1), (2)	18. (1), (2), (4)
19. (1), (2)	20. (1), (3)
21. (2), (3)	22. (1), (3)
23. (1), (2), (4)	24. (2), (4)
25. (1), (3)	26. (1), (3), (4)
27. (2), (3), (4)	28. (2), (3), (4)
29. (1), (2)	30. (2), (3), (4)
31. (1), (2), (3)	32. (2), (3)

Linked Comprehension Type

1. (2)	2. (1)	3. (2)	4. (1)	5. (3)
6. (4)	7. (3)	8. (2)	9. (3)	10. (1)
11. (4)	12. (2)	13. (3)	14. (4)	15. (2)
16. (1)	17. (3)	18. (1)	19. (4)	20. (2)

Matrix Match Type

1. a → p, q, r; b → p, r, s; c → p, r, s; d → p, r, s.
2. a → p, q, r; b → p, s; c → r, s; d → p, q.
3. a → q, s; b → p, s; c → p, r; d → q, s.
4. a → r, s; b → p, q; c → p, q; d → p, r.
5. a → s; b → r; c → p; d → q.
6. a → q, s; b → p, r; c → p, r; d → p, s.
7. (3)

Numerical Value Type

1. (5)	2. (4)	3. (4)	4. (12)	5. (8)
6. (6)	7. (8)	8. (5)	9. (8)	10. (1)
11. (11)	12. (7)	13. (2)	14. (7)	15. (6)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (3)	2. (4)	3. (1)	4. (1)	5. (1)
6. (2)				

JEE Advanced

Single Correct Answer Type

1. (2)

Multiple Correct Answers Type

1. (1), (2), (3), (4)	2. (2), (3)
3. (1), (4)	4. (1), (2)
5. (2), (3)	6. (2), (3), (4)

Matrix Match Type

1. (4) 2. b → p, q. 3. (4)

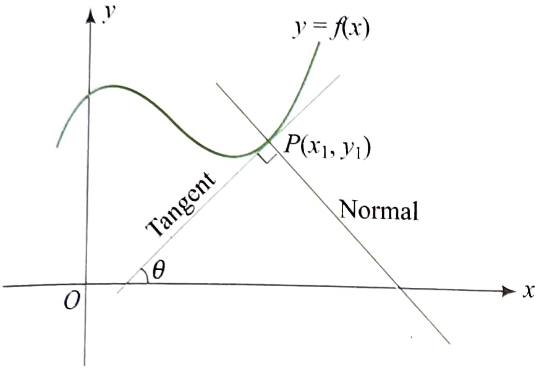
Numerical Value Type

1. (3)

5 Application of Derivatives

TANGENT AND NORMAL

TANGENT AND NORMAL AT A POINT ON THE CURVE



If tangent at P makes an angle θ with positive direction of x-axis, then slope of the tangent is $m = \tan\theta$.

Also, slope of tangent to the curve at any point on it is $\frac{dy}{dx}$.

$$\therefore \quad m = \tan\theta = \frac{dy}{dx}$$

Also, slope of normal is $-\frac{dx}{dy}$.

Therefore, equation of tangent at point $P(x_1, y_1)$ is

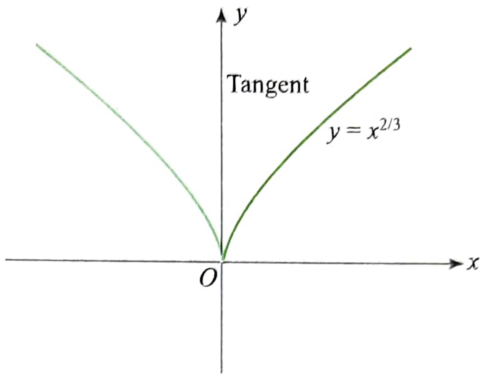
$$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

and equation of normal at point $P(x_1, y_1)$ is

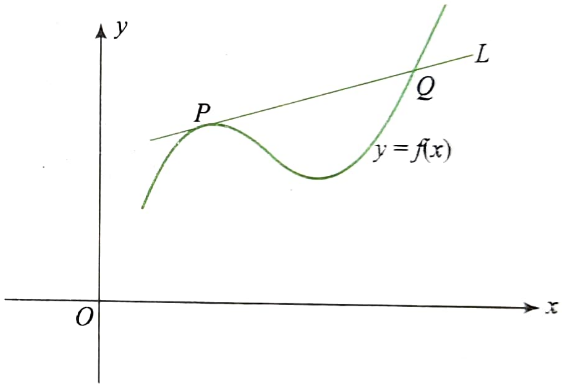
$$(y - y_1) = \left(-\frac{dx}{dy}\right)_{(x_1, y_1)} (x - x_1)$$

Note

- The point $P(x_1, y_1)$ will satisfy the equation of curve, the equation of tangent and the equation of normal line.
- If the tangent at point P on the curve is parallel to x-axis, then $dy/dx = 0$ at point P.
- If the tangent at point P on the curve is parallel to y-axis, then $dy/dx = \infty$ or $dx/dy = 0$ at point P.
- If the tangent at point P on the curve is equally inclined to both the axes, then $dy/dx = \pm 1$ at point P.
- If the tangent at point P makes equal intercepts on the coordinate axes, then $dy/dx = -1$ at point P.
- Tangent to a curve at point P can be drawn even though dy/dx does not exist at point P.
  e.g., $x = 0$ is a tangent to $y = x^{2/3}$ at (0, 0).

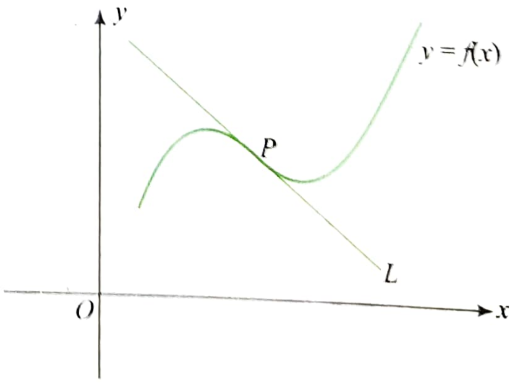


- A tangent to the curve at one point may intersect the curve at other point on the curve.



Thus, a line may meet the curve at several points and may still be tangent to it at some point.
In the figure, line L is tangent to curve at P and it intersects the curve again at Q.

- A line may be tangent to the curve at point P and may cross the curve at the same point also.



In the figure, line L is tangent to the curve at point P. Also, it is crossing the curve at point P.

ILLUSTRATION 5.1

How many tangents to the curves $f_1(x) = x^2 - x + 1$ and $f_2(x) = x^3 - x^2 - 2x + 1$ are parallel?

Sol. We have curves

$$f_1(x) = x^2 - x + 1 \text{ and } f_2(x) = x^3 - x^2 - 2x + 1$$

$$\therefore \quad f'_1(x) = 2x - 1 \text{ and } f'_2(x) = 3x^2 - 2x - 2$$

Let the tangent to curve $y = f_1(x)$ at point at $(x_1, f_1(x_1))$ and that to the curve $y = f_2(x)$ at $(x_2, f_2(x_2))$ be parallel.

$$\therefore \quad 2x_1 - 1 = 3x_2^2 - 2x_2 - 2$$

Clearly, this equation has infinite solutions (x_1, x_2).

So, there will be infinite number of parallel tangents to the curves $f_1(x)$ and $f_2(x)$.

ILLUSTRATION 5.2

Prove that the tangent drawn at any point to the curve $f(x) = x^5 + 3x^3 + 4x + 8$ would make an acute angle with the x-axis.

Sol. $f(x) = x^5 + 3x^3 + 4x + 8$

or $\quad f'(x) = 5x^4 + 9x^2 + 4$

Clearly, $f'(x) > 0 \ \forall \ x \in R$

Thus, the tangent drawn at any point would have positive slope and, hence, would make an acute angle with the x-axis.

ILLUSTRATION 5.3

(i) Find the equation of the normal to the curve $y = |x^2 - |x||$ at $x = -2$.

(ii) Find the equation of tangent to the curve

$$y = \sin^{-1}\frac{2x}{1+x^2} \text{ at } x = \sqrt{3}$$

Sol.

(i) In the neighborhood of $x = -2$, $y = |x^2 - |x|| = |x^2 + x| = x^2 + x$.

Hence, the point on curve is $(-2, 2)$.

$$\frac{dy}{dx} = 2x + 1 \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=-2} = -3$$

So, the slope of the normal at $(-2, 2)$ is $\frac{1}{3}$.

Hence, the equation of the normal is $\frac{1}{3}(x+2) = y - 2$

or $\quad 3y = x + 8$

(ii) $y = \sin^{-1}\frac{2x}{1+x^2} = \pi - 2\tan^{-1}x$, for $x > 1$

or $$\frac{dy}{dx} = -\frac{2}{1+x^2}$$

or $$\left(\frac{dy}{dx}\right)_{x=\sqrt{3}} = -\frac{2}{1+3} = -\frac{1}{2}$$

Also, when $x = \sqrt{3}$, $y = \pi - 2 \times \frac{\pi}{3} = \frac{\pi}{3}$

Hence, equation of tangent is $y - \frac{\pi}{3} = -\frac{1}{2}(x - \sqrt{3})$.

ILLUSTRATION 5.4

Find the equation of the tangent to the curve

$$y = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ at the origin.}$$

Sol. Clearly function is continuous at $x = 0$ as

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0 = f(0)$$

Slope of tangent at point (0, 0)

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} h \cdot \sin\frac{1}{h} = 0$$

So, equation of tangent is $(y - 0) = 0 \cdot (x - 0)$ or $y = 0$.

ILLUSTRATION 5.5

Find the equations of the normal to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

Sol. The equation of the given curve is $y = x^3 + 2x + 6$.

The slope of the tangent to the given curve at any point (x, y) is given by

$$\frac{dy}{dx} = 3x^2 + 2$$

$\therefore$ Slope of the normal to the given curve at any point (x, y)

$$= \frac{-1}{3x^2 + 2}$$

The equation of the given line is $x + 14y + 4 = 0$

Normal is parallel to this line. Therefore,

$$\frac{-1}{3x^2+2} = \frac{-1}{14}$$

$$\therefore \quad x^2 = 4$$

$$\therefore \quad x = \pm 2$$

Putting $x = 2$ in given equation of curve, we have

$$y = (2)^3 + 2(2) + 6 = 18.$$

Putting $x = -2$ in given equation of curve, we have

$$y = (-2)^3 + 2(-2) + 6 = -6.$$

Hence the corresponding points on the curve are (2, 18) and (–2, –6).

Therefore, required equations of normal are

$$y - 18 = \frac{-1}{14}(x - 2) \text{ and } y - (-6) = \frac{-1}{14}(x + 2)$$

or $\quad x + 14y - 254 = 0$ and $x + 14y + 86 = 0$

ILLUSTRATION 5.6

If the equation of tangent to the curve $y^2 = ax^3 + b$ at point (2, 3) is $y = 4x - 5$, then find the values of a and b.

Sol. We have curve $y^2 = ax^3 + b$ (1)

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 3ax^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3ax^2}{2y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{3\times a\times(2)^2}{2\times 3} = 2a$$

Now, line $y = 4x - 5$ is tangent to the curve at (2, 3).

$\therefore \quad 2a = 4$

or $\quad a = 2$

Also, (2, 3) lies on the curve (1).

$\Rightarrow \quad 9 = 8a + b$

$\therefore \quad b = -7$

ILLUSTRATION 5.7

For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangents pass through the origin.

Sol. The equation of the given curve is $y = 4x^3 - 2x^5$.

Therefore, slope of tangent at any point $P(x_0, y_0)$ is

$$\left.\frac{dy}{dx}\right]_{(x_0,y_0)} = 12x_0^2 - 10x_0^4$$

Now, tangent at this point is given by

$$y - y_0 = (12x_0^2 - 10x_0^4)(x - x_0) \quad (1)$$

This tangent is passing through the origin (0, 0). Therefore,

$$0 - y_0 = (12x_0^2 - 10x_0^4)(0 - x_0)$$

or $\quad y_0 = 12x_0^3 - 10x_0^5 \quad (2)$

Also, point (x_0, y_0) lies on the curve. Therefore,

$$y_0 = 4x_0^3 - 2x_0^5 \quad (3)$$

From (2) and (3), we have

$$12x_0^3 - 10x_0^5 = 4x_0^3 - 2x_0^5$$

or $\quad 8x_0^5 - 8x_0^3 = 0$

or $\quad x_0^3(x_0^2 - 1) = 0$

or $\quad x_0 = 0, \pm 1$

When $x_0 = 0$, $y_0 = 4(0)^3 - 2(0)^5 = 0$.

When $x_0 = 1$, $y_0 = 4(1)^3 - 2(1)^5 = 2$.

When $x_0 = -1$, $y_0 = 4(-1)^3 - 2(-1)^5 = -2$.

Hence, the required points are (0, 0), (1, 2), and (–1, –2).

ILLUSTRATION 5.8

Find the equation of tangent to the curve represented parametrically by the equations $x = t^2 - 3t + 1$ and $y = 2t^2 + 3t - 4$ at the point $M(-1, 10)$.

Sol. For point $M(-1, 10)$, we have

$$t^2 - 3t + 1 = -1 \quad (1)$$

and $\quad 2t^2 + 3t - 4 = 10 \quad (2)$

From (1),

$$t^2 - 3t + 2 = 0$$

$\Rightarrow \quad (t-1)(t-2) = 0$

$\Rightarrow \quad t = 1, 2$

From (2),

$$2t^2 + 3t - 14 = 0$$

$\Rightarrow \quad (2t+7)(t-2) = 0$

$\Rightarrow \quad t = \frac{-7}{2}, 2$

So, for the point $M(-1, 10)$, value of t is 2 (common value from the two equations).

$$\therefore \quad \text{Slope of tangent at } M(-1, 10) = \left(\frac{dy}{dx}\right)_{t=2}$$

$$= \left(\frac{dy/dt}{dx/dt}\right)_{t=2}$$

$$= \left(\frac{4t+3}{2t-3}\right)_{t=2} = 11$$

Therefore, equation of tangent is

$$y - 10 = 11(x+1)$$

or $\quad 11x - y + 21 = 0$

ILLUSTRATION 5.9

For the curve $xy = c$, prove that the portion of the tangent intercepted between the coordinate axes is bisected at the point of contact.

Sol. Let the point at which the tangent is drawn be (α, β) on the curve $xy = c$. Then,

$$\left(\frac{dy}{dx}\right) = -\frac{\beta}{\alpha}$$

Thus, the equation of the tangent is

$$y - \beta = -\frac{\beta}{\alpha}(x - \alpha)$$

or $\quad y\alpha - \alpha\beta = -x\beta + \alpha\beta$

or $\quad \frac{x}{2\alpha} + \frac{y}{2\beta} = 1$

Hence, the tangent line cuts x-and y-axes at $A(2\alpha, 0)$ and $B(0, 2\beta)$, respectively, and the point (α, β) bisects AB.

ILLUSTRATION 5.10

If the tangent at any point $(4m^2, 8m^3)$ of $x^3 - y^2 = 0$ is a normal to the curve $x^3 - y^2 = 0$, then find the value of m.

Sol. Here, $y^2 = x^3$ (1)

or $\quad 2y\frac{dy}{dx} = 3x^2$

$$\therefore \quad \text{Slope at } (4m^2, 8m^3) = \left(\frac{3x^2}{2y}\right)_{(4m^2,\, 8m^3)} = 3m$$

Therefore, equation of the tangent at $(4m^2, 8m^3)$ is

$$\frac{y-8m^3}{x-4m^2}=3m \quad \text{or} \quad y=3mx-4m^3 \qquad (2)$$

For another point, solving equations (1) and (2), we get

$$x^3=(3mx-4m^3)^2$$

or $\quad x=4m^2, m^2$

$\therefore \quad A(4m^2, 8m^3)$ and $B(m^2, -m^3)$

Hence, slope of the tangent at B is

$$\left(\frac{dy}{dx}\right)_{(m^2,-m^3)}=\left(\frac{3x^2}{2y}\right)_{(m^2,-m^3)}=\frac{-3}{2}m$$

or $\quad$ Slope of the normal at $B=\dfrac{2}{3m}$

Since tangent at A and normal at B coincide, we get

$$\frac{2}{3m}=3m \quad \text{or} \quad m^2=\frac{2}{9} \quad \text{or } m=\pm\sqrt{\frac{2}{9}}$$

ILLUSTRATION 5.11

Find all the tangents to the curve $y=\cos(x+y)$, where $-2\pi\le x\le 2\pi$, that are parallel to the line $x+2y=0$.

Sol. Slope of the given line $=-\dfrac{1}{2}$ $\qquad$ (1)

$\therefore \quad$ Slope of the tangent $=\left(\dfrac{dy}{dx}\right)=-\dfrac{1}{2}$

The equation of given curve is $y=\cos(x+y)$.

Differentiating the curve w.r.t. x, we get

$$\frac{dy}{dx}=-\sin(x+y)\left\{1+\frac{dy}{dx}\right\}$$

$$\therefore \quad \frac{dy}{dx}=-\frac{\sin(x+y)}{1+\sin(x+y)}$$

$=$ Slope of tangent $\qquad$ (2)

From (1) and (2), $\dfrac{-\sin(x+y)}{1+\sin(x+y)}=-\dfrac{1}{2}$

or $\quad \sin(x+y)=1$ $\qquad$ (3)

or $\quad \cos(x+y)=0$

From the given curve, $y=\cos(x+y)$ or $y=0$ $\qquad$ (4)

and $\quad \sin(x)=1$ $\qquad$ [Using (3) and (4)]

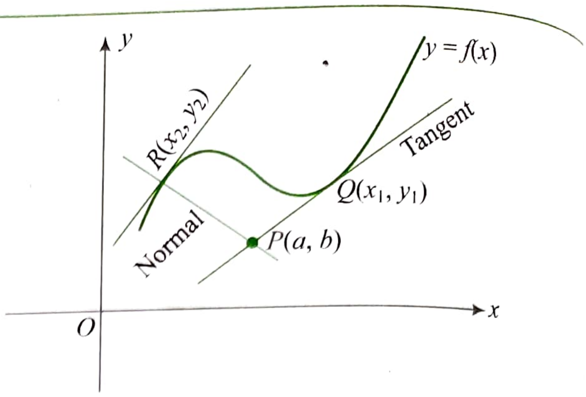
$\therefore \quad x=\dfrac{\pi}{2}, -\dfrac{3\pi}{2}$

Therefore, the points are $P(\pi/2, 0)$, $Q(-3\pi/2, 0)$.

Tangent at P is $x+2y=\dfrac{\pi}{2}$ and tangent at Q is $x+2y=-\dfrac{3\pi}{2}$.

TANGENT AND NORMAL FROM EXTERNAL POINT

For a point $P(a, b)$ which does not lie on the curve $y=f(x)$, the equation of possible tangents to the curve $y=f(x)$, passing through P can be determined by finding the point of contact $Q(x_1, y_1)$ on the curve.



Since Q lies on the given curve,

$$y_1=f(x_1) \qquad (1)$$

Also, slope of PQ is

$$\frac{y_1-b}{x_1-a}=\left(\frac{dy}{dx}\right)_{(x_1,y_1)} \qquad (2)$$

Solving equations (1) and (2), we get the point of contact on the curve and hence, the equation of required tangent.

To find the equation of normal from point P, we solve equations

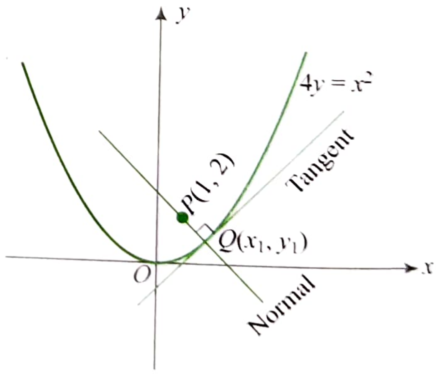
$$y_2=f(x_2) \text{ and } \frac{y_2-b}{x_2-a}=\left(-\frac{dx}{dy}\right)_{(x_2,y_2)}.$$

ILLUSTRATION 5.12

Find the equation of all possible normals to the parabola $x^2=4y$ drawn from point (1, 2).

Sol. We have curve

$$4y=x^2 \qquad (1)$$



Let the normal through $P(1, 2)$ to the curve $4y=x^2$ meet the curve at $Q(x_1, y_1)$.

Point Q lies on the curve.

Differentiating (1), w.r.t. x, we get

$$2x=4\frac{dy}{dx}$$

or $\quad \dfrac{dy}{dx}=\dfrac{x}{2}$

$\therefore \quad 4y_1=x_1^2$ $\qquad$ (1)

Also, slope of PQ,

$$\frac{y_1-2}{x_1-1}=-\frac{2}{x_1} \qquad (2)$$

Solving (1) and (2), we get

$$\frac{\frac{x_1^2}{4}-2}{x_1-1}=-\frac{2}{x_1}$$

$\therefore\quad x_1^3-8x_1=-8x_1+8$

$\Rightarrow\quad x_1^3=8$

$\therefore\quad x_1=2$ and $y_1=1$

So, the coordinates of point Q are (2, 1).

$\therefore\quad$ Slope of $PQ=\dfrac{2-1}{1-2}=-1$

Hence, equation of normal is

$$y-1=-1(x-2)$$

or $\quad x+y=3$

ILLUSTRATION 5.13

Find the points on the curve where tangents to the curve $y^2-2x^3-4y+8=0$ pass through (1, 2).

Sol. We have curve

$$y^2-2x^3-4y+8=0 \qquad (1)$$

Differentiating curve w.r.t. x, we get

$$2y\frac{dy}{dx}-6x^2-4\frac{dy}{dx}=0$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{3x^2}{y-2}$$

Let the tangent through point $P(1, 2)$ meet the curve at point $Q(x_1, y_1)$.

$\therefore\quad y_1^2-2x_1^3-4y_1+8=0$

or $\quad (y_1-2)^2=2x_1^3-4 \qquad (2)$

Slope of tangent PQ,

$$\frac{y_1-2}{x_1-1}=\frac{3x_1^2}{y_1-2}$$

$\therefore\quad (y_1-2)^2=3x_1^2(x_1-1) \qquad (3)$

From (2) and (3), we get

$$2x_1^3-4=3x_1^2(x_1-1)$$

or $\quad x_1^3-3x_1^2+4=0$

$\Rightarrow\quad (x_1-2)(x_1^2-x_1-2)=0$

$\Rightarrow\quad (x_1-2)^2(x_1+1)=0$

When $x_1=2$, $(y_1-2)^2=12$

$\therefore\quad y_1=2\pm2\sqrt{3}$

When $x_1=-1$, $(y_1-2)^2=-6$ (not possible)

Therefore, required points of contact are $(2, 2\pm2\sqrt{3})$.

ILLUSTRATION 5.14

From the point (1, 1) tangents are drawn to the curve represented parametrically as $x=2t-t^2$ and $y=t+t^2$. Find the possible points of contact.

Sol. $y=t+t^2$ and $x=2t-t^2$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1+2t}{2-2t}$$

So, the equation of tangent at any point P(t) on the curve is

$$(y-t-t^2)=\frac{1+2t}{2-2t}(x-2t+t^2)$$

Tangent passes through (1, 1). So,

$$(1-t-t^2)=\frac{1+2t}{2-2t}(1-2t+t^2)$$

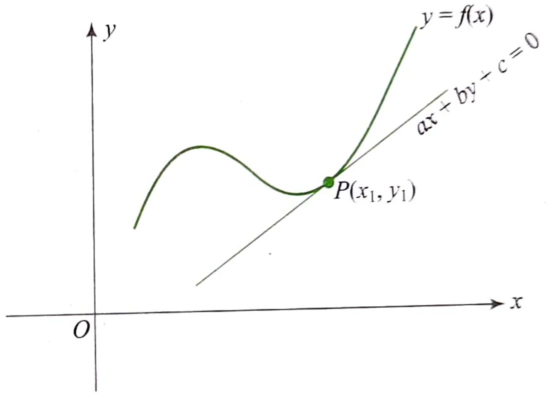
$\Rightarrow\quad 2(1-t-t^2)=(1+2t)(1-t)$ or $t=1$

$\Rightarrow\quad 2-2t-2t^2=-2t^2+t+1$ or $t=1$

$\Rightarrow\quad t=1/3$ or $t=1$

Therefore, points of contact on the curve are $P(1, 2)$ for t = 1 and $Q\left(\frac{5}{9}, \frac{4}{9}\right)$ for $t=\frac{1}{3}$.

Condition for which Given Line is Tangent or Normal to Given Curve



Let the given line be $ax+by+c=0$ and the given curve be $y=f(x)$.

If the line touches the curve at point $Q(x_1, y_1)$, then

$$y_1=f(x_1) \qquad (1)$$

and $\quad ax_1+by_1+c=0 \qquad (2)$

Also,

Slope of line = Slope of tangent to the curve at Q

$$\therefore\quad -\frac{a}{b}=\left(\frac{dy}{dx}\right)_{(x_1,y_1)} \qquad (3)$$

Eliminating x_1 and y_1 from above three equations, we get the condition for line touching the curve.

Similarly, we get the condition if the line is normal to the curve.

ILLUSTRATION 5.15

Show that the straight line $x\cos\alpha+y\sin\alpha=p$ touches the curve $xy=a^2$, if $p^2=4a^2\cos\alpha\sin\alpha$.

Sol. Let the line touches the curve at point $P(x_1, y_1)$ on the curve. Then

$$x_1\cos\alpha+y_1\sin\alpha=p \qquad (1)$$

and $\quad x_1y_1=a^2 \qquad (2)$

Differentiating $xy=a^2$ w.r.t. x, we get $\dfrac{dy}{dx}=-\dfrac{y}{x}$

Now, slope of the line = slope of the tangent to the curve at $P(x_1, y_1)$

or $$-\frac{y_1}{x_1} = -\frac{\cos\alpha}{\sin\alpha} \quad (3)$$

From equations (1) and (3),

$$x_1 \cos\alpha + x_1 \cos\alpha = p$$

or $2\cos\alpha\, x_1 = p$

and $2\sin\alpha\, y_1 = p$

Putting these values in (3), we get

$$(2\cos\alpha)(2\sin\alpha) = p^2$$

or $p^2 = 4a^2 \cos\alpha \sin\alpha$

ILLUSTRATION 5.16

Find the condition that the line $Ax + By = 1$ may be normal to the curve $a^{n-1} y = x^n$.

Sol. We have the line

$$Ax + By = 1 \quad (1)$$

and the curve

$$a^{n-1} y = x^n \quad (2)$$

Let the line be normal to the curve at $P(x_1, y_1)$.

$\therefore \quad Ax_1 + By_1 = 1 \quad (3)$

and $\quad a^{n-1} y_1 = x_1^n \quad (4)$

Differentiating (2), w.r.t. x, we get

$$a^{n-1}\frac{dy}{dx} = nx^{n-1}$$

$\therefore$ Slope of normal $$= -\frac{dx}{dy} = -\frac{a^{n-1}}{nx^{n-1}} = -\frac{a^{n-1}y}{nx^{n-1}y}$$

$$= -\frac{x^n}{nx^{n-1}y} = -\frac{x}{ny}$$

$$\therefore \quad -\frac{x_1}{ny_1} = -\frac{A}{B}$$

$$\therefore \quad Bx_1 = Any_1 \quad (5)$$

Eliminating y_1 from (3) and (5), we get

$$Ax_1 + \frac{B^2 x_1}{An} = 1$$

$$\therefore \quad x_1 = \frac{An}{A^2 n + B^2}$$

So, from (5),

$$y_1 = \frac{B}{A^2 n + B^2}$$

Putting the values of x_1 and y_1 into (4), we get

$$a^{n-1}\frac{B}{A^2 n + B^2} = \left(\frac{An}{A^2 n + B^2}\right)^n$$

$$\therefore \quad a^{n-1} B\left(A^2 n + B^2\right)^{n-1} = n^n A^n$$

This is the required condition.

CONCEPT APPLICATION EXERCISE 5.1

1. Find the equation of the tangent to the curve $(1 + x^2)y = 2 - x$, where it crosses the x-axis.
2. Show that the tangent to the curve $3xy^2 - 2x^2y = 1$ at (1, 1) meets the curve again at the point (– 16/5, – 1/20).
3. Find the equation of tangent and normal to the curve $x = 2at^2/(1 + t^2)$, $y = 2at^3/(1 + t^2)$ at the point for which $t = 1/2$.
4. Find the normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at θ. Prove that it always passes through a fixed point and find that fixed point.
5. Find the equation of the normal to the curve $x^3 + y^3 = 8xy$ at the point where it meets the curve $y^2 = 4x$ other than the origin.
6. If the curve $y = ax^2 - 6x + b$ passes through (0, 2) and has its tangent parallel to the x-axis at $x = \frac{3}{2}$, then find the values of a and b.
7. Find the value of $n \in N$ such that the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ at the point (a, b).
8. If the tangent to the curve $xy + ax + by = 0$ at (1, 1) is inclined at an angle $\tan^{-1} 2$ with x-axis, then find a and b?
9. Does there exists line/lines which is/are tangent to the curve $y = \sin x$ at (x_1, y_1) and normal to the curve at (x_2, y_2)?
10. Find the condition that the line $Ax + By = 1$ may be normal to the curve $a^{n-1} y = x^n$.
11. In the curve $x^a y^b = K^{a+b}$, prove that the portion of the tangent intercepted between the coordinate axes is divided at its point of contact into segments which are in a constant ratio. (All the constants being positive).

ANSWERS

1. $x + 5y = 2$
3. $16x + 13y = 9a$
4. $(a, 0)$
5. $y - x = 0$
6. $a = 2, b = 2$
7. All n
8. $b = -2$ and $a = 1$
9. There is no such line.
10. $a^{n-1}B(nA^2 + B^2)^{n-1} = n^n A^n$

ANGLE BETWEEN CURVES

Let $y = f_1(x)$ and $y = f_2(x)$ be two curves which intersect each other at a point $P(x_1, y_1)$. The angle between these curves at point P is same as the angle between their tangents at point P.

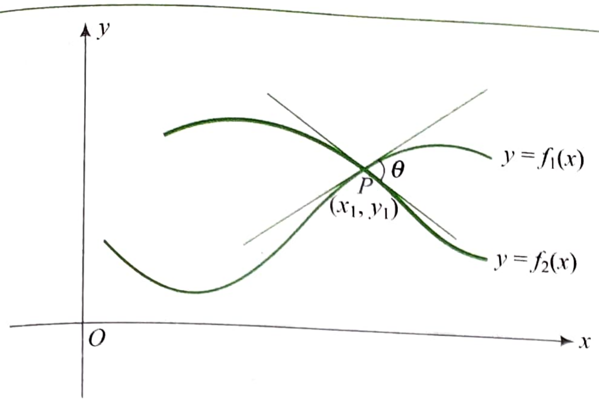
Let $m_1 = \left(\frac{df_1(x)}{dx}\right)_{(x_1, y_1)}$ and $m_2 = \left(\frac{df_2(x)}{dx}\right)_{(x_1, y_1)}$.

Then the acute angle between the curves at point P is given by

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Thus, to find the angle between two curves, first we solve the curves for their point of intersection, say P.

Angle between curves at point P is also same as the angle between normals to the curves at point P.



If curves intersect at more than one point, then we may need to find the angle between curves at each point of intersection. The points of intersection of two curves can be obtained by eliminating one of the variables and solving the resulting equation by some set methods or sometimes by hit and trial method.

ILLUSTRATION 5.17

Find the acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection.

Sol. $y = |x^2 - 1|$ (1)

and $y = |x^2 - 3|$ (2)

They intersect when $|x^2 - 1| = |x^2 - 3|$

or $1 - x^2 = x^2 - 3$ or $x^2 = 2$ or $x = \pm\sqrt{2}$

Therefore, the points of intersection are $(\pm\sqrt{2}, 1)$.

Since the curves are symmetrical about the y-axis, the angle of intersection at $(-\sqrt{2}, 1)$ is same as the angle of intersection at $(\sqrt{2}, 1)$.

At $x = \sqrt{2}$, $y = x^2 - 1$ and $y = 3 - x^2$.

$\therefore$ At $(\sqrt{2}, 1)$, $m_1 = 2x = 2\sqrt{2}$, $m_2 = -2x = -2\sqrt{2}$.

$$\therefore \quad \tan\theta = \left|\frac{4\sqrt{2}}{1-8}\right| = \frac{4\sqrt{2}}{7} \quad \text{or} \quad \theta = \tan^{-1}\frac{4\sqrt{2}}{7}$$

ILLUSTRATION 5.18

Find the angle between the curves $2y^2 = x^3$ and $y^2 = 32x$ at all their points of intersection.

Sol. We have curves

$2y^2 = x^3$ (1)

$y^2 = 32x$ (2)

Solving the curves, we get

$2(32x) = x^3$

$\therefore$ $x(x^2 - 64) = 0$

$\therefore$ $x = 0, 8, -8$

For $x = 0, y = 0$.

For $x = 8, y = \pm 16$ (from (1)).

$x = -8$ is not possible.

So, points of intersection are (0, 0), (8, 16) and (8, −16).

Differentiating (1), w.r.t. x, we get

$$4y\left(\frac{dy}{dx}\right)_1 = 3x^2$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_1 = \frac{3x^2}{4y}$$

Differentiating (2), w.r.t. x, we get

$$2y\left(\frac{dy}{dx}\right)_2 = 32$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_2 = \frac{16}{y}$$

For (8, 16), $\left(\frac{dy}{dx}\right)_1 = 3$ and $\left(\frac{dy}{dx}\right)_2 = 1$.

Therefore, angle between curves at (8, 16) is

$$\theta_1 = \tan^{-1}\left|\frac{3-1}{1+(3)(1)}\right| = \tan^{-1}\frac{1}{2}$$

For (8, −16), $\left(\frac{dy}{dx}\right)_1 = -3$ and $\left(\frac{dy}{dx}\right)_2 = -1$.

Therefore, angle between curves at (8, −16) is

$$\theta_2 = \tan^{-1}\frac{1}{2}$$

$$\text{Now,} \quad \left(\frac{dy}{dx}\right)_1 = \frac{\pm 3x^2}{2\sqrt{2x}\sqrt{x}} = \frac{\pm 3\sqrt{x}}{2\sqrt{2}}$$

For (0, 0), $\left(\frac{dy}{dx}\right)_1 = 0$ and $\left(\frac{dy}{dx}\right)_2 \to \infty$.

So, angle between curves is 90°.

ILLUSTRATION 5.19

Find the angle between curves $f(x) = 2^x \log_e x$ and $g(x) = x^{2x} - 1$.

Sol. We have curves,

$f(x) = 2^x \log_e x$ (1)

$g(x) = x^{2x} - 1$ (2)

Solving curves, we get

$2^x \log_e x = x^{2x} - 1$

There is not any set method to solve this equation.

Using hit and trial, we get point of intersection as (1, 0).

Differentiating (1) w.r.t. x, we get

$$f'(x) = \frac{2^x}{x} + 2^x (\log_e 2)(\log_e x)$$

$\therefore$ $f'(1) = 2$

Differentiating (2) w.r.t. x, we get

$$g'(x) = x^{2x}\left(2x \times \frac{1}{x} + 2\log_e x\right)$$

$\therefore$ $g'(1) = 2$

Thus, slope of tangents to the curves at (1, 0) is the same i.e., 2.

So, two curves touch each other at (1, 0), i.e., angle between curves is 0°.

ORTHOGONAL INTERSECTION OF CURVES

If angle between two curves at their point of intersection is 90°, then we say that curves are intersecting orthogonally.

When two curves $y = f(x)$ and $y = g(x)$ intersect orthogonally, we need not to find the point of intersection of curves (sometimes

it is not feasible to find the point of intersection even using hit and trial method) as from $f'(x) \times g'(x) = -1$, we come to know that curves are orthogonal.

ILLUSTRATION 5.20

Find the angle between the curves $y^2 = 4x$ and $y = e^{-x/2}$.

Sol. Let the curves intersect at point (x_1, y_1).

For $y^2 = 4x$, $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \dfrac{2}{y_1}$

For $y = e^{-x/2}$, $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = -\dfrac{1}{2}e^{-x_1/2} = -\dfrac{y_1}{2}$

Now, $\dfrac{2}{y_1} \times \left(-\dfrac{y_1}{2}\right) = -1$

So, curves intersect orthogonally, i.e., angle between curves is 90°.

ILLUSTRATION 5.21

Find the values of a if the curves $x^2/a^2 + y^2/4 = 1$ and $y^3 = 16x$ cut orthogonally.

Sol. The two curves are

$$x^2/a^2 + y^2/4 = 1 \quad (1)$$

$$y^3 = 16x \quad (2)$$

Differentiating (1), $dy/dx = -4x/(a^2y) = m_1$.

Differentiating (2), $dy/dx = 16/(3y^2) = m_2$.

The two curves cut orthogonally. Therefore,

$m_1 m_2 = -1$

or $[-4x/(a^2y)]\,[16/(3y^2)] = -1$

or $64x = 3a^2y^3$ or $64x = 3a^2(16x)$ [using (2)]

or $a^2 = 4/3$

$\therefore$ $a = \pm 2/\sqrt{3}$

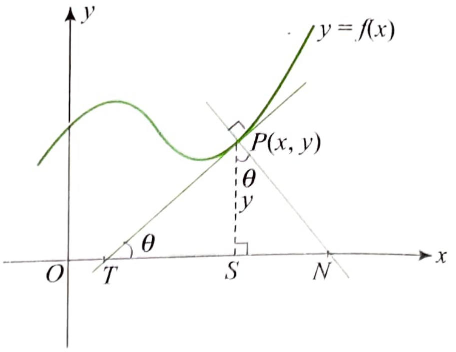
CONCEPT APPLICATION EXERCISE 5.2

1. Find the angle of intersection of $y = a^x$ and $y = b^x$.
2. Find the angle of intersection of the curves $xy = a^2$ and $x^2 - y^2 = 2a^2$.
3. Find the angle at which the curve $y = Ke^{Kx}$ intersects the y-axis.
4. Find the angle between the curves $x^2 - \dfrac{y^2}{3} = a^2$ and $xy^3 = c$.
5. Find the angle at which the two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 + 3 = 0$ intersect each other.
6. If the curves $ay + x^2 = 7$ and $x^3 = y$ cut orthogonally at (1, 1), then find the value a.

ANSWERS

1. $\tan^{-1}\left|\dfrac{\log a/b}{1 + \log a \log b}\right|$
2. The two curves touch each other.
3. $\cot^{-1}K^2$ 4. 90° 5. 90° 6. 6

TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL



Length of Tangent

A segment of tangent (PT) to a curve $y = f(x)$ lying between the point of tangency P (the point at which tangent is drawn to the curve) and the point where the tangent meets x-axis is called tangent (should not be confused with tangent line) or length of tangent.

In ΔPST,

$$PT = |y \operatorname{cosec} \theta| = |y|\sqrt{(1+\cot^2\theta)} = |y|\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

Length of Normal

A segment of normal (PN) to the curve $y = f(x)$ at point P, lying between the point P and the point where the normal meets x-axis is called normal (should not be confused with normal line) or length of normal.

In ΔPSN,

$$PN = |y \operatorname{cosec}(90° - \theta)| = |y \sec \theta|$$

$$= |y|\sqrt{1+\tan^2\theta} = |y|\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Length of Subtangent

The projection of length of tangent on the x-axis is called subtangent or length of subtangent.

In ΔPST,

$$ST = |y \cot \theta| = \left|\frac{y}{\tan\theta}\right| = \left|y\frac{dx}{dy}\right|$$

Length of Subnormal

The projection of length of normal on the x-axis is called subnormal or length of subnormal.

In ΔPSN,

$$SN = |y \cot(90° - \theta)| = |y \tan \theta| = \left|y\frac{dy}{dx}\right|$$

ILLUSTRATION 5.22

Prove that subtangent to the curve $y = e^{x/a}$ is constant.

Sol. We have curve, $y = e^{x/a}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{x/a} \times \frac{1}{a} = \frac{y}{a}$$

$$\therefore \quad y\frac{dx}{dy} = a$$

So, subtangent is constant.

ILLUSTRATION 5.23

Determine the value of p such that the subtangent and subnormal are equal for the curve $y = e^{px} + px$ at the point (0, 1).

Sol. We have $y = e^{px} + px$

$$\therefore \quad \frac{dy}{dx} = pe^{px} + p$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,0)} = 2p$$

Now, Subtangent = Subnormal

$$\therefore \quad y\frac{dx}{dy} = y\frac{dy}{dx}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm 1$$

$$\Rightarrow \quad 2p = \pm 1$$

$$\Rightarrow \quad p = \pm\frac{1}{2}$$

ILLUSTRATION 5.24

Find the length of normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$.

Sol. Here, $\frac{dx}{d\theta} = a(1+\cos\theta)$ and $\frac{dy}{d\theta} = a(\sin\theta)$

or $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

or $$\left(\frac{dy}{dx}\right)_{\theta=\pi/2} = \frac{\sin\pi/2}{1+\cos\pi/2} = 1$$

and the length of normal is

$$\left\{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right\}_{\theta=\pi/2} = a\left(1-\cos\frac{\pi}{2}\right)\sqrt{1+1^2} = \sqrt{2}a$$

ILLUSTRATION 5.25

For the curve $x^{m+n} = a^{m-n}y^{2n}$, prove that the mth power of the subtangent varies as the nth power of the subnormal.

Sol. Given

$$x^{m+n} = a^{m-n}y^{2n} \qquad (1)$$

Taking logarithm on both sides, we get

$$(m+n)\log_e x = (m-n)\log_e a + 2n\log_e y$$

Differentiating both sides w.r.t. x, we get

$$\frac{(m+n)}{x} = 0 + \frac{2n}{y}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(m+n)}{2n}\cdot\frac{y}{x}$$

Now, $$\frac{(\text{Subtangent})^m}{(\text{Subnormal})^n} = \frac{\left(y\frac{dx}{dy}\right)^m}{\left(y\frac{dy}{dx}\right)^n} = \frac{y^{m-n}}{\left(\frac{dy}{dx}\right)^{m+n}}$$

$$= \frac{y^{m-n}}{\left\{\frac{(m+n)}{2n}\cdot\frac{y}{x}\right\}^{m+n}} = \frac{x^{m+n}}{\left(\frac{m+n}{2n}\right)^{m+n}\cdot y^{2n}}$$

$$= \frac{a^{m-n}}{\left(\frac{m+n}{2n}\right)^{m+n}} \qquad \text{(From (1))}$$

= Constant (independent of x and y)

$\therefore$ $(\text{Subtangent})^m \propto (\text{Subnormal})^n$

CONCEPT APPLICATION EXERCISE 5.3

1. Find the length of the tangent for the curve $y = x^3 + 3x^2 + 4x - 1$ at point $x = 0$.
2. For the curve $y = a\ln(x^2 - a^2)$, show that the sum of lengths of tangent and sub-tangent at any point is proportional to product of coordinates of point of tangency.
3. For the curve $y = f(x)$, prove that

$$\frac{(\text{length of normal})^2}{(\text{length of tangent})^2} = \frac{\text{sub-normal}}{\text{sub-tangent}}.$$

4. If the sub-normal at any point on $y = a^{1-n}x^n$ is of constant length, then find the value of n.

ANSWERS

1. $\frac{\sqrt{17}}{4}$ 4. $n = 1/2$

APPLICATIONS OF TANGENT AND NORMAL

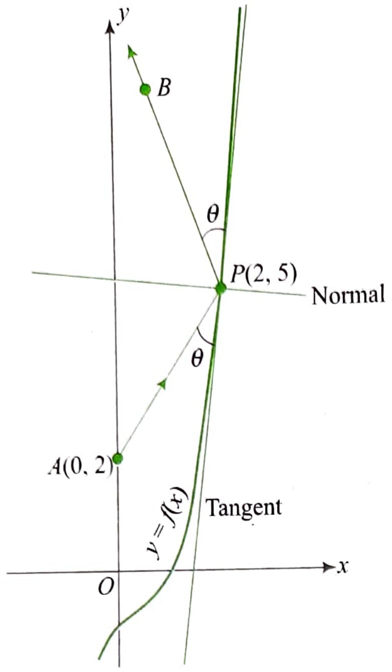
There is a big scope for the applications of tangent and normal in calculus like studying the roots of the equation, finding locus, determining shortest distance between two curves, reflection property of the curves etc. Let us discuss some of the applications with the help of few illustrations.

ILLUSTRATION 5.26

If a ray passing through $A(0, 2)$ strikes the curve $y = x^3 - x^2 + x - 1$ at point $P(2, 5)$, then find the equation of the reflected ray.

Sol. Incident ray is AP.

Reflection of ray AP at point P by the curve is same as reflection of AP by the tangent to the curve at P.



Now, for $y = x^3 - x^2 + x - 1$,

$$\frac{dy}{dx} = 3x^2 - 2x + 1$$

So, $\left(\frac{dy}{dx}\right)_{x=2} = 12 - 4 + 1 = 9$

Now, incident ray and reflected ray (PB) make same angle with tangent line at P.

Let the slope of reflected ray be m.

Slope of $AP = \frac{5-2}{2-0} = \frac{3}{2}$

So, $\tan\theta = \left|\frac{9-\frac{3}{2}}{1+9\times\frac{3}{2}}\right| = \left|\frac{9-m}{1+9\times m}\right|$

$\therefore \quad \frac{15}{29} = \frac{m-9}{1+9m}$ (For reflected ray)

$\Rightarrow \quad 15 + 135m = 29m - 261$

$\Rightarrow \quad 106m = -276$

$\Rightarrow \quad m = -138/53$

Therefore, equation of reflected ray is $138(x - 2) + 53(y - 5) = 0$.

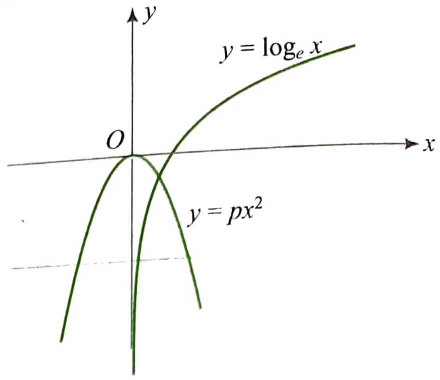
ILLUSTRATION 5.27

Find the possible values of p such that the equation $px^2 = \log_e x$ has exactly one solution.

Sol. For one solution to given equation, two curves $y = px^2$ and $y = \log_e x$ must intersect at only one point.

Case I: $p < 0$

In this case, graph of $y = px^2$ is downward parabola.



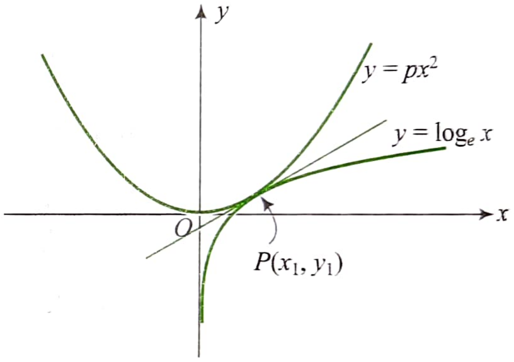
From the figure, it is clear that graphs intersect at exactly one point. So, there is only one solution.

Case II: $p > 0$

In this case, graph of $y = px^2$ is upward parabola.

Here, two curves must touch each other for given equation to have only one solution.

So, tangents to curves at their point of intersection must coincide or tangents must have same slope.



Differentiating the given curves w.r.t. x, we get

$$\frac{dy}{dx} = 2px \text{ and } \frac{dy}{dx} = \frac{1}{x}$$

At point $P(x_1, y_1)$,

$$2px_1 = \frac{1}{x_1}$$

$\therefore \quad 2px_1^2 = 1$ (1)

Also, P lies on the curves.

$\therefore \quad y_1 = px_1^2$ (2)

From (1) and (2),

$$y_1 = \frac{1}{2}$$

Also, $y_1 = \log_e x_1$

$\therefore \quad \frac{1}{2} = \log_e x_1$

$\therefore \quad x_1 = e^{1/2}$

So, from $y = px^2$,

$$\frac{1}{2} = pe$$

or $\quad p = \frac{1}{2e}$

Case III: $p = 0$

In this case, we have curves $y = 0$ and $y = \log_e x$.

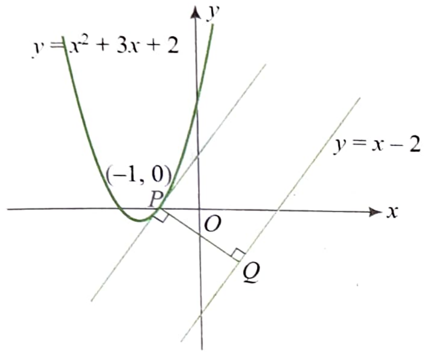
Clearly, x-axis intersects $y = \log_e x$ at only one point i.e., (1, 0).

From the all the cases, possible values of p are $(-\infty, 0] \cup \left\{\frac{1}{2e}\right\}$.

ILLUSTRATION 5.28

Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

Sol. We know that shortest distance between two curves occur along the line which is common normal to the curves.



Let point $P(x_1, y_1)$ be a point on the parabola closest to the line $y = x - 2$.

Then tangent at point P must be parallel to the given line so that PQ is normal to parabola and perpendicular to the given line.

Differentiating curve parabola w.r.t. x, we get

$$\frac{dy}{dx} = 2x + 3$$

At point $P(x_1, y_1)$,

$$2x_1 + 3 = 1 \qquad \text{(Slope of given line)}$$

$\therefore \quad x_1 = -1$

$\therefore \quad y_1 = (-1)^2 + 3(-1) + 2 = 0$

So, point P lies at (–1, 0).

Distance of point P from the given line is the shortest distance.

$$\therefore \quad \text{Shortest distance} = \frac{|-1-0-2|}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

ILLUSTRATION 5.29

Find the minimum value of

$$(x_1 - x_2)^2 + \left(\frac{x_1^2}{20} - \sqrt{(17-x_2)(x_2-13)}\right)^2,$$

where $x_1 \in R^+$ and $x_2 \in (13, 17)$.

Sol. In the given expression,

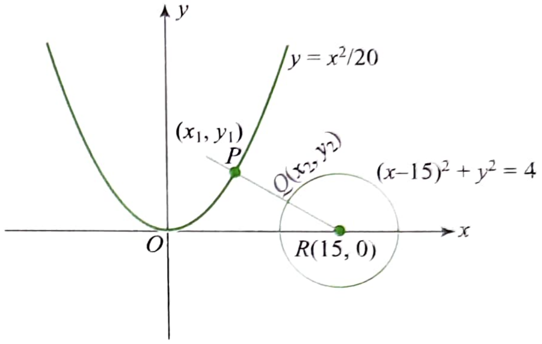
let $\quad y_1 = \frac{x_1^2}{20}$ and $y_2 = \sqrt{(17-x_2)(x_2-13)}$.

So, given expression is $(x_1 - x_2)^2 + (y_1 - y_2)^2$, which represents the square of the distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Point P lies on the parabola $y = \frac{x^2}{20}$ or $x^2 = 20y$ and point Q lies on the curve $y = \sqrt{(17-x)(x-13)}$ or $(x - 15)^2 + y^2 = 4$, which is semi-circle above x-axis.

For minimum value of the square of the distance PQ, P and Q must lie on the common normal to the curves.

Normal to the circle always passes through its centre.



So, we have to find the normal to parabola which goes through point $R(15, 0)$.

For parabola, $\frac{dy}{dx} = \frac{x}{10}$

$$\therefore \quad \text{Slope of PQ} = -\frac{10}{x_1} = \frac{y_1 - 0}{x_1 - 15}$$

$$\therefore \quad -\frac{10}{x_1} = \frac{\frac{x_1^2}{20}}{x_1 - 15} \qquad \text{(as P lies on the curve)}$$

$\Rightarrow \quad x_1^3 + 200x_1 - 3000 = 0$

$\therefore \quad x_1 = 10$

and $\quad y_1 = 5$

$\therefore \quad PQ = PR - QR$

$$= \sqrt{(10-15)^2 + 5^2} - 2$$

$$= (5\sqrt{2} - 2)$$

Therefore, the minimum value of given expression is $(5\sqrt{2} - 2)^2$.

ILLUSTRATION 5.30

Prove that all the points on the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$, where tangents are parallel to the axis of x lie on the curve $y^2 = 4ax$.

Sol. We have the curve,

$$y^2 = 4a\left(x + a\sin\frac{x}{a}\right) \qquad (1)$$

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a\left[1+\cos\frac{x}{a}\right]$$

For points at which the tangents are parallel to x-axis,

$$\frac{dy}{dx} = 0$$

or $$4a\left(1+\cos\frac{x}{a}\right) = 0$$

$\Rightarrow$ $$\cos\frac{x}{a} = -1$$

$\Rightarrow$ $$\sin\frac{x}{a} = 0$$

Putting this value into equation (1), we get

$$y^2 = 4ax$$

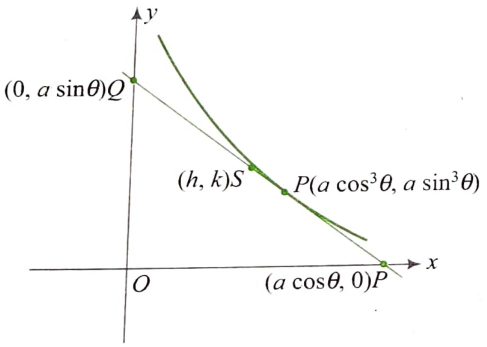
So, required points on the given curve lie on the curve $y^2 = 4ax$.

ILLUSTRATION 5.31

The tangent at any point on the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ meets the axes in P and Q. Prove that the locus of the midpoint of PQ is a circle.

Sol. The given curve is $x = a\cos^3\theta$, $y = a\sin^3\theta$.

$\therefore$ $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta(-\sin\theta)} = -\tan\theta$$



Therefore, equation of tangent at any point P on the curve is

$$y - a\sin^3\theta = -\tan\theta\,(x - a\cos^3\theta)$$

$\Rightarrow$ $$\frac{y}{\sin\theta} - a\sin^2\theta = -\frac{x}{\cos\theta} + a\cos^2\theta$$

$\Rightarrow$ $$\frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1$$

This tangent meets the axes at $P(a\cos\theta, 0)$ and $Q(0, a\sin\theta)$.

Let midpoint of PQ be $S(h, k)$.

So, $2h = a\cos\theta$ and $2k = a\sin\theta$

Squaring these and then adding, we get

$\therefore$ $$(2h)^2 + (2k)^2 = a^2$$

or $$h^2 + k^2 = a^2/4$$

So, the locus of the midpoint of PQ is $x^2 + y^2 = a^2/4$, which is a circle.

CONCEPT APPLICATION EXERCISE 5.4

1. Find the least positive integral value of c for which equation $e^x = cx^2$ has three distinct real roots.
2. Find the point on the curve $3x^2 - 4y^2 = 72$ which is nearest to the line $3x + 2y + 1 = 0$.
3. Find the possible values of a such that the inequality $3 - x^2 > |x - a|$ has at least one negative solution.
4. Tangents are drawn from the origin to curve $y = \sin x$. Prove that points of contact lie on $y^2 = \dfrac{x^2}{1+x^2}$.
5. Find the distance of the point on $y = x^4 + 3x^2 + 2x$ which is nearest to the line $y = 2x - 1$.
6. The graphs $y = 2x^3 - 4x + 2$ and $y = x^3 + 2x - 1$ intersect in exactly 3 distinct points. Then find the slope of the line passing through two of these points.

ANSWERS

1. $c = 2$ **2.** $(-6, 3)$ **3.** $a \in (-13/4, 3)$

5. $\dfrac{1}{\sqrt{5}}$ **6.** 8

DERIVATIVE AS RATE MEASURE

If a quantity y depends on and varies with a quantity x, then the rate of change of y with respect to x is $\dfrac{dy}{dx}$.

For example, the rate of change of pressure p with respect to height h is $\dfrac{dp}{dh}$.

A rate of change with respect to time is simply called as the rate of change.

For example, rate of change of displacement (s) of an object is velocity $v = \dfrac{ds}{dt}$ and rate of change of velocity (v) of an object is acceleration $a = \dfrac{dv}{dt}$ etc.

If variable x increases with time, $\dfrac{dx}{dt} > 0$ and if x decreases with time, $\dfrac{dx}{dt} < 0$.

ILLUSTRATION 5.32

Displacement s of a particle at time t is expressed as $s = \dfrac{1}{2}t^3 - 6t$. Find the acceleration at the time when the velocity vanishes (i.e., velocity tends to zero).

Sol. $s = \dfrac{1}{2}t^3 - 6t$

Thus velocity, $v = \dfrac{ds}{dt} = \left(\dfrac{3t^2}{2} - 6\right)$

and acceleration, $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = 3t$

Velocity vanishes when $\frac{3t^2}{2} - 6 = 0$

or $t^2 = 4$ or $t = 2$

Thus, the acceleration when the velocity vanishes is $a = 3t = 6$ units.

ILLUSTRATION 5.33

On the curve $x^3 = 12y$, find the interval of values of x for which the abscissa changes at a faster rate than the ordinate?

Sol. Given $x^3 = 12y$; differentiating w.r.t. y, we get

$$3x^2 \frac{dx}{dy} = 12$$

$$\therefore \quad \frac{dx}{dy} = \frac{12}{3x^2}$$

Now, if abscissa changes at a faster rate than the ordinate, then we must have $\left|\frac{dx}{dy}\right| > 1$

or $\left|\frac{12}{3x^2}\right| > 1$

or $x^2 < 4, x \neq 0$

or $-2 < x < 2, x \neq 0$

or $x \in (-2, 2) - (0)$

ILLUSTRATION 5.34

The length x of a rectangle is decreasing at the rate of 5 cm/min and the width y is increasing at the rate of 4 cm/min. When $x = 8$ cm and $y = 6$ cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

Sol. Since the length (x) is decreasing at the rate of 5 cm/min and the width (y) is increasing at the rate of 4 cm/min, we have

$$\frac{dx}{dt} = -5 \text{ cm/min and } \frac{dy}{dt} = 4 \text{ cm/min}$$

(a) The perimeter (P) of a rectangle is given by,

$$P = 2(x + y)$$

$$\therefore \quad \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$= 2(-5 + 4)$$

$$= -2 \text{ cm/min}$$

Hence, the perimeter is decreasing at constant rate of 2 cm/min.

(b) The area (A) of a rectangle is given by

$$A = x \times y$$

$$\therefore \quad \frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt} = -5y + 4x$$

When $x = 8$ cm and $y = 6$ cm,

$$\frac{dA}{dt} = (-5 \times 6 + 4 \times 8) \text{ cm}^2/\text{ min}$$

$$= 2 \text{ cm}^2/\text{min}$$

Hence, the area of the rectangle is increasing at the rate of 2 cm^2/min.

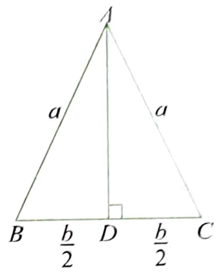
ILLUSTRATION 5.35

The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm/s. How fast is the area decreasing when the two equal sides are equal to the base?

Sol. Let ΔABC be isosceles where BC is the base of fixed length b.

Let the length of the two equal sides of ΔABC be a.

Draw $AD \perp BC$.



$$AD = \sqrt{a^2 - \frac{b^2}{4}}.$$

$$\therefore \quad \text{Area of triangle } (A) = \frac{1}{2} b \sqrt{a^2 - \frac{b^2}{4}}$$

The rate of change of the area with respect to time (t) is given by

$$\frac{dA}{dt} = \frac{1}{2} b \cdot \frac{2a}{2\sqrt{a^2 - \frac{b^2}{4}}} \frac{da}{dt} = \frac{ab}{\sqrt{4a^2 - b^2}} \frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm/s. Therefore,

$$\frac{da}{dt} = -3 \text{ cm/s}$$

$$\therefore \quad \frac{dA}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}}$$

When $a = b$, we have

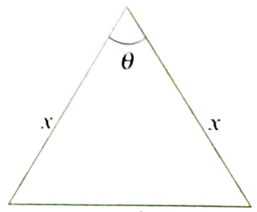
$$\frac{dA}{dt} = \frac{-3b^2}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3}b$$

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}b$ cm^2/s.

ILLUSTRATION 5.36

Let x be the length of one of the equal sides of an isosceles triangle, and let θ be the angle between them. If x is increasing at the rate (1/12) m/h, and θ is increasing at the rate of $\pi/180$ radians/h, then find the rate in m^2/h at which the area of the triangle is increasing when $x = 12$ m and $\theta = \pi/4$.

Sol.



Area of triangle is given by

$$A = \frac{1}{2}x^2 \sin\theta \quad \text{or} \quad 2A = x^2 \sin\theta$$

or $$2\frac{dA}{dt} = x^2 \cos\theta \frac{d\theta}{dt} + \sin\theta\, 2x\frac{dx}{dt}$$

When $x = 12$ and $\theta = \frac{\pi}{4}$,

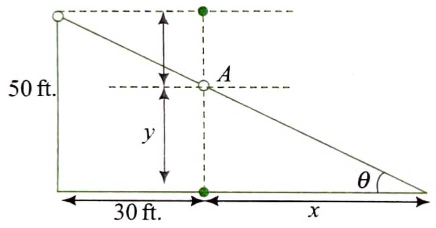
or $$2\frac{dA}{dt} = (144)\left(\frac{1}{\sqrt{2}}\right)\frac{\pi}{180} + \frac{1}{\sqrt{2}} \times 2 \times 12 \times \frac{1}{12} = \frac{12\pi}{15\sqrt{2}} + \frac{2}{\sqrt{2}}$$

or $$\frac{dA}{dt} = \frac{2\pi}{5\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}\pi}{5} + \frac{\sqrt{2}}{2} = \sqrt{2}\left(\frac{\pi}{5} + \frac{1}{2}\right) \text{ m}^2\text{/h.}$$

ILLUSTRATION 5.37

A lamp is 50 ft. above the ground. A ball is dropped from the same height from a point 30 ft. away from the light pole. If ball falls a distance $s = 16t^2$ ft. in t seconds, then how fast is the shadow of the ball moving along the ground 0.5 second later?

Sol.



At time t, ball drops $16t^2$ ft. distance. Therefore,

$$y = 50 - 16t^2 \qquad (1)$$

Point A is the position of the falling ball at some time t. So,

$$\frac{dy}{dt} = -32t$$

From the figure, $\tan\theta = \frac{y}{x} = \frac{50}{30 + x}$ (2)

or $$y = \left(\frac{50}{30+x}\right) \cdot x$$

$\therefore$ $$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{50x}{30+x}\right) = \frac{1500}{(30+x)^2} \cdot \frac{dx}{dt}$$

or $$\frac{dx}{dt} = \frac{(30+x)^2}{1500}(-32t)$$

When $t = \frac{1}{2}$, $y = 46$ [using (1)]

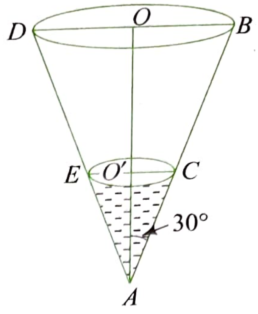
and $x = 345$ [using (2)]

$\therefore$ $$\frac{dx}{dt} = -16\frac{(375)^2}{1500} = -1500 \text{ ft/s}$$

ILLUSTRATION 5.38

If water is poured into an inverted hollow cone whose semi-vertical angle is 30°, such that its depth (measured along the axis) increases at the rate of 1 cm/s. Find the rate at which the volume of water increases when the depth is 24 cm.

Sol.



Let A be the vertex and AO the axis of the cone.

Let $O'A = h$ be the depth of water in the cone.

In $\Delta AO'C$,

$$\tan 30° = \frac{O'C}{h} \quad \text{or} \quad O'C = \frac{h}{\sqrt{3}} = \text{radius}$$

V = Volume of water in the cone

$$= \frac{1}{3}\pi(O'C)^2 \times AO' = \frac{1}{3}\pi\left(\frac{h^2}{3}\right) \times h$$

$$V = \frac{\pi}{9}h^3$$

or $$\frac{dV}{dt} = \frac{\pi}{3}h^2\frac{dh}{dt} \qquad (1)$$

But given that depth of water increases at the rate of 1 cm/s. So,

$$\frac{dh}{dt} = 1 \text{ cm/s} \qquad (2)$$

From (1) and (2), $\frac{dV}{dt} = \frac{\pi h^2}{3}$

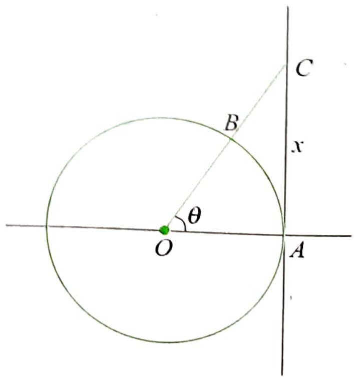
When $h = 24$ cm, the rate of increase of volume is

$$\frac{dV}{dt} = \frac{\pi(24)^2}{3} = 192 \text{ cm}^3\text{/s}$$

ILLUSTRATION 5.39

A horse runs along a circle with a speed of 20 km/hr. A lantern is at the centre of the circle. A fence is along the tangent to the circle at the point at which the horse starts. Then find the speed with which the shadow of the horse moves along the fence at the moment when it covers 1/8 of the circle in km/hr.

Sol. In the figure, lantern is at the centre (O) of the circle of radius r.



At time t, horse is at point B on the circle and its shadow is at point C on the fence at distance x from point A (initial point from where horse starts).

Now, in triangle OAC, $x = r\tan\theta$

$$\therefore \quad \frac{dx}{dt} = \sec^2\theta\left(r\frac{d\theta}{dt}\right)$$

$$= r\omega\sec^2\theta = v\sec^2\theta \quad \text{(where } \omega \text{ is angular velocity)}$$

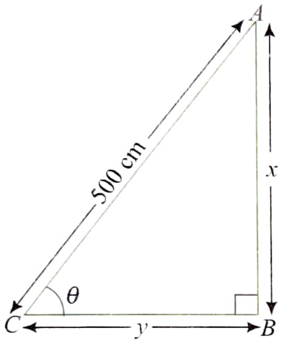
Now when $\theta = \frac{2\pi}{8}$, $\frac{dx}{dt} = 20 \times \sec^2\frac{\pi}{4} = 40$ km/hr

Therefore, shadow is moving at the rate of 40 km/hr.

ILLUSTRATION 5.40

A ladder, 5 meters long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then find the rate at which the angle between the floor and the ladder is decreasing when lower end of the ladder is 2 meters away from the wall.

Sol. Let the angle between the floor and the ladder be θ.



Let at any time t, $AB = x$ cm and $BC = y$ cm.

$$\therefore \quad \sin\theta = \frac{x}{500} \text{ and } \cos\theta = \frac{y}{500}$$

$$\Rightarrow \quad x = 500\sin\theta \text{ and } y = 500\cos\theta$$

Also, it is given that $\frac{dx}{dt} = 10$ cm/s

$$\Rightarrow \quad 500 \times \cos\theta \times \frac{d\theta}{dt} = 10$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{50\cos\theta} = \frac{1}{50 \times \frac{y}{500}} = \frac{10}{y}$$

For $y = 2$ m $= 200$ cm,

$$\frac{d\theta}{dt} = \frac{10}{200} = \frac{1}{20} \text{ rad/s}$$

ILLUSTRATION 5.41

The radius of the base of a cone is increasing at the rate of 3 cm/minute and the altitude is decreasing at the rate of 4 cm/minute. At what rate, lateral surface is changing when the radius is 7 cm and altitude is 24 cm?

Sol. Let r, l and h, respectively, denote the radius, slant height and height of the cone at any time t.

Then, $l^2 = r^2 + h^2$

$$\Rightarrow \quad 2l\frac{dl}{dt} = 2r\frac{dr}{dt} + 2h\frac{dh}{dt} \quad \Rightarrow \quad l\frac{dl}{dt} = r\frac{dr}{dt} + h\frac{dh}{dt}$$

When $r = 7$, $h = 24$

$$\Rightarrow \quad l\frac{dl}{dt} = 7 \times 3 + 24 \times (-4) \qquad \left[\because \frac{dh}{dt} = -4 \text{ and } \frac{dr}{dt} = 3\right]$$

$$\Rightarrow \quad l\frac{dl}{dt} = -75$$

When $r = 7$ and $h = 24$, then we have

$$l^2 = 7^2 + 24^2 \Rightarrow l = 25$$

$$\therefore \quad \frac{dl}{dt} = -3$$

Let S denote the lateral surface area of the cone. Then

$$\frac{dS}{dt} = \pi\frac{d(rl)}{dt} = \pi\left\{\frac{dr}{dt}l + r\frac{dl}{dt}\right\}$$

$$= \pi\{3 \times 25 + 7 \times (-3)\} = 54\pi \text{ cm}^2\text{/min}$$

CONCEPT APPLICATION EXERCISE 5.5

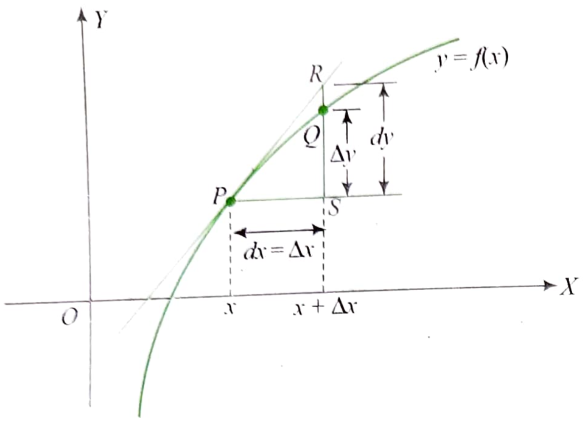
1. The distance covered by a particle moving in a straight line from a fixed point on the line is s, where $s^2 = at^2 + 2bt + c$. Then prove that acceleration is proportional to s^{-3}.
2. Two cyclists start from the junction of two perpendicular roads, their velocities being $3u$ m/min and $4u$ m/min, respectively. Find the rate at which the two cyclists separate.
3. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm³/min. When the thickness of ice is 5 cm, then find the rate at which the thickness of ice decreases.
4. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of the change of the area of the second square with respect to the first square.
5. Two men P and Q start with velocity u at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are being separated.
6. Sand is pouring from a pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always 1/6th of the radius of the base. How fast does the height of the sand cone increase when the height is 4 cm?
7. A swimming pool is to be drained for cleaning. Let L represent the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10 - t)^2$. Then how fast is the water running out at the end of 5 seconds? Also, what is the average rate at which the water flows out during the first 5 seconds?
8. An aeroplane is flying horizontally at a height of $\frac{2}{3}$ km with a velocity of 500 km/h. Find the rate at which it is receding from a fixed point on the ground which it passed over 2 min ago.

ANSWERS

2. $5u$
3. $\frac{1}{18\pi}$ cm/s
4. $(2x^2 - 3x + 1)$
5. $u\sqrt{(2-\sqrt{2})}$
6. $\frac{1}{48\pi}$ cm/s
7. 2000 L/s; 3000 L/s
8. 9 km/h

APPROXIMATION USING DERIVATIVE

Consider the following figure to recall the defintion of derivative of function $y = f(x)$ at some point $P(x, f(x))$ w.r.t. x.



From the defintion of derivative, we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

In the above figure, when point Q moves closer to point P on the curve, then though Δx does not tends to zero, dy is good approximation of Δy.

In that case,

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\therefore \quad f(x+\Delta x) = f(x) + \Delta x \dfrac{dy}{dx}$

ILLUSTRATION 5.42

Find the approximate value of $\sqrt{36.6}$.

Sol. First, we choose the appropriate function to calculate $\sqrt{36.6}$.

Consider function $y = \sqrt{x}$.

Now, we consider the value of x which is close to 36.6 for which square root can be easily calculated.

That is $x = 36$.

So, $\quad x + \Delta x = 36.6$, where $\Delta x = 0.6$

Then $\quad \Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{36.6} - \sqrt{36} = \sqrt{36.6} - 6$

or $\quad \sqrt{36.3} = 6 + \Delta y$

Now, dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)_{x=36} \Delta x = \left(\frac{1}{2\sqrt{x}}\right)_{x=36} \Delta x = \frac{1}{2\sqrt{36}}(0.6) = 0.05$$

Thus, the approximate value of $\sqrt{36.3}$ is $6 + 0.05 = 6.05$.

ILLUSTRATION 5.43

Find the approximate value of sin 3.

Sol. Consider the function $y = \sin x$.

Let $\quad x = 3.14$ and $\Delta x = -0.14$

So, $\quad x + \Delta x = 3.$

Then $\quad \Delta y = \sin(x + \Delta x) - \sin x$

$\quad = \sin 3 - \sin 3.14$

$\therefore \quad \sin 3 = \sin 3.14 + \Delta y$

Now, $\quad dy = \left(\dfrac{dy}{dx}\right)_{x=3.14} \Delta x = (\cos 3.14)\Delta x = -1(-0.14) = 0.14$

$\therefore \quad \sin 3 = \sin 3.14 + \Delta y = 0.14$

ILLUSTRATION 5.44

Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm, respectively.

Sol. Volume of hollow spherical shell,

$$V = \frac{4}{3}\pi[(3.0005)^3 - (3)^3] \quad (1)$$

Now, approximate value of $(3.0005)^3$ can be obtained using differentiation.

Let $(3.0005)^3 = y + \Delta y$, $x = 3$ and $\Delta x = 0.0005$.

Also, let $y = x^3$

$\Rightarrow \quad \dfrac{dy}{dx} = 3x^2$

$\therefore \quad \Delta y = \dfrac{dy}{dx} \times \Delta x$

$\quad = 3x^2 \times 0.0005 = 3 \times 3^2 \times 0.0005$

$\quad = 27 \times 0.0005 = 0.0135$

$\therefore \quad (3.0005)^3 = y + \Delta y$

$\quad = 3^3 + 0.0135 = 27.0135$

$\therefore \quad V = \dfrac{4}{3}\pi[27.0135 - 27.000]$

$\quad = \dfrac{4}{3}\pi[0.0135]$

$\quad = 4\pi \times (0.0045)$

$\quad = 0.0180\,\pi\text{ cm}^3$

ILLUSTRATION 5.45

Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$.

Sol. Let $x = 5$ and $\Delta x = 0.001$. Then, we have

$f(5.001) = f(x + \Delta x)$

$\quad = (x + \Delta x)^3 - 7(x + \Delta x)^2 + 15$

Now, $\Delta y = f(x + \Delta x) - f(x)$

$\therefore \quad f(x + \Delta x) = f(x) + \Delta y$

or $\quad \approx f(x) + f'(x) \cdot \Delta x \quad$ (as $dx = \Delta x$)

or $\quad f(5.001) = [(5)^3 - 7(5)^2 + 15] + [3(5)^2 - 14(5)](0.001)$

$\quad = (125 - 175 + 15) + (75 - 70)(0.001)$

$\quad = -35 + 0.005$

$\quad = -34.995$

ILLUSTRATION 5.46

Find the approximate change in the volume V of a cube of side x meters caused by increasing side by 1%.

Sol. The volume of a cube (V) of side x is given by $V = x^3$.

Therefore,

$$dV = \left(\frac{dV}{dx}\right)\Delta x$$

$$= (3x^2)\Delta x$$

$$= (3x^2)(0.01x) \quad \text{[as 1\% of } x \text{ is } 0.01\,x]$$

$$= 0.03x^3$$

Hence, the approximate change in the volume of the cube is $0.03x^3$ m^3.

CONCEPT APPLICATION EXERCISE 5.6

1. Find the approximate value of $(26)^{1/3}$.
2. Find the approximate value of $(1.999)^6$.
3. If $1° = \alpha$ radians, then find the approximate value of $\cos(60°\ 1')$.
4. Find the approximate value of $f(3.02)$, where $f(x) = 3x^2 + 5x + 3$.
5. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

ANSWERS

1. 2.963 **2.** 63.808 **3.** $\frac{1}{2} - \frac{\alpha\sqrt{3}}{120}$ **4.** 45.46

5. $9.72\,\pi$ cm^3

ROLLE'S THEOREM

Rolle's theorem states that any real-valued differentiable function that attains equal values at two distinct points must have at least one stationary point (point where tangent is parallel to x-axis) somewhere between these two points.

This property was known in the 12th century in ancient India. The outstanding Indian astronomer and mathematician Bhaskara II (1114–1185) mentioned it in his writings.

In a strict form, this theorem was proved in 1691 by the French mathematician Michel Rolle (1652–1719).

The theorem is as stated below:

If a function $f(x)$ is

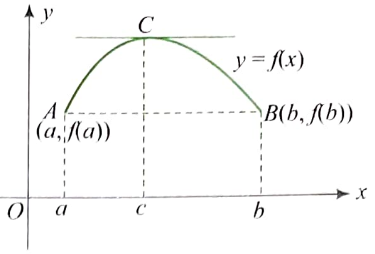
(i) continuous in the closed interval $[a, b]$, i.e., continuous at each point in the interval $[a, b]$,

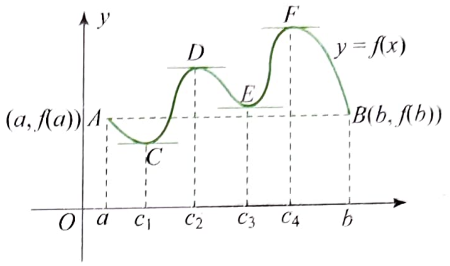
(ii) differentiable in an open interval (a, b), i.e., differentiable at each point in the open interval (a, b), and

(iii) such that $f(a) = f(b)$. Then there exists at least one c, in the interval (a, b) such that $f'(c) = 0$.

GRAPHICAL INTERPRETATION OF ROLLE'S THEOREM

We have following possible graphs of function $y = f(x)$ satisfying above conditions.





Clearly, for $f(a) = f(b)$, graph must have turned at least once in the interval (a, b) creating at least one stationary point.

In the first graph, there is only one such point C. In second graph, we have more than one such points, viz., C, D, E and F.

Note:

- Converse of Rolle's theorem is not true, i.e., if a function $f(x)$ is such that $f'(c) = 0$ for at least one c in the open interval (a, b) then it is not necessary that
  (i) $f(x)$ is continuous in $[a, b]$
  (ii) $f(x)$ is differentiable in (a, b)
  (iii) $f(a) = f(b)$
  For example, if we consider the function $f(x) = x^3 - x^2 - x + 1$ in the interval $[-1, 2]$, then $f'(x) = 3x^2 - 2x - 1$. So, $f'(1) = 3 - 2 - 1 = 0$, but $f(-1) \neq f(2)$.
- If function $y = f(x)$ is continuous as well as differentiable and if $f(x) = 0$ has two roots α and β, then from the argument of Rolle's theorem, we can say that there exists at least one root of the equation $f'(x) = 0$ in the interval (α, β). Similarly, for twice differentiable function $y = f(x)$, between any two roots of the equation $f'(x) = 0$, there exists at least one root of the equation $f''(x) = 0$ and so on.

ILLUSTRATION 5.47

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals:

(i) $f(x) = |x|$ in $[-1, 1]$

(ii) $f(x) = 3 + (x - 2)^{2/3}$ in $[1, 3]$

(iii) $f(x) = \tan x$ in $[0, \pi]$

(iv) $f(x) = \log\left\{\frac{x^2 + ab}{x(a+b)}\right\}$ in $[a, b]$, where $0 < a < b$

Sol.

(i) $f(x) = |x|$ is continuous but non-differentiable in $[-1, 1]$. Hence, Rolle's theorem is not applicable.

(ii) $f(x) = 3 + (x - 2)^{2/3}$ or $f'(x) = \frac{2}{3(x-2)^{1/3}}$.

Thus, $f(x)$ is continuous but derivative does not exist at $x = 2$. Hence, Rolle's theorem is not applicable.

(iii) $f(x) = \tan x$ in $[0, \pi]$ is discontinuous at $x = \pi/2$.

Hence, Rolle's theorem is not applicable.

(iv) $f(x) = \log\left\{\dfrac{x^2+ab}{x(a+b)}\right\}$ in $[a, b]$, where $0 < a < b$.

For $0 < a < b$, $f(x)$ is continuous and differentiable.

$$f(a) = \log\left\{\frac{a^2+ab}{a(a+b)}\right\} = \log\left\{\frac{a(a+b)}{a(a+b)}\right\} = \log 1 = 0$$

and $$f(b) = \log\left\{\frac{b^2+ab}{b(a+b)}\right\} = \log\left\{\frac{b(a+b)}{b(a+b)}\right\} = \log 1 = 0$$

Hence, $f(a) = f(b)$, and Rolle's theorem is applicable.

ILLUSTRATION 5.48

If the function $f(x) = x^3 - 6x^2 + ax + b$ defined on $[1, 3]$ satisfies Rolle's theorem for $c = \dfrac{2\sqrt{3}+1}{\sqrt{3}}$, then find the values of a and b.

Sol. Since $f(x)$ satisfies conditions of Rolle's theorem on $[1, 3]$, we have

$f(1) = f(3)$

$\therefore \quad 1 - 6 + a + b = 27 - 54 + 3a + b$

i.e., $\quad 2a = 22$ or $a = 11$

Since $f(1) = f(3)$ is independent of b, we have

$a = 11$ and $b \in R$

ILLUSTRATION 5.49

Show that between any two roots of $e^{-x} - \cos x = 0$, there exists at least one root of $\sin x - e^{-x} = 0$.

Sol. Let $f(x) = e^{-x} - \cos x$.

Also, let α and β be two of many roots of the equation $e^{-x} - \cos x = 0$.

Thus, $f(\alpha) = 0$ and $f(\beta) = 0$.

Also, $f(x)$ is continuous and differentiable.

So, according to the Rolle's Theorem, there exists at least one $c \in (\alpha, \beta)$ such that $f'(c) = 0$.

This implies that $e^{-c} + \sin c$ vanishes or c is root of the equation $e^{-x} - \sin x = 0$.

Therefore, between any two roots of the equation $e^{-x} - \cos x = 0$, there exists at least one root of $\sin x - e^{-x} = 0$.

APPLICATION OF ROLLE'S THEOREM AND SELECTION OF FUNCTION

One of the important applications of Rolle's theorem is in the study of the roots of the equation and particularly, in locating the roots. If we have to verify that equation $f(x) = 0$ ($f(x)$ is continuous and differentiable) has root or roots in the interval $[a, b]$, then we consider $f(x)$ as derivative of some function say $g(x)$, i.e., $f(x) = g'(x)$. Then by elementary integration, we get $g(x)$. If $g(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$, then we can say that there is at least one $c \in (a, b)$ for which $g'(c) = f(c) = 0$, i.e., $f(x) = 0$ has at least one root in the interval (a, b).

ILLUSTRATION 5.50

How many roots of the equation $(x-1)(x-2)(x-3) + (x-1)(x-2)(x-4) + (x-2)(x-3)(x-4) + (x-1)(x-3)(x-4) = 0$ are positive?

Sol. Let the given equation be $f'(x) = 0$, where $f(x) = (x-1)(x-2)(x-3)(x-4)$.

Since $f(x)$ is continuous and differentiable and $f(1) = f(2) = f(3) = f(4) = 0$, according to the Rolle's theorem, there exists at least one solution of the equation $f'(x) = 0$ in each of the intervals $(1, 2)$, $(2, 3)$ and $(3, 4)$.

Now, $f'(x)$ is a cubic function, so equation $f'(x) = 0$ has exactly one root in each of the intervals $(1, 2)$, $(2, 3)$ and $(3, 4)$.

Thus, $f'(x) = 0$ has three positive roots.

ILLUSTRATION 5.51

If $2a + 3b + 6c = 0$, then prove that at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval $(0, 1)$.

Sol. We have $ax^2 + bx + c = 0$.

Let $\quad f'(x) = ax^2 + bx + c$.

$\therefore \quad f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx + d$.

Now, $\quad f(0) = d$

and $\quad f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c + d = \dfrac{2a+3b+6c}{6} + d = d$

$(\because 2a + 3b + 6c = 0)$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. So, there exists at least one $\alpha \in (0, 1)$, for which $f'(\alpha) = 0$. So, equation $ax^2 + bx + c = 0$ has at least one root in the interval $(0, 1)$.

ILLUSTRATION 5.52

Let $f(x)$ be differentiable function and $g(x)$ be twice differentiable function. Zeros of $f(x)$ and $g'(x)$ be a and b $(a < b)$, respectively. Show that there exists at least one root of equation $f'(x)\, g'(x) + f(x)\, g''(x) = 0$ in (a, b).

Sol. We have $f'(x)\, g'(x) + f(x)\, g''(x) = 0$.

Clearly, L.H.S. is derivative of $(f(x)\, g'(x))$.

Let $\quad h(x) = f(x)\, g'(x)$.

Now, $\quad h(a) = 0 = h(b) \quad$ [As $f(a) = 0$ and $g'(b) = 0$]

So, by Rolle's theorem in the interval $[a, b]$, we have $h'(c) = 0$, for at least one $c \in (a, b)$.

or $\quad f'(c)\, g'(c) + f(c)\, g''(c) = 0$.

ILLUSTRATION 5.53

If $f(x)$ is a thrice differentiable function such that $f'(x) = x^2 - x + 1$ for $x = -1, 0, 1$ then prove that

(i) $f''(x) - 2x + 1$ is not one–one in $(-1, 1)$

(ii) $f'''(x) - 2 = 1$ has at least one root in $(-1, 1)$

Sol. Given, $f'(x) - (x^2 - x + 1) = 0$ for $x = -1, 0, 1$

Now, consider $g(x) = f'(x) - x^2 + x - 1$

$g(-1) = g(0) = g(1) = 0$

Also, $g(x)$ is continuous and differentiable in $[-1, 1]$.

Applying Rolle's theorem for $g(x)$, we get $c_1 \in (-1, 0)$ and $c_2 \in (0, 1)$ such that $g'(c_1) = 0$ and $g'(c_2) = 0$.

or $f''(x) - 2x + 1 = 0$ has one root c_1 in $(-1, 0)$ and one root c_2 in $(0, 1)$.

So, there exists at least one $c_3 \in (c_1, c_2) \subset (-1, 1)$ such that $f'''(c_3) - 2 = 0$.

Clearly, $f''(x) - 2x + 1$ is many–one as $g'(c_1) = 0$ and $g'(c_2) = 0$.

ILLUSTRATION 5.54

Let $P(x)$ be a polynomial with real coefficients. Let $a, b \in R$, $a < b$, be two consecutive roots of $P(x)$. Show that there exists c such that $a \le c \le b$ and $P'(c) + 100\, P(c) = 0$.

Sol. We have $P'(c) + 100\, P(c) = 0$.

Now, multiply both the sides by e^{100c}.

$\therefore \quad e^{100c} P'(c) + 100\, e^{100c} P(c) = 0$

$\therefore \quad f'(c) = 0$, where $f(x) = e^{100x} P(x)$

Now, $f(a) = f(b) = 0$ [As $P(a) = P(b) = 0$]

Also, $P(x)$ is a polynomial function.

$P(x)$ and hence, $f(x)$ is continuous and differentiable.

So, according to the Rolle's theorem, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

$f'(x) = e^{100x}(P'(x) + 100P(x))$

$\therefore \quad e^{100c}\{P'(c) + 100\, P(c)\} = 0$

$\therefore \quad P'(c) + 100\, P(c) = 0$ (As $e^{100c} \ne 0$)

CONCEPT APPLICATION EXERCISE 5.7

1. Let $0 < a < b < \frac{\pi}{2}$. If $f(x) = \begin{vmatrix} \tan x & \tan a & \tan b \\ \sin x & \sin a & \sin b \\ \cos x & \cos a & \cos b \end{vmatrix}$, then find the minimum possible number of roots of $f'(x) = 0$ in (a, b).

2. Find the condition if the equation $3x^2 + 4ax + b = 0$ has at least one root in $(0, 1)$.

3. Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 2$ such that $f(0) = 2$, $g(0) = 1$ and $f(2) = 8$. Also, let there exist a real umber c in $(0, 2)$ such that $f'(c) = 3g'(c)$. Then find the value of $g(2)$.

4. Prove that if $2a_0^2 < 15a$, all roots of $x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d = 0$ cannot be real. It is given that $a_0, a, b, c, d \in R$.

5. Let $f(x)$ be continuous on $[a, b]$, differentiable in (a, b) and $f(x) \ne 0$ for all $x \in [a, b]$. Then prove that there exists one $c \in (a, b)$ such that $\frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}$.

6. Let f and g be functions, continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) = 0$. Then prove that there exists at least one $c \in (a, b)$ such that $g'(c) f(c) + f'(c) = 0$.

7. If $\phi(x)$ is a differentiable function $\forall\, x \in R$ and $a \in R^+$ such that $\phi(0) = \phi(2a)$, $\phi(a) = \phi(3a)$ and $\phi(0) \ne \phi(a)$, then show that there is at least one root of equation $\phi'(x + a) = \phi'(x)$ in $(0, 2a)$.

8. Suppose that f is continuous on $[a, b]$, differentiable on (a, b) and satisfies $f^2(a) - f^2(b) = a^2 - b^2$. Then prove that equation $f(x) f'(x) = x$ has at least one root in (a, b).

ANSWERS

1. one root **2.** $1 + 2a + b = 0$ **3.** 3

MEAN VALUE THEOREMS

The mean value theorem states that for a given section of a smooth curve, there is a point on that section at which the derivative (or slope) of the curve is equal to the average derivative of the section.

This theorem can be understood concretely by applying it to motion. If a car travels 100 kilometres in 1 hour, so that its average speed during that time is 100 km/hr, then at some point of time its instantaneous speed must have been exactly 100 km/hr.

LAGRANGE'S MEAN VALUE THEOREM (LMVT)

If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$, i.e., continuous at each point in the interval $[a, b]$, and

(ii) differentiable in the open interval (a, b), i.e., differentiable at each point in the interval (a, b), then there exists at least one c in the interval (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Proof: Using Rolle's theorem:

Let $F(x) = Ax + f(x)$ (1)

Here, A is a constant.

Now, we choose some A such that $F(a) = F(b)$.

$\therefore \quad Aa + f(a) = Ab + f(b)$

$\therefore \quad A = -\frac{f(b) - f(a)}{b - a}$

So, $F(x) = -\frac{f(b) - f(a)}{b - a}x + f(x)$

Since $f(x)$ is continuous and differentiable and $F(a) = F(b)$, function $F(x)$ satisfies all the conditions of Rolle's theorem.

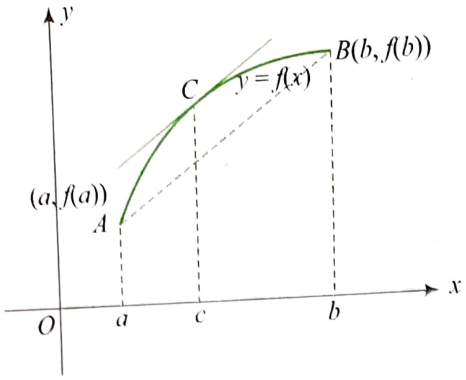
Therefore, there exists at least one $c \in (a, b)$ such that $F'(c) = 0$.

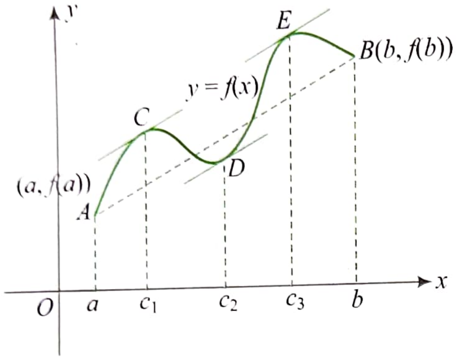
$\therefore \quad -\frac{f(b) - f(a)}{b - a} + f'(c) = 0$

$\therefore \quad f'(c) = \frac{f(b) - f(a)}{b - a}$

Graphical Interpretation of LMVT

Consider the following possible graphs of function $y = f(x)$ which is continuous and differentiable in $[a, b]$.





In both the graphs, slope of AB is $\dfrac{f(b)-f(a)}{b-a}$.

In the first figure, there is exactly one point C $(c, f(c))$ on the graph where tangent line to the graph is parallel to AB, i.e., $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

In the second figure, there are three points D, E and F on the graph where tangent lines to the graph are parallel to AB,

i.e., $f'(c_1)=f'(c_2)=f'(c_3)=\dfrac{f(b)-f(a)}{b-a}$.

Graphically, line AB can be translated in such a way that it touches the graph at least once.

Another Form of LMVT

If a function $f(x)$ is

(i) continuous in the closed interval $[a, a+h]$ and

(ii) differentiable in the open interval $(a, a+h)$, then there exists at least one value θ, $0<\theta<1$ such that $f(a+h)=f(a)+hf'(a+\theta h)$.

Proof:

Putting $b=a+h$ in above theorem, there exists at least one c in the interval $(a, a+h)$ such that

$$f'(c)=\frac{f(a+h)-f(a)}{a+h-a}=\frac{f(a+h)-f(a)}{h} \qquad (1)$$

Let $c=a+\theta h$, where

$a<a+\theta h<a+h$

or $0<\theta h<h$

or $0<\theta<1$

Then from (1),

$$f'(a+\theta h)=\frac{f(a+h)-f(a)}{h}$$

or $f(a+h)=f(a)+hf'(a+\theta h)$, where $0<\theta<1$

ILLUSTRATION 5.55

The function $f(x)=x^3-6ax^2+5x$ satisfies the conditions of Lagrange's mean value theorem over the interval [1, 2] and the value of c (that of LMVT) is $\dfrac{7}{4}$, then find the value of a.

Sol. Given, $f(x)=x^3-6ax^2+5x$

Now, $f(2)=8-24a+10=18-24a$

and $f(1)=1-6a+5=6-6a$

Also, $f'(x)=3x^2-12ax+5$

From LMVT, we have

$$f'\left(\frac{7}{4}\right)=\frac{f(2)-f(1)}{2-1}$$

$$\therefore \quad 3\times\frac{49}{16}-12a\times\frac{7}{4}+5=\frac{(18-24a)-(6-6a)}{1}$$

$$\Rightarrow \quad 147+80-336a=192-288a$$

$$\Rightarrow \quad a=\frac{35}{48}$$

ILLUSTRATION 5.56

If $f:[-5, 5]\to R$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5)\neq f(5)$.

Sol. It is given that $f:[-5, 5]\to R$ is a differentiable function. Since every differentiable function is a continuous function. f is continuous on $[-5, 5]$.

Therefore, by the mean value theorem, there exists $c\in(-5, 5)$ such that

$$f'(c)=\frac{f(5)-f(-5)}{5-(-5)}$$

or $10f'(c)=f(5)-f(-5)$

It is also given that $f'(x)$ does not vanish anywhere. Therefore.

$f'(c)\neq 0$

or $10f'(c)\neq 0$

or $f(5)-f(-5)\neq 0$

or $f(5)\neq f(-5)$

ILLUSTRATION 5.57

Let f be differentiable for all x. If $f(1)=-2$ and $f'(x)\geq 2$ for all $x\in(1, 6)$, then find the range of values of $f(6)$.

Sol. Since $f(x)$ is differentiable function, by LMVT there exists at least one $c\in(1, 6)$ such that

$$f'(c)=\frac{f(6)-f(1)}{6-1}.$$

It is given that $f'(x)\geq 2$ for all $x\in(1, 6)$.

$\therefore \quad f'(c)\geq 2$

$$\Rightarrow \quad \frac{f(6)+2}{5}\geq 2$$

$\Rightarrow \quad f(6)\geq 8$

ILLUSTRATION 5.58

Let f: $[2, 7] \to [0, \infty)$ be continuous in $[2, 7]$ and differentiable in $(2, 7)$. Then prove that there exists at least one $c \in (2, 7)$ such that

$$(f(7)-f(2))\frac{(f(7))^2+(f(2))^2+f(2)f(7)}{3}=5f^2(c)f'(c).$$

Sol. After rearranging the statement to be proved, we have

$$\frac{(f(7))^3-(f(2))^3}{7-2}=3f^2(c)f'(c)$$

$$=[(f^3(x))']_{x=c} \text{ for at least one } c \in (2, 7)$$

Clearly, this suggests that we have to consider the function $g(x) = (f(x))^3$.

Since $g(x)$ is also continuous in $[2, 7]$ and differentiable in $(2, 7)$, there exists at least one $c \in (2, 7)$ for which

$$g'(c)=\frac{g(7)-g(2)}{7-2}$$

or $$\frac{(f(7))^3-(f(2))^3}{7-2}=3f^2(c)f'(c)$$

ILLUSTRATION 5.59

Let $f(x)$ and $g(x)$ be differentiable functions in (a, b), continuous at a and b, and $g(x) \neq 0$ in $[a, b]$. Then prove that $\frac{g(a)f(b)-f(a)g(b)}{g(c)f'(c)-f(c)g'(c)}=\frac{(b-a)g(a)g(b)}{(g(c))^2}$ for at least one $c \in (a, b)$.

Sol. We have to prove

$$\frac{g(a)f(b)-f(a)g(b)}{g(c)f'(c)-f(c)g'(c)}=\frac{(b-a)g(a)g(b)}{(g(c))^2}$$

After rearranging, we have

$$\frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2}=\frac{\frac{f(b)}{g(b)}-\frac{f(a)}{g(a)}}{(b-a)}$$

Let $$h(x)=\frac{f(x)}{g(x)}$$

As $f(x)$ and $g(x)$ are differentiable functions in (a, b), $h(x)$ will also be differentiable in (a, b).

So, according to Lagrange's mean value theorem, there exists at least one $c \in (a, b)$ such that $h'(c)=\frac{h(b)-h(a)}{b-a}$, which proves the required result.

Inequalities Using LMVT

ILLUSTRATION 5.60

Using mean value theorem prove that $|\cos a - \cos b| \le |a - b|$.

Sol. Consider $f(x) = \cos x$ in $[a, b]$ which is continuous and differentiable.

Hence, according to LMVT, there exists at least one $c \in (a, b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad -\sin c=\frac{\cos b-\cos a}{b-a}$$

$$\Rightarrow \quad \left|\frac{\cos b-\cos a}{b-a}\right|=|-\sin c| \le 1$$

$$\Rightarrow \quad |\cos a-\cos b| \le |a-b|$$

ILLUSTRATION 5.61

Using mean value theorem, show that $\frac{\beta-\alpha}{1+\beta^2}<\tan^{-1}\beta-\tan^{-1}\alpha<\frac{\beta-\alpha}{1+\alpha^2}$, $\beta>\alpha>0$.

Sol. Consider function $f(x) = \tan^{-1} x$, $x \in [\alpha, \beta]$.

Since $f(x)$ is continuous and differentiable, there exists at least one $c \in (\alpha, \beta)$ such that

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha}=f'(c)$$

Now, $$f'(x)=\frac{1}{1+x^2}$$

$$\therefore \quad f'(c)=\frac{1}{1+c^2}$$

Now, $\alpha < c < \beta$

$$\Rightarrow \quad \alpha^2<c^2<\beta^2$$

$$\Rightarrow \quad 1+\alpha^2<1+c^2<1+\beta^2$$

$$\Rightarrow \quad \frac{1}{1+\alpha^2}>\frac{1}{1+c^2}>\frac{1}{1+\beta^2}$$

$$\Rightarrow \quad \frac{1}{1+\beta^2}<f'(c)<\frac{1}{1+\alpha^2}$$

$$\Rightarrow \quad \frac{1}{1+\beta^2}<\frac{f(\beta)-f(\alpha)}{\beta-\alpha}<\frac{1}{1+\alpha^2}$$

$$\Rightarrow \quad \frac{\beta-\alpha}{1+\beta^2}<f(\beta)-f(\alpha)<\frac{\beta-\alpha}{1+\alpha^2}$$

$$\Rightarrow \quad \frac{\beta-\alpha}{1+\beta^2}<\tan^{-1}\beta-\tan^{-1}\alpha<\frac{\beta-\alpha}{1+\alpha^2}$$

ILLUSTRATION 5.62

Prove that $\frac{1}{28}<(28)^{1/3}-3<\frac{1}{27}$.

Sol. Let $f(x) = x^{1/3}$, $x \in [27, 28]$.

$$\therefore \quad f'(x)=\frac{1}{3x^{2/3}}$$

Now, by LMVT, we have

$$\frac{f(28)-f(27)}{28-27}=f'(c) \text{ for } c \in (27, 28)$$

$\therefore \quad \dfrac{28^{\frac{1}{3}} - 27^{\frac{2}{3}}}{28-27} = 3c^{2/3}$

$\Rightarrow \quad (28)^{1/3} - 3 = \dfrac{1}{3c^{2/3}}$ (1)

Now, $27 < c < 28$

$\Rightarrow \quad 9 < c^{2/3} < (28)^{2/3}$

$\Rightarrow \quad \dfrac{1}{27} > \dfrac{1}{3.c^{\frac{2}{3}}} > \dfrac{1}{3(28)^{\frac{2}{3}}}$

Also, $\dfrac{1}{3(28)^{\frac{2}{3}}} = \dfrac{(28)^{\frac{1}{3}}}{3} \cdot \dfrac{1}{28} > \dfrac{1}{28}$

$\Rightarrow \quad \dfrac{1}{27} > \dfrac{1}{3.c^{\frac{2}{3}}} > \dfrac{1}{28}$

$\Rightarrow \quad \dfrac{1}{28} < (28)^{1/3} - 3 < \dfrac{1}{27}$ (Using (1))

CAUCHY'S MEAN VALUE THEOREM

Cauchy's mean value theorem, also known as the extended mean value theorem, is the more general form of the mean value theorem. It states that if functions $f(x)$ and $g(x)$ are both continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $g'(x)$ is not zero on that open interval, then there exists some c in (a, b), such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Proof:

The proof of Cauchy's mean value theorem is based on the same idea that we used to prove the mean value theorem. First, we define a new function h(x) and then we aim to transform this function so that it satisfies the conditions of Rolle's theorem.

We have to prove that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

or $\quad f'(c) - g'(c)\dfrac{f(b)-f(a)}{g(b)-g(a)} = 0$

Let $\quad h(x) = f(x) - g(x)\dfrac{f(b)-f(a)}{g(b)-g(a)}$

Now, $\quad h(a) = h(b) = \dfrac{f(a)g(b)-g(a)f(b)}{g(b)-g(a)}$

So, according to the Rolle's theorem there exists at least one $c \in (a, b)$ such that

$$h'(c) = 0$$

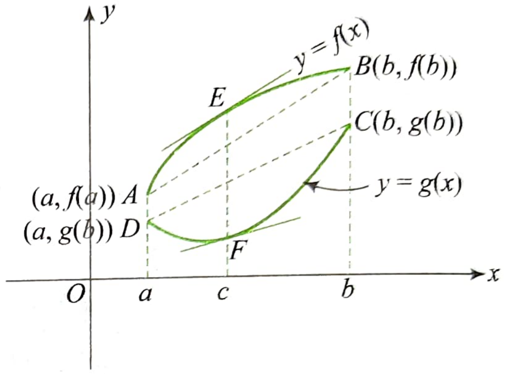
or $\quad f'(c) - g'(c)\dfrac{f(b)-f(a)}{g(b)-g(a)} = 0$

or $\quad \dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Note:

From the statement $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$, one may tend to rearrange it as $\dfrac{f'(c)}{g'(c)} = \dfrac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}}$ and may think that it is the ratio of LMVT, i.e., $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ and $g'(c) = \dfrac{g(b)-g(a)}{b-a}$ for function $y = f(x)$ and $y = g(x)$ at $x = c$.

This is erroneous which is evident from the following figure.



In the above figure, we observe that at point $E(c, f(c))$ on the graph of $y = f(x)$, tangent line to the curve $y = f(x)$ is parallel to AB, i.e., $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

But at point $F(c, g(c))$, tangent line to the curve $y = g(x)$ is not parallel to CD, i.e., $g'(c) \neq \dfrac{g(b)-g(a)}{b-a}$.

ILLUSTRATION 5.63

Let $f(x)$ and $g(x)$ be two differentiable functions in R and $f(2) = 8$, $g(2) = 0$, $f(4) = 10$, and $g(4) = 8$. Then prove that $g'(x) = 4f'(x)$ for at least one $x \in (2, 4)$.

Sol. Consider $h(x) = g(x) - 4f(x)$ in $[2, 4]$.

Now, $\quad h(2) = g(2) - 4f(2) = -32$,

and $h(4) = g(4) = -4f(4) = 8 - 4 \times 10 = -32$

Therefore, $h'(x) = 0$ for at least one $x \in (2, 4)$ using Rolle's theorem.

Alternatively, using Cauchy's mean value theorem, there exists at least one $c \in (2, 4)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(4)-f(2)}{g(4)-g(2)} = \frac{10-8}{8-0} = \frac{1}{4}$$

or $\quad 4f'(c) = g'(c)$

or $\quad 4f'(x) = g'(x)$ (replacing c by x)

ILLUSTRATION 5.64

Suppose α, β, and θ are angles satisfying $0 < \alpha < \theta < \beta < \frac{\pi}{2}$

Then prove that $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = -\cot\theta$.

Sol. Let $f(x) = \sin x$ and $g(x) = \cos x$. Then f and g are continuous and derivable.

Also, $\sin x \neq 0$ for any $x \in \left(0, \frac{\pi}{2}\right)$.

So, by Cauchy's mean value theorem,

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}$$

or $$\frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha} = \frac{\cos\theta}{-\sin\theta}$$

ILLUSTRATION 5.65

Let f be continuous on $[a, b]$, $a > 0$, and differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$\frac{bf(a)-af(b)}{b-a} = f(c)-cf'(c).$$

Sol. $$\frac{bf(a)-af(b)}{b-a} = f(c)-cf'(c)$$

i.e., $$\frac{\frac{f(a)}{a}-\frac{f(b)}{b}}{\frac{1}{a}-\frac{1}{b}} = \frac{\frac{f(c)-cf'(c)}{c^2}}{\frac{1}{c^2}}$$

This suggests we have to consider the functions

$$g(x) = \frac{f(x)}{x} \text{ and } h(x) = \frac{1}{x}$$

Both the functions $g(x)$ and $h(x)$ are continuous and differentiable in (a, b).

Therefore, there exists at least one $c \in (a, b)$ for which

$$\frac{g(a)-g(b)}{h(a)-h(b)} = \frac{g'(c)}{f'(c)}$$

or $$\frac{\frac{f(a)}{a}-\frac{f(b)}{b}}{\frac{1}{a}-\frac{1}{b}} = \frac{\frac{f(c)-cf'(c)}{c^2}}{\frac{1}{c^2}}$$

or $$\frac{bf(a)-af(b)}{b-a} = f(c)-cf'(c)$$

Hence proved.

CONCEPT APPLICATION EXERCISE 5.8

1. Find c of Lagrange's mean value theorem for the function $f(x) = 3x^2 + 5x + 7$ in the interval $[1, 3]$.
2. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then prove that there exists at least one $c \in (a, b)$ such that $$\frac{f'(c)}{3c^2} = \frac{f(b)-f(a)}{b^3-a^3}.$$
3. If $a, b \in R$ and $a < b$, then prove that there exists at least one real number $c \in (a, b)$ such that $\frac{b^2+a^2}{4c^2} = \frac{c}{a+b}$.
4. If $f(x)$ and $g(x)$ are continuous functions in $[a, b]$ and are differentiable in (a, b), then prove that there exists at least one $c \in (a, b)$ for which $$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}, \text{ where } a < c < b.$$
5. Prove that $|\tan^{-1}x - \tan^{-1}y| \le |x-y|\ \forall x, y \in R$.
6. Using Lagrange's mean value theorem, prove that $$\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}, \text{ where } 0 < a < b.$$
7. If $a > b > 0$, with the aid of Lagrange's mean value theorem, prove that
   (a) $nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)$, if $n > 1$.
   (b) $nb^{n-1}(a-b) > a^n - b^n > na^{n-1}(a-b)$, if $0 < n < 1$.
8. Let $f(x)$ and $g(x)$ be two functions which are defined and differentiable for all $x \ge x_0$. If $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$ for all $x > x_0$, then prove that $f(x) > g(x)$ for all $x > x_0$.
9. If $f(x)$ is differentiate in $[a, b]$, then prove that there exists at least one $c \in (a, b)$ such that $(a^2 - b^2) f'(c) = 2c(f(a) - f(b))$.

ANSWERS

1. $c = 2$

Solved Examples

EXAMPLE 5.1

Prove that the equation of the normal to $x^{2/3} + y^{2/3} = a^{2/3}$ is $y \cos\theta - x\sin\theta = a\cos 2\theta$, where θ is the angle which the normal makes with the axis of x.

Sol. We have $x^{2/3} + y^{2/3} = a^{2/3}$. (1)

Differentiating, we get $dy/dx = -y^{1/3}/x^{1/3}$

$\therefore$ Slope of the normal $= -dx/dy = x^{1/3}/y^{1/3} = \tan\theta$

or $$\frac{x^{1/3}}{\sin\theta} = \frac{y^{1/3}}{\cos\theta} = \frac{\sqrt{(x^{1/3})^2 + (y^{1/3})^2}}{\sqrt{\sin^2\theta + \cos^2\theta}}$$

$$= \sqrt{(a^{2/3})}$$

$$= a^{1/3} \qquad \text{[Using (1)]}$$

$\therefore$ $x = a\sin^3\theta,\ y = a\cos^3\theta$

Therefore, equation of the normal whose slope is $\tan\theta$ is

$$y - a\cos^3\theta = \tan\theta\,(x - a\sin^3\theta)$$

or $$y\cos\theta - a\cos^4\theta = x\sin\theta - a\sin^4\theta$$

or $$y\cos\theta - x\sin\theta = a(\cos^4\theta - \sin^4\theta)$$

$$= a(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$

$$= a\cos 2\theta$$

Example 5.2

If the area of the triangle included between the axes and any tangent to the curve $x^n y = a^n$ is constant, then find the value of n.

Sol. Let $P(x_1, y_1)$ be any point on the curve $x^n y = a^n$.

Then, $x_1^n y_1 = a^n$ (1)

Now, $x^n y = a^n$ or $nx^{n-1} y + x^n \frac{dy}{dx} = 0$ (differentiate w.r.t. x)

or $\frac{dy}{dx} = -n\frac{y}{x}$ or $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-ny_1}{x_1}$

or $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -n\frac{a^n}{x_1^{n+1}}$ [Using (1)]

The equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{na^n}{x_1^{n+1}}(x - x_1)$$

This meets the coordinate axes at

$$A\left(\frac{x_1^{n+1} y_1}{na^n} + x_1, 0\right) \text{ and } B\left(0, y_1 + \frac{na^n}{x_1^n}\right)$$

$\therefore$ Area of $\Delta AOB = \frac{1}{2}(OA \times OB)$

$$= \frac{1}{2}\left(\frac{x_1^{n+1} y_1}{na^n} + x_1\right)\left(y_1 + \frac{na^n}{x_1^n}\right)$$

$$= \frac{1}{2}\left(\frac{x_1}{n} + x_1\right)\left(\frac{a^n}{x_1^n} + \frac{na^n}{x_1^n}\right) \quad \text{[Using (1)]}$$

$$= \frac{1}{2}\frac{(n+1)^2}{n} a^n x_1^{1-n}$$

Clearly, this area is constant if $n = 1$.

Example 5.3

Show that the segment of the tangent to the curve $y = \frac{a}{2}\ln\left(\frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}}\right) - \sqrt{a^2 - x^2}$ contained between the y-axis and the point of tangency has a constant length.

Sol. Given $y = \frac{a}{2}\ln\left(\frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}}\right) - \sqrt{a^2 - x^2}$

Let $x = a \sin\phi$. Therefore, (1)

$$y = \frac{a}{2}\ln\left(\frac{1 + \cos\phi}{1 - \cos\phi}\right) - a\cos\phi = -a\ln\tan(\phi/2) - a\cos\phi$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{d\phi}\right)}{\left(\frac{dx}{d\phi}\right)} = \frac{-a\,\text{cosec}\,\phi + a\sin\phi}{a\cos\phi} = -\cot\phi$$

The equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = -\cot\phi(x - x_1)$$

Tangent meets y-axis at $Q(0, y_1 + x_1\cot\phi)$ [From (1)]

$$\therefore \quad PQ = \sqrt{x_1^2 + x_1^2\cot^2\phi} = x_1\,\text{cosec}\,\phi = a = \text{constant}$$

Example 5.4

If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again in (x_2, y_2), then prove that $\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1$.

Sol. Slope of the tangent at $(x_1, y_1) = -\frac{x_1^2}{y_1^2}$

The tangent cuts the curve again at (x_2, y_2). Therefore,

Slope of the tangent $= \frac{y_2 - y_1}{x_2 - x_1}$

or $-\frac{x_1^2}{y_1^2} = \frac{y_2 - y_1}{x_2 - x_1}$

Also, $x_1^3 + y_1^3 = a^3$ and $x_2^3 + y_2^3 = a^3$

$\therefore$ $x_1^3 + y_1^3 = x_2^3 + y_2^3$

or $\frac{y_2^3 - y_1^3}{x_1^3 - x_2^3} = 1$

or $\frac{y_2 - y_1}{x_1 - x_2} = \frac{x_1^2 + x_2^2 + x_1x_2}{y_1^2 + y_2^2 + y_1y_2}$

or $\frac{x_1^2}{y_1^2} = \frac{x_1^2 + x_2^2 + x_1x_2}{y_1^2 + y_2^2 + y_1y_2}$

or $x_1^2y_1^2 + x_1^2y_2^2 + x_1^2y_1y_2 = y_1^2x_1^2 + x_2^2y_1^2 + y_1^2x_1x_2$

or $x_2^2y_1^2 - y_2^2x_1^2 = x_1y_1[x_1y_2 - x_2y_1]$

or $x_2y_1 + y_2x_1 = -x_1y_1$

or $\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1$

Example 5.5

Find the condition for the line $y = mx$ to cut at right angles the conic $ax^2 + 2hxy + by^2 = 1$.

Sol. $ax^2 + 2hxy + by^2 = 1$ (1)

Differentiating w.r.t. x, we get

$$2ax + 2h\left[x\frac{dy}{dx} + y\right] + 2by\frac{dy}{dx} = 0$$

or $\frac{dy}{dx} = -\frac{ax + hy}{hx + by}$

Now, line $y = mx$ and curve intersect at right angle. Therefore,

$$m\left(\frac{ax + hy}{hx + by}\right) = 1$$

Putting $y = mx$, we have

$$m\left[\frac{ax + h \cdot mx}{hx + b \cdot mx}\right] = 1$$

or $m(a + hm) = h + bm$

or $m^2h + (a - b)m - h = 0$

EXAMPLE 5.6

Show the condition that the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ intersect orthogonally is $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$.

Sol. The given curves are

$$ax^2 + by^2 = 1 \quad (1)$$

$$a'x^2 + b'y^2 = 1 \quad (2)$$

Differentiating (1), $\frac{dy}{dx} = -\frac{ax}{by} = m_1$ (say)

Differentiating (2), $\frac{dy}{dx} = -\frac{a'x}{b'y} = m_2$ (say)

If the curves (1) and (2) intersect at $P(x_1, y_1)$, then at this point P,

$$m_1 = -\frac{ax_1}{by_1} \text{ and } m_2 = -\frac{a'x_1}{b'y_1}$$

If the curves (1) and (2) intersect orthogonally at P, then

$$m_1 m_2 = -1 \text{ or } \frac{aa'x_1^2}{bb'y_1^2} = -1 \quad (3)$$

Since point $P(x_1, y_1)$ lies on both (1) and (2),

$$ax_1^2 + by_1^2 = 1 \text{ and } a'x_1^2 + b'y_1^2 = 1$$

Subtracting, we get $(a - a')x_1^2 + (b - b')y_1^2 = 0$

or $\frac{x_1^2}{y_1^2} = -\frac{b-b'}{a-a'}$

Substituting in equation (3), we get

$$\left(\frac{aa'}{bb'}\right)\left(-\frac{b-b'}{a-a'}\right) = -1$$

or $\frac{b-b'}{bb'} = \frac{a-a'}{aa'}$

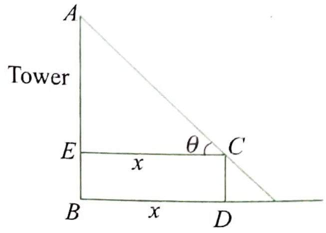
or $\frac{1}{b'} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{a}$

or $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$

EXAMPLE 5.7

A man is moving away from a tower 41.6 m high at a rate of 2 m/s. If the eye level of the man is 1.6 m above the ground, then find the rate at which the angle of elevation of the top of the tower is decreasing when he is at a distance of 30 m from the foot of the tower.

Sol. Let CD be the position of the man at any time t, $BD = x = EC$ and $\angle ACE = \theta$.



Given, $AB = 41.6$ m, $CD = 1.6$ m and $\frac{dx}{dt} = 2m/\sec$.

$$AE = AB - EB = AB - CD = 41.6 - 1.6 = 40 \text{ m}$$

We have to find $\frac{d\theta}{dt}$ when $x = 30$ m.

From ΔAEC,

$$\tan\theta = \frac{AE}{EC} = \frac{40}{x} \quad (1)$$

Differentiating w.r.t. to t, we get

$$\sec^2\theta\frac{d\theta}{dt} = -\frac{40}{x^2}\frac{dx}{dt}$$

$$\Rightarrow \sec^2\theta\frac{d\theta}{dt} = -\frac{40}{x^2}\times 2$$

$$\Rightarrow \frac{d\theta}{dt} = -\frac{80}{x^2}\cos^2\theta = -\frac{80}{x^2}\times\frac{x^2}{x^2+40^2}$$

$$\left[\because \cos\theta = \frac{x}{\sqrt{x^2+40^2}}\right]$$

$$\Rightarrow \frac{d\theta}{dt} = -\frac{80}{x^2+40^2} \quad (2)$$

When $x = 30$ m, $\frac{d\theta}{dt} = -\frac{80}{30^2+40^2} = -\frac{4}{125}$ radian/sec

EXAMPLE 5.8

If f is continuous and differentiable function and $f(0) = 1$, $f(1) = 2$, then prove that there exists at least one $c \in [0, 1]$ for which $f'(c)\,(f(c))^{n-1} > \sqrt{2^{n-1}}$, where $n \in N$.

Sol. Let $g(x) = (f(x))^n$. Given $f(x)$ is continuous and differentiable. Then $g(x)$ is also continuous and differentiable. Then from Lagrange's mean value theorem, there exists at least one $c \in (0, 1)$ for which

$$g'(c) = nf'(c)\,(f(c))^{n-1} = \frac{(f(1))^n - (f(0))^n}{1-0} = 2^n - 1$$

or $$nf'(c)\,(f(c))^{n-1} = \frac{2^n - 1}{2-1} = (1 + 2 + 2^2 + \cdots + 2^{n-1})$$

or $$f'(c)\,(f(c))^{n-1} = \frac{1+2+2^2+\cdots+2^{n-1}}{n} > (1\cdot 2\cdots 2^{n-1})^{\frac{1}{n}}$$

(as A.M. > G.M.)

$$> \sqrt{2^{n-1}}.$$

EXAMPLE 5.9

Let a, b, c be three real numbers such that $a < b < c$. $f(x)$ is continuous in $[a, c]$ and differentiable in (a, c). Also, $f'(x)$ is strictly increasing in (a, c). Prove that $(b-c)f(a) + (c-a)f(b) + (a-b)f(c) < 0$.

Sol. By Lagrange's mean value theorem in $[a, b]$ for f,

$$\frac{f(b)-f(a)}{b-a} = f'(u), \text{ where } a < u < b$$

Also, in $[b, c]$,

$$\frac{f(c)-f(b)}{c-b} = f'(v), \quad \text{where } b < v < c$$

Since $f'(x)$ is strictly increasing,

$$f'(u) < f'(v)$$

or $$\frac{f(b)-f(a)}{b-a} < \frac{f(c)-f(b)}{c-b}$$

or $f(b)(c-b+b-a) - f(a)(c-b) - f(c)(b-a) < 0$

or $(b-c)f(a) + (c-a)f(b) + (a-b)f(c) < 0$

EXAMPLE 5.10

Use the mean value theorem to prove $e^x \geq 1 + x \ \forall\, x \in R$

Sol. Consider the function $f(x) = e^x - 1$ in $[0, x]\ \forall\ x$, where $x > 0$. Therefore, f is continuous and differentiable. Hence, using LMVT, for at least one $c \in (0, x)$ such that

$$f'(c) = \frac{(e^x-1)-0}{x} = \frac{e^x-1}{x} \quad \left[\because f'(c) = \frac{f(x)-f(0)}{x-0}\right]$$

But $f'(c) = e^c$

Hence, $\frac{e^x-1}{x} = e^c > 1$, for $x > 0$

$\therefore \quad e^x - 1 > x$

$\therefore \quad e^x > x + 1$ for $x > 0$ (1)

Again, consider the function

$f(x) = e^x - 1$ in $[x, 0]$ where $x < 0$

Using LMVT, for at least one $c \in (x, 0)$ such that

$$f'(c) = \frac{0-(e^x-1)}{-x} = \frac{e^x-1}{x}$$

But $f'(c) = e^c$. Hence $\frac{e^x-1}{x} = e^c < 1$ for $c < 0$. Therefore,

$$\frac{(e^x-1)}{x} < 1 \text{ for } x < 0$$

or $(e^x - 1) > x$ (As x is –ve) (2)

From (1) and (2),

$e^x > x + 1$ for $x \neq 0$

For $x = 0$, $e^x = x + 1$.

$\therefore \quad e^x \geq x + 1$ for $x \in R$

EXAMPLE 5.11

Show that the square roots of two successive natural numbers greater than N^2 differ by less than $\frac{1}{2N}$.

Sol. Let n be a natural number greater than N^2.

Consider the function $f(x) = \sqrt{x}$ defined on the interval $[n, n+1]$.

Clearly, $f(x)$ satisfies all the conditions of Lagrange's mean value theorem on $[n, n+1]$.

So, there exists $c \in (n, n+1)$ such that

$$f'(c) = \frac{f(n+1)-f(n)}{(n+1)-n}$$

or $$\frac{1}{2\sqrt{c}} = \sqrt{n+1} - \sqrt{n}$$

Now, $n > N^2$ and $c \in (n, n+1)$

$\therefore \quad c > N^2$

or $\sqrt{c} > N$

or $2\sqrt{c} > 2N$

or $\frac{1}{2\sqrt{c}} < \frac{1}{2N}$

or $\sqrt{n+1} - \sqrt{n} < \frac{1}{2N}$

EXAMPLE 5.12

Using Rolle's theorem, prove that there is at least one root in $(45^{1/100}, 46)$ of the equation $P(x) = 51x^{101} - 2323(x)^{100} - 45x + 1035 = 0$.

Sol. Given that

$$P(x) = 51x^{101} - 2323x^{100} - 45x + 1035$$

To show that at least one root of $P(x)$ lies in $(45^{1/100}, 46)$, using Rolle's theorem, we consider anti-derivative of $P(x)$,

i.e., consider $F(x) = \frac{x^{102}}{2} - \frac{2323x^{101}}{101} - \frac{45x^2}{2} + 1035\,x + c$

Then being a polynomial function, $F(x)$ is continuous and differentiable. Now,

$$F(45^{1/100}) = \frac{(45)^{\frac{102}{100}}}{2} - \frac{2323(45)^{\frac{101}{100}}}{101} - \frac{45\,(45)^{\frac{2}{100}}}{2} + 1035(45)^{\frac{1}{100}} + c$$

$$= \frac{45}{2}(45)^{\frac{2}{100}} - 23 \times 45(45)^{\frac{1}{100}} - \frac{45(45)^{\frac{2}{100}}}{2} + 1035(45)^{\frac{1}{100}} + c$$

$$= c$$

and $$F(46) = \frac{(46)^{102}}{2} - \frac{2323(46)^{101}}{101} - \frac{45(46)^2}{2} + 1035\,(46) + c$$

$$= 23(46)^{101} - 23(46)^{101} - 23 \times 45 \times 46 + 1035 \times 46 + c$$

$$\therefore \quad F\left(45^{\frac{1}{100}}\right) = F(46) = c$$

Therefore, Rolle's theorem is applicable.

Hence, there must exist at least one root of $F'(x) = 0$ i.e., of $P(x) = 0$ in the interval $(45^{1/100}, 46)$.

EXAMPLE 5.13

For a twice differentiable function $f(x)$, $g(x)$ is defined as $g(x) = (f'(x))^2 + f''(x)f(x)$ on $[a, e]$. If for $a < b < c < d < e$, $f(a) = 0$, $f(b) = 2$, $f(c) = -1$, $f(d) = 2$, $f(e) = 0$, then find the minimum number of zeros of $g(x)$.

Sol. $g(x) = (f'(x))^2 + f''(x)f(x) = \frac{d}{dx}(f(x)f'(x))$

Let $h(x) = f(x)f'(x)$

Since $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$,

$f(x) = 0$ has at least four roots, namely, a, α, β, e, where $b < \alpha < c$ and $c < \beta < d$, and $f'(x) = 0$ for at least three points k_1, k_2, k_3 where $a < k_1 < \alpha$, $\alpha < k_2 < \beta$, $\beta < k_3 < c$ [since between any two roots of $f(x) = 0$, there lies at least one root of $f'(x) = 0$].

Therefore, there are at least seven roots of $f(x)f'(x) = 0$.

Therefore, there are at least six roots of $\frac{d}{dx}(f(x)f'(x)) = 0$, i.e., of $g(x) = 0$.

EXAMPLE 5.14

If $f''(x)$ exists for all points in $[a, b]$ and $\frac{f(c)-f(a)}{c-a} = \frac{f(b)-f(c)}{b-c}$, where $a < c < b$, then show that there exists a number 'k' such that $f''(k) = 0$.

Sol. Since $f(x)$ satisfies the conditions of LMVT in $[a, b]$, we have

$$k_1 \in (a, c) \text{ such that } \frac{f(c)-f(a)}{c-a} = f'(k_1)$$

and $$k_2 \in (c, b) \text{ such that } \frac{f(b)-f(c)}{b-c} = f'(k_2)$$

As $f'(x)$ satisfies the conditions of Rolle's theorem in (k_1, k_2), we have $k \in (k_1, k_2) \subset (a, b)$ such that $f''(k) = 0$

EXAMPLE 5.15

For all $x \in [0, 1]$, let the second derivative $f''(x)$ of a function $f(x)$ exists and satisfies $|f''(x)| < 1$. If $f(0) = f(1)$, then show that $|f'(x)| < 1$ for all x in $[0, 1]$.

Sol. Since $f''(x)$ exists for all x in $[0, 1]$, $f(x)$ and $f'(x)$ are differentiable and continuous in $[0, 1]$.

Now, $f(x)$ is continuous in $[0, 1]$ and differentiable in $(0, 1)$ and $f(0) = f(1)$.

Therefore, by Rolle's theorem, there is at least one c such that $f'(c) = 0$, where $0 < c < 1$.

Case I: Let $x = c$. Then,

$$f'(x) = f'(c) = 0 \text{ or } |f'(x)| = |0| = 0 < 1$$

Case II: Let $x > c$. By Lagrange's mean value theorem for $f'(x)$ in $[c, x]$,

$$\frac{f'(x)-f'(c)}{x-c} = f''(\alpha) \text{ for at least one } \alpha, c < \alpha < x$$

or $f'(x) = (x-c)f''(\alpha)$ $\quad [\because f'(c) = 0]$

or $|f'(x)| = |x-c|\,|f''(\alpha)|$

But $x \in [0, 1]$, $c \in (0, 1)$

or $|x-c| < 1 - 0$ or $|x-c| < 1$

and given $|f''(x)| \le 1 \; \forall x \in [0, 1]$

$\therefore$ $|f''(\alpha)| \le 1$

$\therefore$ $|f'(x)| < 1 \times 1$ $\quad (\because |f'(x)| = |x-c|\,|f''(\alpha)|)$

or $|f'(x)| < 1 \; \forall x \in [0, 1]$

Case III: Let $x < c$. Then

$$\frac{f'(c)-f'(x)}{c-x} = f''(\alpha)$$

or $|-f'(x)| = |c-x|\,|f''(\alpha)|$

or $|f'(x)| < 1 \times 1$ or $|f'(x)| < 1$

Combining all cases, we get $|f'(x)| < 1 \; \forall x \in [0, 1]$.

Single Correct Answer Type

1. The number of tangents to the curve $x^{3/2}+y^{3/2}=2a^{3/2}, a>0$, which are equally inclined to the axes, is
   (1) 2 (2) 1 (3) 0 (4) 4

2. The angle made by the tangent of the curve $x = a(t + \sin t \cos t)$, $y = a(1 + \sin t)^2$ with the x-axis at any point on it is
   (1) $\frac{1}{4}(\pi+2t)$ (2) $\frac{1-\sin t}{\cos t}$
   (3) $\frac{1}{4}(2t-\pi)$ (4) $\frac{1+\sin t}{\cos 2t}$

3. If m is the slope of a tangent to the curve $e^y = 1 + x^2$, then
   (1) $|m| > 1$ (2) $m > 1$
   (3) $m > -1$ (4) $|m| \le 1$

4. If at each point of the curve $y = x^3 - ax^2 + x + 1$, the tangent is inclined at an acute angle with the positive direction of the x-axis, then
   (1) $a > 0$ (2) $a \le \sqrt{3}$
   (3) $-\sqrt{3} \le a \le \sqrt{3}$ (4) none of these

5. The slope of the tangent to the curve $y = \sqrt{4-x^2}$ at the point where the ordinate and the abscissa are equal is
   (1) -1 (2) 1
   (3) 0 (4) none of these

6. The curve given by $x + y = e^{xy}$ has a tangent parallel to the y-axis at the point
   (1) (0, 1) (2) (1, 0)
   (3) (1, 1) (4) none of these

7. If the line joining the points (0, 3) and (5, –2) is a tangent to the curve $y = \frac{c}{x+1}$, then the value of c is
   (1) 1 (2) -2
   (3) 4 (4) none of these

8. A differentiable function $y = f(x)$ satisfies $f'(x) = (f(x))^2 + 5$ and $f(0) = 1$. Then the equation of tangent at the point where the curve crosses y-axis is
   (1) $x - y + 1 = 0$ (2) $x - 2y + 1 = 0$
   (3) $6x - y + 1 = 0$ (4) $x - 2y - 1 = 0$

9. The distance between the origin and the tangent to the curve $y = e^{2x} + x^2$ drawn at the point $x = 0$ is
   (1) $\frac{1}{\sqrt{5}}$ (2) $\frac{2}{\sqrt{5}}$
   (3) $\frac{-1}{\sqrt{5}}$ (4) $\frac{2}{\sqrt{3}}$

10. The point on the curve $3y = 6x - 5x^3$ the normal at which passes through the origin is
   (1) (1, 1/3) (2) (–1, –1/3)
   (3) (2, – 28/3) (4) none of these

11. The normal to the curve $2x^2 + y^2 = 12$ at the point (2, 2) cuts the curve again at
   (1) $\left(-\frac{22}{9}, -\frac{2}{9}\right)$ (2) $\left(\frac{22}{9}, \frac{2}{9}\right)$
   (3) (–2, –2) (4) none of these

12. At what points of curve $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$, the tangent makes equal angle with the axis?
   (1) $\left(\frac{1}{2}, \frac{5}{24}\right)$ and $\left(-1, -\frac{1}{6}\right)$
   (2) $\left(\frac{1}{2}, \frac{4}{9}\right)$ and (–1, 0)
   (3) $\left(\frac{1}{3}, \frac{1}{7}\right)$ and $\left(-3, \frac{1}{2}\right)$
   (4) $\left(\frac{1}{3}, \frac{4}{47}\right)$ and $\left(-1, -\frac{1}{3}\right)$

13. The equation of the tangent to the curve $y = be^{-x/a}$ at the point where it crosses the y-axis is
   (1) $\frac{x}{a} - \frac{y}{b} = 1$ (2) $ax + by = 1$
   (3) $ax - by = 1$ (4) $\frac{x}{a} + \frac{y}{b} = 1$

14. The angle of intersection of the normals at the point $\left(-\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ of the curves $x^2 - y^2 = 8$ and $9x^2 + 25y^2 = 225$ is
   (1) 0 (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

15. A function $y = f(x)$ has a second-order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2, 1) and at that point tangent to the graph is $y = 3x - 5$, then the value of $f(0)$ is
   (1) 1 (2) –1 (3) 2 (4) 0

16. The curve $x + y - \log_e(x + y) = 2x + 5$ has a vertical tangent at the point (α, β). Then $\alpha + \beta$ is equal to
   (1) –1 (2) 1 (3) 2 (4) –2

17. A curve is defined parametrically by $x = e^{\sqrt{t}}$, $y = 3t - \log_e(t^2)$, where t is a parameter. Then the equation of the tangent line drawn to the curve at $t = 1$ is
   (1) $y = \frac{2}{e}x + 1$ (2) $y = \frac{2}{e}x - 1$
   (3) $y = \frac{e}{2}x + 1$ (4) $y = \frac{e}{2}x - 1$

18. If $x + 4y = 14$ is a normal to the curve $y^2 = \alpha x^3 - \beta$ at (2, 3), then the value of $\alpha + \beta$ is
   (1) 9 (2) – 5 (3) 7 (4) –7

19. In the curve represented parametrically by the equations $x = 2 \ln \cot t + 1$ and $y = \tan t + \cot t$,
(1) tangent and normal intersect at the point (2, 1)
(2) normal at $t = \pi/4$ is parallel to the y-axis
(3) tangent at $t = \pi/4$ is parallel to the line $y = x$
(4) tangent at $t = \pi/4$ is parallel to the x-axis

20. The abscissas of points P and Q on the curve $y = e^x + e^{-x}$ such that tangents at P and Q make 60° with the x-axis are
(1) $\ln\left(\frac{\sqrt{3}+\sqrt{7}}{7}\right)$ and $\ln\left(\frac{\sqrt{3}+\sqrt{5}}{2}\right)$
(2) $\ln\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$
(3) $\ln\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)$
(4) $\pm \ln\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$

21. If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b on x- and y-axes, respectively, then the value of a^2b is
(1) $27c^3$ (2) $\frac{4}{27}c^3$ (3) $\frac{27}{4}c^3$ (4) $\frac{4}{9}c^3$

22. Let C be the curve $y = x^3$ (where x takes all real values). The tangent at A meets the curve again at B. If the gradient at B is K times the gradient at A, then K is equal to
(1) 4 (2) 2 (3) -2 (4) $\frac{1}{4}$

23. The equation of the line tangent to the curve $x \sin y + y \sin x = \pi$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is
(1) $3x + y = 2\pi$ (2) $x - y = 0$
(3) $2x - y = \pi/2$ (4) $x + y = \pi$

24. The x-intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to
(1) square of the abscissa of the point of tangency
(2) square root of the abscissa of the point of tangency
(3) cube of the abscissa of the point of tangency
(4) cube root of the abscissa of the point of tangency

25. At any point on the curve $2x^2y^2 - x^4 = c$, the mean proportional between the abscissa and the difference between the abscissa and the sub-normal drawn to the curve at the same point is equal to
(1) ordinate (2) radius vector
(3) x-intercept of tangent (4) sub-tangent

26. Given $g(x) = \frac{x+2}{x-1}$ and the line $3x + y - 10 = 0$. Then the line is
(1) tangent to $g(x)$ (2) normal to $g(x)$
(3) chord of $g(x)$ (4) none of these

27. If the length of sub-normal is equal to the length of sub-tangent at any point (3, 4) on the curve $y = f(x)$ and the tangent at (3, 4) to $y = f(x)$ meets the coordinate axes at A and B, then the maximum area of the triangle OAB, where O is origin, is
(1) 45/2 (2) 49/2 (3) 25/2 (4) 81/2

28. The number of points in the rectangle $\{(x, y) | -12 \le x \le 12$ and $-3 \le y \le 3\}$ which lie on the curve $y = x + \sin x$ and at which the tangent to the curve is parallel to the x-axis is
(1) 0 (2) 2 (3) 4 (4) 8

29. Tangent of acute angle between the curves $y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$ at their points of intersection is
(1) $\frac{5\sqrt{3}}{2}$ (2) $\frac{3\sqrt{5}}{2}$ (3) $\frac{5\sqrt{3}}{4}$ (4) $\frac{3\sqrt{5}}{4}$

30. The lines tangent to the curves $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to
(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

31. The two curves $x = y^2$, $xy = a^3$ cut orthogonally at a point. Then a^2 is equal to
(1) $\frac{1}{3}$ (2) 3 (3) 2 (4) $\frac{1}{2}$

32. The tangent to the curve $y = e^{kx}$ at a point (0, 1) meets the x-axis at $(a, 0)$, where $a \in [-2, -1]$. Then $k \in$
(1) $[-1/2, 0]$ (2) $[-1, -1/2]$
(3) $[0, 1]$ (4) $[1/2, 1]$

33. The curves $4x^2 + 9y^2 = 72$ and $x^2 - y^2 = 5$ at (3, 2)
(1) touch each other (2) cut orthogonally
(3) intersect at 45° (4) intersect at 60°

34. The coordinates of a point on the parabola $y^2 = 8x$ whose distance from the circle $x^2 + (y + 6)^2 = 1$ is minimum is
(1) (2, 4) (2) (2, −4)
(3) (18, −12) (4) (8, 8)

35. At the point $P(a, a^n)$ on the graph of $y = x^n$, $(n \in N)$, in the first quadrant, a normal is drawn. The normal intersects the y-axis at the point $(0, b)$. If $\lim_{a \to 0} b = \frac{1}{2}$, then n equals
(1) 1 (2) 3 (3) 2 (4) 4

36. Let f be a continuous and differentiable function. If the tangent to $y = f(x)$ at $x = a$ is also the normal to $y = f(x)$ at $x = b$, then there exists at least one $c \in (a, b)$ such that
(1) $f'(c) = 0$ (2) $f'(c) > 0$
(3) $f'(c) < 0$ (4) none of these

37. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is
(1) (2, 6) (2) (2, −6)
(3) $\left(\frac{9}{8}, -\frac{9}{2}\right)$ (4) $\left(\frac{9}{8}, \frac{9}{2}\right)$

38. The rate of change of the volume of a sphere w.r.t. its surface area, when the radius is 2 cm, is

(1) 1 (2) 2

(3) 3 (4) 4

39. If there is an error of k% in measuring the edge of a cube, then the percent error in estimating its volume is

(1) k (2) $3k$

(3) $\frac{k}{3}$ (4) none of these

40. A lamp of negligible height is placed on the ground ℓ_1 away from a wall. A man ℓ_2 m tall is walking at a speed of $\frac{\ell_1}{10}$ m/s from the lamp to the nearest point on the wall. When he is midway between the lamp and the wall, the rate of change in the length of this shadow on the wall is

(1) $-\frac{5\ell_2}{2}$ m/s (2) $-\frac{2\ell_2}{5}$ m/s

(3) $-\frac{\ell_2}{2}$ m/s (4) $-\frac{\ell_2}{5}$ m/s

41. The function $f(x) = x(x+3)e^{-x/2}$ satisfies all the conditions of Rolle's theorem on [–3, 0]. The value of c which verifies Rolle's theorem is

(1) –2 (2) –1

(3) 0 (4) 3

42. The radius of a right circular cylinder increases at the rate of 0.1 cm/min, and the height decreases at the rate of 0.2 cm/min. The rate of change of the volume of the cylinder, in cm³/min, when the radius is 2 cm and the height is 3 cm is

(1) -2π (2) $-\frac{8\pi}{5}$

(3) $-\frac{3\pi}{5}$ (4) $\frac{2\pi}{5}$

43. A cube of ice melts without changing its shape at the uniform rate of 4 cm³/min. The rate of change of the surface area of the cube, in cm²/min, when the volume of the cube is 125 cm³, is

(1) – 4 (2) – 16/5

(3) – 16/6 (4) – 8/15

44. The radius of the base of a cone is increasing at the rate of 3 cm/min and the altitude is decreasing at the rate of 4 cm/min. The rate of change of lateral surface when the radius is 7 cm and altitude is 24 cm is

(1) 108π cm²/min (2) 7π cm²/min

(3) 27π cm²/min (4) none of these

45. If $f(x) = x^3 + 7x - 1$, then $f(x)$ has a zero between $x = 0$ and $x = 1$. The theorem that best describes this is

(1) mean value theorem

(2) maximum-minimum value theorem

(3) intermediate value theorem

(4) none of these

46. Consider the function $f(x) = \begin{cases} x\sin\frac{\pi}{x}, & \text{for } x>0 \\ 0, & \text{for } x=0 \end{cases}$

Then, the number of points in (0, 1) where the derivative $f'(x)$ vanishes is

(1) 0 (2) 1 (3) 2 (4) infinite

47. Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 1$, such that $f(0) = 0, g(0) = 0, f(1) = 6$. Let there exists a real number c in (0, 1) such that $f'(c) = 2g'(c)$. Then the value of $g(1)$ must be

(1) 1 (2) 3 (3) – 2 (4) –1

48. If $3(a + 2c) = 4(b + 3d)$, then the equation $ax^3 + bx^2 + cx + d = 0$ will have

(1) no real solution

(2) at least one real root in (–1, 0)

(3) at least one real root in (0, 1)

(4) none of these

49. A value of C for which the conclusion of mean value theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is

(1) $\frac{1}{2}\log_e 3$ (2) $\log_3 e$

(3) $\log_e 3$ (4) $2\log_3 e$

50. For $f(x) = 4x^3 + 3x^2 - x - 1$, the range of values of $\frac{f(x_1) - f(x_2)}{x_1 - x_2}$ is

(1) $\left(-\infty, -\frac{5}{4}\right)$ (2) $\left(-\infty, -\frac{7}{4}\right]$

(3) $\left[-\frac{7}{4}, \infty\right)$ (4) $\left[-\frac{5}{4}, \infty\right)$

51. Let $f(x)$ be a twice differentiable function for all real values of x and satisfies $f(1) = 1, f(2) = 4, f(3) = 9$. Then which of the following is definitely true?

(1) $f''(x) = 2\ \forall\ x \in (1, 3)$

(2) $f''(x) = f'(x) = 5$ for some $x \in (2, 3)$

(3) $f''(x) = 3\ \forall\ x \in (2, 3)$

(4) $f''(x) = 2$ for some $x \in (1, 3)$

52. The value of c in Lagrange's theorem for the function $f(x) = \log\sin x$ in the interval $[\pi/6, 5\pi/6]$ is

(1) $\pi/4$ (2) $\pi/2$

(3) $2\pi/3$ (4) none of these

53. In which of the following functions is Rolle's theorem applicable?

(1) $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$ on [0, 1]

(2) $f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$ on $[-\pi, 0]$

(3) $f(x)=\dfrac{x^2-x-6}{x-1}$ on $[-2, 3]$

(4) $f(x)=\begin{cases}\dfrac{x^3-2x^2-5x+6}{x-1}, & \text{if } x\neq 1,\\ -6, & \text{if } x=1\end{cases}$ on $[-2, 3]$

54. Let $f'(x)=e^{x^2}$ and $f(0)=10$. If $A<f(1)<B$ can be concluded from the mean value theorem, then the largest value of $(A-B)$ equals

(1) e (2) $1-e$ (3) $e-1$ (4) $1+e$

55. If $f(x)$ and $g(x)$ are differentiable functions for $0\le x\le 1$ such that $f(0)=10$, $g(0)=2$, $f(1)=2$, $g(1)=4$, then in the interval $(0, 1)$,

(1) $f'(x)=0$ for all x
(2) $f'(x)+4g'(x)=0$ for at least one x
(3) $f'(x)=2g'(x)$ for at most one x
(4) none of these

56. A continuous and differentiable function $y=f(x)$ is such that its graph cuts line $y=mx+c$ at n distinct points. Then the minimum number of points at which $f''(x)=0$ is/are

(1) $n-1$ (2) $n-3$
(3) $n-2$ (4) cannot say

57. Given $f'(1)=1$ and $f(2x)=f(x)\ \forall\ x>0$. If $f'(x)$ is differentiable, then there exists a number $c\in(2, 4)$ such that $f''(c)$ equals

(1) $\dfrac{1}{4}$ (2) $\dfrac{-1}{2}$ (3) $-1\dfrac{1}{4}$ (4) $-\dfrac{1}{8}$

58. If $f(x)$ is differentiable in $[a, b]$ such that $f(a)=2$, $f(b)=6$, then there exists at least one c, $a<c<b$, such that $(b^3-a^3)f'(c)=$

(1) c^2 (2) $2c^2$ (3) $-3c^2$ (4) $12c^2$

Multiple Correct Answers Type

1. Points on the curve $f(x)=\dfrac{x}{1-x^2}$ where the tangent is inclined at an angle of $\dfrac{\pi}{4}$ to the x-axis are

(1) $(0, 0)$ (2) $\left(\sqrt{3}, -\dfrac{\sqrt{3}}{2}\right)$
(3) $\left(-2, \dfrac{2}{3}\right)$ (4) $\left(-\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$

2. In the curve $y=ce^{x/a}$, the

(1) sub-tangent is constant
(2) sub-normal varies as the square of the ordinate
(3) tangent at (x_1, y_1) on the curve intersects the x-axis at a distance of (x_1-a) from the origin
(4) equation of the normal at the point where the curve cuts y-axis is $cy+ax=c^2$

3. Let the parabolas $y=x(c-x)$ and $y=x^2+ax+b$ touch each other at the point $(1, 0)$. Then

(1) $a+b+c=0$ (2) $a+b=2$
(3) $b-c=1$ (4) $a+c=-2$

4. The angle formed between the y-axis and the tangent to $y=x^2+4x-17$ at $(5/2, -3/4)$ is

(1) $\tan^{-1}(9)$ (2) $\dfrac{\pi}{2}-\tan^{-1}(9)$
(3) $\dfrac{\pi}{2}+\tan^{-1}(9)$ (4) none of these

5. Which of the following pair(s) of curves is/are orthogonal?

(1) $y^2=4ax;\ y=e^{-x/2a}$ (2) $y^2=4ax;\ x^2=4ay$ at $(0, 0)$
(3) $xy=a^2;\ x^2-y^2=b^2$ (4) $y=ax;\ x^2+y^2=c^2$

6. The coordinates of the point(s) on the graph of the function $f(x)=\dfrac{x^3}{3}-\dfrac{5x^2}{2}+7x-4$, where the tangent drawn cuts off intercepts from the coordinate axes which are equal in magnitude but opposite in sign, are

(1) $(2, 8/3)$ (2) $(3, 7/2)$
(3) $(1, 5/6)$ (4) none of these

7. The abscissa of a point on the curve $xy=(a+x)^2$, the normal at which cuts off numerically equal intercepts from the coordinate axes, is

(1) $-\dfrac{a}{\sqrt{2}}$ (2) $\sqrt{2}a$
(3) $\dfrac{a}{\sqrt{2}}$ (4) $-\sqrt{2}a$

8. The angle between the tangents at any point P and the line joining P to the origin, where P is a point on the curve $\ln(x^2+y^2)=c\tan^{-1}\dfrac{y}{x}$, c is a constant, is

(1) independent of x
(2) independent of y
(3) independent of x but dependent on y
(4) independent of y but dependent on x

9. If OT and ON are perpendiculars dropped from the origin to the tangent and normal to the curve $x=a\sin^3 t$, $y=a\cos^3 t$ at an arbitrary point, then

(1) $4OT^2+ON^2=a^2$
(2) the length of the tangent is $\left|\dfrac{y}{\cos t}\right|$
(3) the length of the normal is $\left|\dfrac{y}{\sin t}\right|$
(4) none of these

10. Let $C_1: y=x^2\sin 3x$, $C_2: y=x^2$ and $C_3: y=-x^2$, then

(1) C_1 touches C_2 at infinite points
(2) C_1 touches C_3 at infinite points
(3) C_1 and C_2 and C_1 and C_3 meet at alternate points
(4) none of these

11. If the line $x \cos\theta + y \sin\theta = P$ is the normal to the curve $(x + a)y = 1$, then θ may lie in
(1) I quadrant (2) II quadrant
(3) III quadrant (4) IV quadrant

12. Common tangent(s) to $y = x^3$ and $x = y^3$ is/are
(1) $x - y = \frac{1}{\sqrt{3}}$ (2) $x - y = -\frac{1}{\sqrt{3}}$
(3) $x - y = \frac{2}{3\sqrt{3}}$ (4) $x - y = \frac{-2}{3\sqrt{3}}$

13. Given $f(x) = 4 - \left(\frac{1}{2} - x\right)^{2/3}$, $g(x) = \begin{cases} \frac{\tan[x]}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$,
$h(x) = \{x\}$, $k(x) = 5^{\log_2(x+3)}$.
Then in [0, 1], Lagrange's mean value theorem is not applicable to (where [·] and {·} represents the greatest integer functions and fractional part functions, respectively)
(1) f (2) g (3) k (4) h

14. Let $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x$, where a_i's are real and $f(x) = 0$ has a positive root α_0. Then
(1) $f'(x) = 0$ has a root α_1 such that $0 < \alpha_1 < \alpha_0$
(2) $f'(x) = 0$ has at least one real root
(3) $f''(x) = 0$ has at least one real root
(4) none of these

15. Which of the following is/are correct?
(1) Between any two roots of $e^x \cos x = 1$, there exists at least one root of $\tan x = 1$.
(2) Between any two roots of $e^x \sin x = 1$, there exists at least one root of $\tan x = -1$.
(3) Between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x = 1$.
(4) Between any two roots of $e^x \sin x = 1$, there exists at least one root of $e^x \cos x = 1$.

16. Among the following, the function(s) on which LMVT theorem is applicable in the indicated intervals is/are
(1) $f(x) = x^{\frac{1}{3}}$ in [–1, 1]
(2) $f(x) = x + \frac{1}{x}$ in $\left[\frac{1}{2}, 3\right]$
(3) $f(x) = (x - 1)\,|(x - 1)(x - 2)|$ in [–1, 2]
(4) $f(x) = e^{|(x-1)(x-3)|}$ in [1, 3]

17. Let $f(x)$ be a differentiable function and $f(a) = f(b) = 0$ $(\alpha < \beta)$, then in the interval (a, b),
(1) $f(x) + f'(x) = 0$ has at least one root
(2) $f(x) - f'(x) = 0$ has at least one real root
(3) $f(x) \times f'(x) = 0$ has at least one real root
(4) none of these

Linked Comprehension Type

For Problems 1–3

Tangent at a point P_1 [other than (0, 0)] on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve again at P_3 and so on.

1. If P_1 has coordinates (1, 1), then the sum $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{x_n}$ is (where $x_1, x_2, \ldots$ are abscissas of $P_1, P_2, \ldots$, respectively)
(1) 2/3 (2) 1/3 (3) 1/2 (4) 3/2

2. If P_1 has co-ordinates (1, 1), then the sum $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{y_n}$ is (where $y_1, y_2, \ldots$ are ordinates of $P_1, P_2, \ldots$, respectively)
(1) 1/8 (2) 1/9 (3) 8/9 (4) 9/8

3. The ratio of area of $\Delta P_1P_2P_3$ to that of $\Delta P_2P_3P_4$ is
(1) 1/4 (2) 1/2 (3) 1/8 (4) 1/16

For Problems 4–6

Consider the curve $x = 1 - 3t^2$, $y = t - 3t^3$. A tangent at point $(1 - 3t^2, t - 3t^3)$ is inclined at an angle θ to the positive x-axis and another tangent at point $P(-2, 2)$ cuts the curve again at Q.

4. The value of $\tan\theta + \sec\theta$ is equal to
(1) 3t (2) t (3) $t - t^2$ (4) $t^2 - 2t$

5. The point Q will be
(1) (1, – 2) (2) $\left(-\frac{1}{3}, -\frac{2}{9}\right)$
(3) (– 2, 1) (4) none of these

6. The angle between the tangents at P and Q will be
(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{3}$

For Problems 7 and 8

A spherical balloon is being inflated so that its volume increases uniformly at the rate of 40 cm^3/min.

7. At $r = 8$, its surface area increases at the rate
(1) 8 cm^2/min (2) 10 cm^2/min
(3) 20 cm^2/min (4) none of these

8. When $r = 8$, then the increase in radius in the next 1/2 min is
(1) 0.025 cm (2) 0.050 cm
(3) 0.075 cm (4) 0.01 cm

For Problems 9–11

A conical paper cup 20 cm across the top and 15 cm deep is full of water. The cup springs a leak at the bottom and loses water at 5 cu. cm per minute.

9. How fast is the water level dropping at the instant when the water is exactly 7.5 cm deep?
(1) $\frac{1}{\pi}$ cm/min (2) $\frac{1}{5\pi}$ cm/min
(3) $\frac{1}{2\pi}$ cm/min (4) $\frac{2}{3\pi}$ cm/min

10. The amount of water (in cm^3) when the height of water is 3 cm is
(1) 4π (2) 3π (3) 27π (4) 2π

11. The value of $\frac{d^2h}{dt^2}$ (in cm/min^2) when the water is exactly 7.5 cm deep and $\frac{d^2V}{dt^2} = -\frac{4}{9}$ cm^3/min^2 is
(1) $-\frac{2}{5}$ (2) $-\frac{2}{125\pi^3}$
(3) $-\frac{2}{5\pi^3}$ (4) none of these

For Problems 12 and 13

Let $A(0, 0)$ and $B(8, 2)$ be two fixed points on the curve $y^3 = |x|$. A point C (abscissa is less than 0) starts moving from origin along the curve such that rate of change in the ordinate is 2 cm/sec. After t_0 seconds, triangle ABC becomes a right triangle.

12. The value of t_0 is
(1) 1 sec (2) 2 sec (3) $\frac{1}{4}$ sec (4) $\frac{1}{2}$ sec

13. After t_0 seconds, tangent is drawn to the curve at point C to intersect it again at (α, β). Then the value of $4\alpha + 3\beta$ is
(1) $\frac{4}{3}$ (2) $\frac{3}{4}$ (3) 2 (4) 1

Matrix Match Type

1. Match the following lists:

List I	List II
a. The sides of a triangle ABC vary slightly in such a way that its circum-radius remains constant. If $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} + 1 = \|m\|$, then the value of m is	**p.** 1
b. The length of sub-tangent to the curve $x^2y^2 = 16$ at the point $(-2, 2)$ is $\|k\|$. Then the value of k is	**q.** -1
c. The curve $y = 2e^{2x}$ intersects the y-axis at an angle $\cot^{-1} \|(8n-4)/3\|$. Then the value of n is	**r.** 2
d. The area of a triangle formed by normal at the point $(1, 0)$ on the curve $x = e^{\sin y}$ with axes is $\|2t+1\|/6$ sq. units. Then the value of t is	**s.** -2

2. Match the following lists:

List I	List II
a. A circular plate is expanded by heat from radius 6 cm to 6.06 cm. Approximate increase in the area is	**p.** 5
b. If an edge of a cube increases by 2%, then the percentage increase in the volume is	**q.** 0.72π
c. If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is thrice the rate of decrease of x, then x is equal to (rate of decrease is nonzero)	**r.** 6
d. The rate of increase in the area of an equailateral triangle of side 30 cm, when each side increases at the rate of 0.1 cm/s, is	**s.** $\frac{3\sqrt{3}}{2}$

3. Match the following lists:

List I: Curves	List II: Angle between the curves
a. $y^2 = 4x$ and $x^2 = 4y$	**p.** $90°$
b. $2y^2 = x^3$ and $y^2 = 32x$	**q.** Any one of $\tan^{-1}\frac{3}{4}$ or $\tan^{-1}(16^{1/3})$
c. $xy = a^2$ and $x^2 + y^2 = 2a^2$	**r.** $0°$
d. $y^2 = x$ and $x^3 + y^3 = 3xy$ at other than origin	**s.** $\tan^{-1}\frac{1}{2}$

4. The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$ and cuts the y-axis at point Q, where its gradient is 3. Now, match the following lists and then choose the correct code.

List I	List II
a. The value of a is	**p.** 3
b. The value of b is	**q.** 0
c. The value of c is	**r.** $-\frac{3}{4}$
d. The value of $y'(1)$ is	**s.** $-\frac{1}{2}$

Codes:

	a	b	c	d
(1)	s	r	q	p
(2)	q	s	r	p
(3)	s	r	p	q
(4)	s	p	q	r

5. Match the following lists and then choose the correct code.

List I	List II
a. The equation $x \log x = 3 - x$ has at least one root in	**p.** $(0, 1)$
b. If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one root in	**q.** $(1, 3)$
c. If $c = \sqrt{3}$ and $f(x) = x + \frac{1}{x}$, then interval of x in which LMVT is applicable for $f(x)$ is	**r.** $(0, 3)$
d. If $c = \frac{1}{2}$ and $f(x) = 2x - x^2$, then interval of x in which LMVT is applicable for $f(x)$ is	**s.** $(-1, 1)$

Codes:

	a	b	c	d
(1)	s	r	q	p
(2)	q	r	q	p
(3)	s	r	p	q
(4)	s	p	q	r

Numerical Value Type

1. There is a point (p, q) on the graph of $f(x) = x^2$ and a point (r, s) on the graph of $g(x) = \frac{-8}{x}$, where $p > 0$ and $r > 0$. If the line through (p, q) and (r, s) is also tangent to both the curves at these points, respectively, then the value of $p + r$ is______.

2. A curve is defined parametrically by the equations $x = t^2$ and $y = t^3$. A variable pair of perpendicular lines through the origin O meet the curve at P and Q. If the locus of the point of intersection of the tangents at P and Q is $ay^2 = bx - 1$, then the value of $(a + b)$ is______.

3. If d is the minimum distance between the curves $f(x) = e^x$ and $g(x) = \log_e x$, then the value of d^6 is ______.

4. Let $f(x)$ be a non-constant thrice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(6 - x)$ and $f'(0) = 0 = f'(2) = f'(5)$. If n is the minimum number of roots of $(f''(x)^2) + f'(x) f'''(x) = 0$ in the interval $[0, 6]$, then the value of n is______.

5. At the point $P(a, a^n)$ on the graph of $y = x^n$, $(n \in N)$, in the first quadrant, a normal is drawn. The normal intersects the y-axis at the point $(0, b)$. If $\lim\limits_{a \to 0} b = \frac{1}{2}$, then n equals______.

6. A curve is given by the equations $x = \sec^2 \theta$, $y = \cot \theta$. If the tangent at P where $\theta = \pi/4$ meets the curve again at Q, then $[PQ]$ is, where $[\cdot]$ represents the greatest integer function.______.

7. Water is dropped at the rate of 2 m^3/s into a cone of semi-vertical angle 45°. If the rate at which periphery of water surface changes when the height of the water in the cone is 2 m is d, then the value of $5d$ is______.

8. If the slope of line through the origin which is tangent to the curve $y = x^3 + x + 16$ is m, then the value of m is______.

9. Let $y = f(x)$ be drawn with $f(0) = 2$ and for each real number, the tangent to $y = f(x)$ at $(a, f(a))$ has x-intercept $(a - 2)$. If $f(x)$ is of the form of $k\,e^{px}$, then $\left(\frac{k}{p}\right)$ has the value equal to______.

10. Suppose a, b, c are such that the curve $y = ax^2 + bx + c$ is tangent to $y = 3x - 3$ at $(1, 0)$ and is also tangent to $y = x + 1$ at $(3, 4)$. Then the value of $(2a - b - 4c)$ equals______.

11. Let C be a curve defined by $y = e^{a+bx^2}$. The curve C passes through the point $P(1, 1)$ and the slope of the tangent at P is (-2). Then the value of $2a - 3b$ is______.

12. If the curve C in the xy plane has the equation $x^2 + xy + y^2 = 1$, then the fourth power of the greatest distance of a point on C from the origin is______.

13. If a, b are two real numbers with $a < b$, then a real number c can be found between a and b such that the value of $\frac{a^2 + ab + b^2}{c^2}$ is______.

14. Let f: $[1, 3] \to [0, \infty)$ be continuous and differentiable function. If $(f(3) - f(1))\,(f^2(3) + f^2(1) + f(3)f(1)) = kf^2(c)\,f'(c)$ where $c \in (1, 3)$, then the value of k is ______.

15. The x intercept of the tangent to a curve $f(x, y) = 0$ is equal to the ordinate of the point of contact. Then the value of $\frac{d^2x}{dy^2}$ at the point $(1, 1)$ on the curve is ______.

16. If $f(x)$ is a differentiable function such that $f(1) = \sin 1$, $f(2) = \sin 4$ and $f(3) = \sin 9$, then the minimum number of distinct roots of $f'(x) = 2x \cos x^2$ in $(1, 3)$ is ______.

17. Let $f(x) = x(x^2 + mx + n) + 2$, for all $x \in R$ and $m, n \in R$. If Rolle's theorem holds for $f(x)$ at $x = 4/3$ in $x \in [1, 2]$, then $(m + n)$ equals ______.

18. If length of the perpendicular from the origin upon the tangent drawn to the curve $x^2 - xy + y^2 + \alpha(x - 2) = 4$ at $(2, 2)$ is equal to 2 then α equals ______.

19. If $f(x) = \begin{cases} x \log_e x, & x > 0 \\ 0, & x = 0 \end{cases}$ and conclusion of LMVT holds at $x = 1$ in the interval $[0, a]$ for $f(x)$, then $[a^2]$ is equal to (where [.] denotes the greatest integer)______.

Archives

JEE MAIN

Single Correct Answer Type

1. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is
   (1) $\frac{3\sqrt{2}}{8}$ (2) $\frac{2\sqrt{3}}{8}$ (3) $\frac{3\sqrt{2}}{5}$ (4) $\frac{\sqrt{3}}{4}$
   (AIEEE 2009)

2. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, which is parallel to the x-axis, is
   (1) $y = 8$ (2) $y = 0$ (3) $y = 3$ (4) $y = 2$
   (AIEEE 2010)

3. Consider the function, $f(x) = |x - 2| + |x - 5|$, $x \in R$.
   Statement 1: $f'(4) = 0$
   Statement 2: f is continuous in $[2, 5]$, differentiable in $(2, 5)$ and $f(2) = f(5)$.
   (1) Statement 1 is false, statement 2 is true.
   (2) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.
   (3) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1.
   (4) Statement 1 is true, statement 2 is false.
   (AIEEE 2012)

4. If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in (0, 1)$

(1) $2f'(c) = g'(c)$ (2) $2f'(c) = 3g'(c)$

(3) $f'(c) = g'(c)$ (4) $f'(c) = 2g'(c)$

(JEE Main 2014)

5. The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at (1, 1)

(1) does not meet the curve again.

(2) meets the curve again in the second quadrant.

(3) meets the curve again in the third quadrant.

(4) meets the curve again in the fourth quadrant.

(JEE Main 2015)

6. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \frac{\pi}{2}\right)$. A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point

(1) $\left(0, \frac{2\pi}{3}\right)$ (2) $\left(\frac{\pi}{6}, 0\right)$

(3) $\left(\frac{\pi}{4}, 0\right)$ (4) (0, 0)

(JEE Main 2016)

7. The normal to the curve $y(x - 2)(x - 3) = x + 6$ at the point where the curve intersects the y-axis passes through the point

(1) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (2) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

(3) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (4) $\left(\frac{1}{2}, -\frac{1}{3}\right)$

(JEE Main 2017)

8. If the curves $y^2 = 6x$ and $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is

(1) 9/2 (2) 6 (3) 7/2 (4) 4

(JEE Main 2018)

JEE ADVANCED

Multiple Correct Answers Type

1. Let $f, g : [-1, 2] \to R$ be continuous functions which are twice differentiable on the interval (–1, 2). Let the values of f and g at the points –1, 0 and 2 be as given in the following table:

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	–1

In each of the intervals (–1, 0) and (0, 2), the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is (are)

(1) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(2) $f'(x) - 3g'(x) = 0$ has exactly one solution in (–1, 0)

(3) $f'(x) - 3g'(x) = 0$ has exactly one solution in (0, 2)

(4) $f'(x) - 3g'(x) = 0$ has exactly two solutions in (–1, 0) and exactly two solutions in (0, 2)

(JEE Advanced 2015)

2. For every twice differentiable function $f : R \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) **TRUE**?

(1) There exist $r, s \in R$, where $r < s$, such that f is one-one on the open interval (r, s)

(2) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \leq 1$

(3) $\lim_{x \to \infty} f(x) = 1$

(4) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (2)	2. (1)	3. (4)	4. (3)	5. (1)
6. (2)	7. (3)	8. (3)	9. (1)	10. (1)
11. (1)	12. (1)	13. (4)	14. (2)	15. (2)
16. (2)	17. (1)	18. (1)	19. (4)	20. (2)
21. (3)	22. (1)	23. (4)	24. (3)	25. (1)
26. (1)	27. (2)	28. (1)	29. (3)	30. (4)
31. (4)	32. (4)	33. (2)	34. (2)	35. (3)
36. (1)	37. (4)	38. (1)	39. (2)	40. (2)
41. (1)	42. (4)	43. (2)	44. (1)	45. (3)
46. (4)	47. (2)	48. (2)	49. (4)	50. (3)
51. (4)	52. (2)	53. (4)	54. (2)	55. (2)
56. (3)	57. (4)	58. (4)		

Multiple Correct Answers Type

1. (1), (2), (4)	2. (1), (2), (3), (4)
3. (1), (3), (4)	4. (2), (3)
5. (1), (2), (3), (4)	6. (1), (2)
7. (1), (3)	8. (1), (2)
9. (1), (2), (3)	10. (1), (2)
11. (2), (4)	12. (3), (4)
13. (1), (2), (4)	14. (1), (2), (3)
15. (1), (2), (3)	16. (2), (3), (4)
17. (1), (2), (3)	

Linked Comprehension Type

1. (1)	2. (3)	3. (4)	4. (1)	5. (2)
6. (3)	7. (2)	8. (1)	9. (2)	10. (1)
11. (4)	12. (3)	13. (4)		

Matrix Match Type

1. a → p, q; b → r, s; c → r, q; d → p, s.
2. a → q; b → r; c → p; d → s.
3. a → p, q; b → p, s; c → r; d → q.
4. (3) 5. (2)

Numerical Value Type

1. (5) 2. (7) 3. (8) 4. (12) 5. (2)
6. (3) 7. (5) 8. (13) 9. (4) 10. (9)
11. (5) 12. (4) 13. (3) 14. (6) 15. (−1)
16. (2) 17. (3) 18. (−2) 19. (7)

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (1) 2. (3) 3. (3) 4. (4) 5. (4)
6. (1) 7. (3) 8. (1)

JEE ADVANCED

Multiple Correct Answers Type

1. (2), (3) 2. (1), (2), (4)

6 Monotonicity and Maxima-Minima of Functions

INTRODUCTION

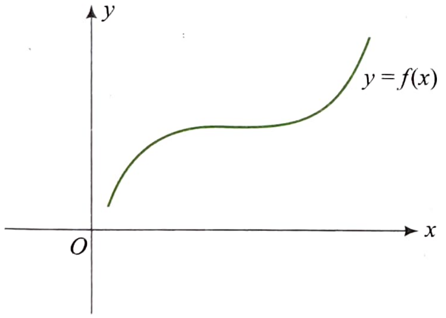
In calculus, we study functions for different aims. We have already learned about the continuity, discontinuity, differentiability and non-differentiability of the function. One of the most important tools in analysing the functions is their monotonic behaviour. Functions like $f(x) = 2x + 3$, $g(x) = \log_e x$ and $h(x) = x^3$ always increase with the increase in the input value of the independent variable in their entire domain, i.e., graph of the function rises from left to right. Functions like $f(x) = \cos^{-1} x$ and $g(x) = 0.5^x$ always decrease with increase in the value of x. Both of the above categories of functions are called monotonic functions. Functions like $f(x) = x^2 - x + 1$ and $g(x) = \sin x$ have dual nature, i.e., increase in some intervals and decrease in some other intervals. Such functions are called non-monotonic functions. Such functions often have turning points on the graph and give point of maxima and minima.

CLASSIFICATION OF FUNCTIONS BASED ON MONOTONICITY

INCREASING FUNCTION

A function $f(x)$ is increasing or non-decreasing on an interval I if $f(x_2) \geq f(x_1)$ for all $x_2 > x_1$, where $x_1, x_2 \in I$.

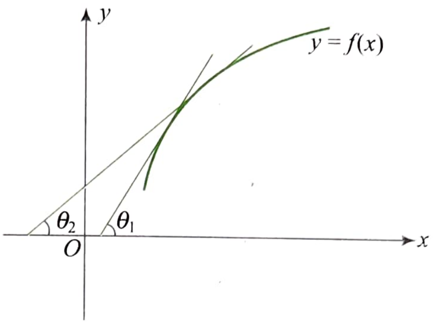
If function is differentiable in I then $\frac{df(x)}{dx} \geq 0$ for all x. So, tangent to the curve at each point on it either makes an acute angle with positive x-axis or parallel to x-axis.



STRICTLY INCREASING FUNCTION

A function $f(x)$ is strictly increasing on an interval I if $f(x_2) > f(x_1)$ for all $x_2 > x_1$, where $x_1, x_2 \in I$.

If function is differentiable in I then $\frac{df(x)}{dx} > 0$ for all x, i.e., tangent to the curve at each point on it makes an acute angle with positive x-axis as shown in the following figure.

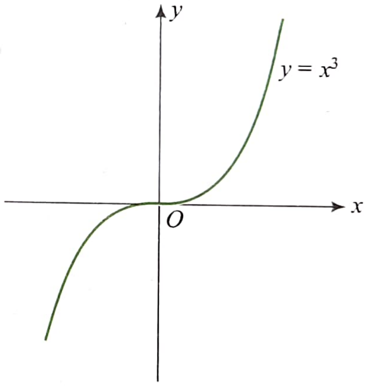


Functions e^x and $x^3 + x$ are strictly increasing functions.

Here $\frac{d}{dx}(e^x) = e^x > 0$ and $\frac{d}{dx}(x^3 + x) = 3x^2 + 1 > 0$ for all x.

Now consider $f(x) = x^3$.

$\therefore \quad f'(x) = 3x^2 \geq 0.$

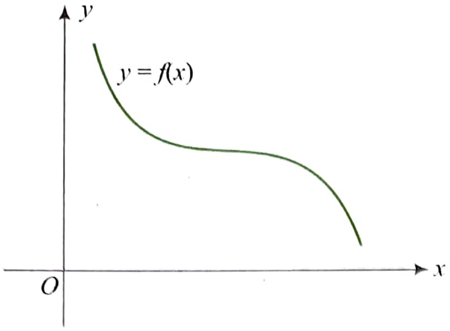


Here $f'(x) = 0$ at $x = 0$ only, but not in the neighbourhood of $x = 0$, i.e., in some interval. So, this function is also strictly increasing function. In other words, the function remains to be strictly increasing if the derivative is zero at discrete points but not in the interval. Consider one more function $f(x) = x + \sin x$. Here $f'(x) = 1 + \cos x \geq 0$. Now $1 + \cos x = 0$, if $\cos x = -1$ or $x = (2n + 1)\pi$, $n \in Z$, which is a set of discrete points. So, function is strictly increasing.

DECREASING FUNCTION

A function $f(x)$ is decreasing or non-increasing on an interval I if $f(x_2) \leq f(x_1)$ for all $x_2 > x_1$, where $x_1, x_2 \in I$.

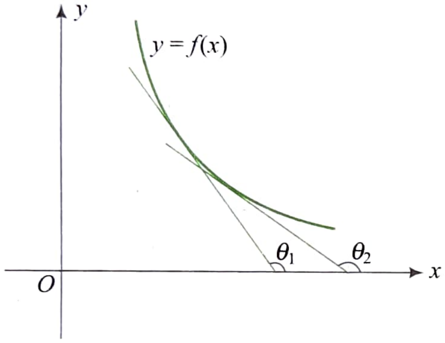
If function is differentiable in I then $\frac{df(x)}{dx} \leq 0$ for all x, i.e., tangent to the curve at each point on it either makes an obtuse angle with positive x-axis or parallel to x-axis.



STRICTLY DECREASING FUNCTION

A function $f(x)$ is strictly decreasing on an interval I if $f(x_2) < f(x_1)$ for all $x_2 > x_1$, where $x_1, x_2 \in I$.

If function is differentiable in I then $\frac{df(x)}{dx} < 0$ for all x, i.e., tangent to the curve at each point on it makes an obtuse angle with positive x-axis.



Functions 0.5^x and $\cos^{-1}x$ are strictly decreasing functions.

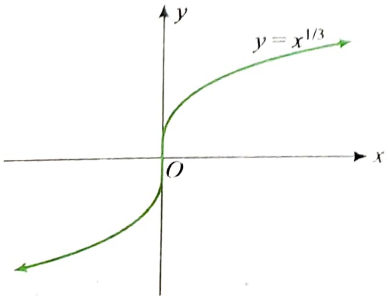
Here $\frac{d}{dx}(0.5^x) = 0.5^x \log_e 0.5 < 0$ for all $x > 0$ and

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} < 0$ for all $x \in (-1, 1)$.

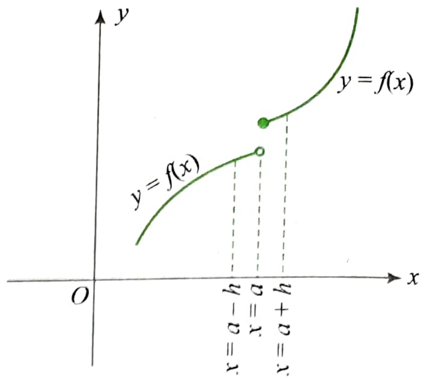
Functions $f(x) = -x^5$ and $g(x) = \cos x - x$ also qualify for strictly decreasing function as $f'(x) = 0$ for $x = 0$ only and $g'(x) = 0$ for $x = (4n-1)\frac{\pi}{2}$, $n \in Z$, which forms a set of discrete points.

Note:

A function may increase on an interval without having a derivative defined at all points. For example, the function $f(x) = x^{1/3}$ is increasing everywhere, including the origin $x = 0$, despite the fact that the derivative is not defined at this point as function has vertical tangent at this point.



Also, it is not necessary that function is continuous at all the points. e.g. consider the function having following graph.

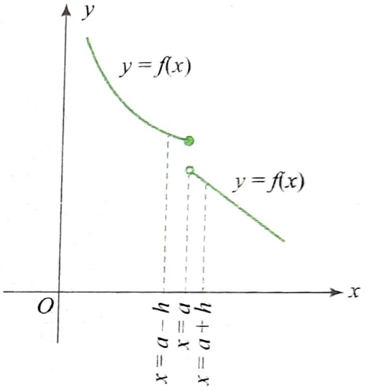


Here function is increasing for all x including $x = a$, where function is discontinuous.

In the neighbourhood of $x = a$, we have $f(a-h) < f(a) < f(a+h)$.

Thus $f(x)$ is increasing at $x = a$. This also defines the increasing nature of function at a point.

Now consider the following graph of the function.



In the neighbourhood of $x = a$, we have $f(a-h) > f(a) > f(a+h)$.

So, function is decreasing for all x including $x = a$, where it is discontinuous.

ILLUSTRATION 6.1

Check the nature of the following differentiable functions

(i) $f(x) = e^x + \sin x, x \in R^+$

(ii) $f(x) = \sin x + \tan x - 2x, x \in (0, \pi/2)$

(iii) $f(x) = \cos^{-1}x + \log_e\left(x + \frac{1}{x}\right)$

Sol. Since the functions are differentiable, we check the nature of the functions using differentiation.

(i) $f(x) = e^x + \sin x, x \in R^+$

$\therefore \quad f'(x) = e^x + \cos x$

Clearly $f'(x) > 0 \ \forall x \in R^+$ (as $e^x > 1, x \in R^+$ and $-1 \le \cos x \le 1, x \in R^+$)

Hence $f(x)$ is strictly increasing.

(ii) $f(x) = \sin x + \tan x - 2x, x \in (0, \pi/2)$

$\Rightarrow \quad f'(x) = \cos x + \sec^2 x - 2$

as $\cos x > \cos^2 x, x \in (0, \pi/2)$, we have

$$f'(x) > \cos^2 x + \sec^2 x - 2$$
$$= (\cos x - \sec x)^2 > 0, x \in (0, \pi/2)$$

Hence $f(x)$ is strictly increasing in $(0, \pi/2)$

(iii) $f(x) = \cos^{-1}x + \log_e\left(x + \frac{1}{x}\right)$

$\cos^{-1}x$ is defined if $x \in [-1, 1]$

$\log_e\left(x + \frac{1}{x}\right)$ is defined if $x > 0$

So, domain of $f(x)$ is $(0, 1]$.

Now, $f'(x) = -\frac{1}{\sqrt{1-x^2}} + \frac{1-\frac{1}{x^2}}{x+\frac{1}{x}}$

For $x \in (0, 1]$, $1 - \frac{1}{x^2} < 0$

$\therefore \quad f'(x) < 0$

Hence, $f(x)$ is decreasing function.

ILLUSTRATION 6.2

Prove that function $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$ is strictly increasing $\forall\, x \in R$.

Sol. $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$

$\therefore \quad f'(x) = \frac{2x}{1+x^2} + e^{-x} = e^{-x} + \frac{2}{x + \frac{1}{x}}$

For $x < 0$, $x + \frac{1}{x} < -2$

$\therefore \quad -\frac{1}{2} < \frac{1}{x+\frac{1}{x}} < 0$

$\Rightarrow \quad -1 < \frac{2}{x+\frac{1}{x}} < 0$

Also, $e^{-x} > 1$

$\therefore \quad e^{-x} + \frac{2x}{1+x^2} > 0$

$\therefore \quad f(x)$ is strictly increasing function $\forall\, x \in R$

ILLUSTRATION 6.3

Find the least value of k for which the function $x^2 + kx + 1$ is an increasing function in the interval $(1, 2)$.

Sol. We have $f(x) = x^2 + kx + 1$

Now $f(x)$ is increasing in $(1, 2)$

$\therefore \quad f'(x) \ge 0$ for all $x \in (1, 2)$

$\Rightarrow \quad 2x + k \ge 0$ for all $x \in (1, 2)$

$\Rightarrow \quad k \ge -2x$ for all $x \in (1, 2)$

$\Rightarrow \quad k \ge$ greatest value of $-2x$ in $(1, 2)$

$\Rightarrow \quad k \ge -2.$

Thus, least value of k is -2.

ILLUSTRATION 6.4

Find the values of 'a' if $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ is increasing for all values of x.

Sol. $f'(x) = 2e^x + ae^{-x} + 2a + 1$

$= e^{-x}(2e^{2x} + (2a + 1)\, e^x + a)$

$= e^{-x}(e^x + a)(2e^x + 1)$

For $f(x)$ to be increasing $f'(x) \ge 0\ \forall\, x \in R$

$\Rightarrow \quad e^x + a \ge 0\ \forall\, x \in R$

$\Rightarrow \quad a \ge -e^x\ \forall\, x \in R$

$\Rightarrow \quad a \ge 0$

ILLUSTRATION 6.5

If $f(x)$ and $g(x)$ are differentiable and increasing functions then which of the following functions always increases?

(i) $f(x) + g(x)$ **(ii)** $f(x)\,g(x)$

(iii) $f(x) - g(x)$ **(iv)** $\frac{f(x)}{g(x)}$

Sol. Given that $f(x)$ and $g(x)$ are increasing functions.

$\therefore \quad f'(x) \ge 0$ and $g'(x) \ge 0$.

(i) $(f(x) + g(x))' = f'(x) + g'(x) \ge 0$

$\therefore \quad f(x) + g(x)$ increases.

(ii) $(f(x)\,g(x))' = f'(x)g(x) + f(x)g'(x)$, the sign of which depends upon the sign of $f(x)$ and $g(x)$ also.

So, $(f(x)\,g(x))$ may be increasing or decreasing.

(iii) $(f(x) - g(x))' = f'(x) - g'(x)$, may be positive or negative.

So, $f(x) - g(x)$ may be increasing or decreasing.

(iv) $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$, may be positive or negative.

So, $\frac{f(x)}{g(x)}$ may be increasing or decreasing.

ILLUSTRATION 6.6

Let $f(x)$ be a differentiable function and $g(x) = (f(x))^3 - 3(f(x))^2 + 4f(x) + 5x + 3\sin x + 4\cos x\ \forall\, x \in R$, then prove that g is increasing whenever f is increasing.

Sol. $g(x) = (f(x))^3 - 3(f(x))^2 + 4f(x) + 5x + 3\sin x + 4\cos x$

$\Rightarrow \quad g'(x) = (3(f(x))^2 - 6f(x) + 4)\,f'(x) + 5 + 3\cos x - 4\sin x \quad \ldots(1)$

Now $\quad 3(f(x))^2 - 6(f(x)) + 4 = 3(f(x) - 1)^2 + 1 > 0$

and $\quad -5 \le 3\cos x - 4\sin x \le 5$

$\therefore \quad 0 \le 3\cos x - 4\sin x + 5 \le 10$

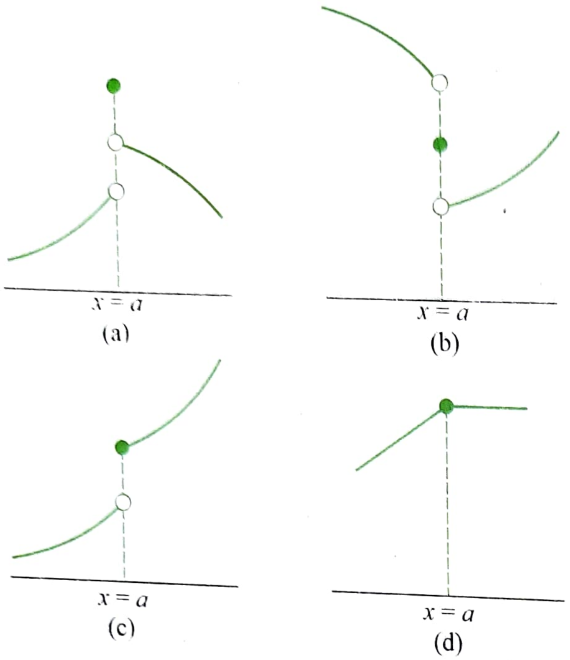
When $f(x)$ increases, $f'(x) \ge 0$

So, from (1), $g'(x) \ge 0$.

So, $g(x)$ in increasing increases whenever $f(x)$ increases.

ILLUSTRATION 6.7

For each of the following graph, comment whether $f(x)$ is increasing or decreasing or neither increasing nor decreasing at $x = a$.



Sol.

(a) Neither increasing nor decreasing as $f(a-h) < f(a)$ and $f(a+h) < f(a)$

(b) Decreasing as $f(a-h) > f(a) > f(a+h)$

(c) Increasing as $f(a-h) < f(a) < f(a+h)$

(d) Neither increasing nor decreasing as $f(a-h) < f(a)$ but $f(a+h) = f(a)$

ILLUSTRATION 6.8

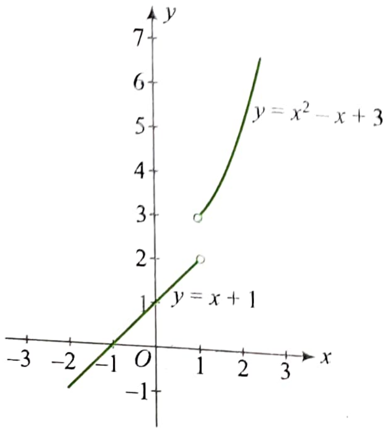
Find the complete set of values of λ, for which the function

$$f(x) = \begin{cases} x+1, & x<1 \\ \lambda, & x=1 \\ x^2-x+3, & x>1 \end{cases}$$ is strictly increasing at $x = 1$.

Sol. $x + 1$ is increasing function.

Also $\frac{d}{dx}(x^2 - x + 3) = 2x - 1 > 0$ for $x > 1$.

$\therefore$ $x^2 - x + 1$ is increasing function for $x > 1$.



Hence $f(x)$ is increasing function at $x = 1$,

if $\lim_{x\to 1^-}(x+1) \le \lambda \le \lim_{x\to 1^+}(x^2 - x + 3)$

$\therefore$ $2 \le \lambda \le 3$

ILLUSTRATION 6.9

Prove that function

$$f(x) = \begin{cases} -2x^3 + 3x^2 - 6x + 5, & x < 0 \\ -x^2 - x + 1, & x \ge 0 \end{cases}$$ decreases for all real x.

Sol. Since $\lim_{x\to 0^-} f(x) = 5$ and $\lim_{x\to 0^+} f(x) = 1$, $f(x)$ is discontinuous at $x = 1$.

Now, let us check the nature of the branch functions.

$$f'(x) = \begin{cases} -6(x^2 - x + 1), & x < 0 \\ -2x - 1, & x > 0 \end{cases}$$

$$= \begin{cases} -6((x - 1/2)^2 + 3/4), & x < 0 \\ -2x - 1, & x > 0 \end{cases}$$

Clearly derivative of each branch function is negative for corresponding values of x.

Also, $\lim_{x\to 0^-} f(x) < \lim_{x\to 0^+} f(x)$

So, $f(x)$ is decreasing for all real x including $x = 0$.

NATURE OF THE COMPOSITE FUNCTIONS

The nature of the composite functions $f(g(x))$ and $g(f(x))$ depends upon the nature of the functions $f(x)$ and $g(x)$.

If $f(x)$ is increasing and $g(x)$ is decreasing then for $x_2 > x_1$ we have $f(x_2) \ge f(x_1)$ and $g(x_2) \le g(x_1)$.

So, for $x_2 > x_1$, we have $g(f(x_2)) \le g(f(x_1))$ and $f(g(x_2)) \le f(g(x_1))$.

Thus $f(g(x))$ is decreasing function.

Similarly, $g(f(x))$ will be also decreasing function.

If $f(x)$ and $g(x)$ are differentiable functions then

$f'(x) \ge 0$ and $g'(x) \le 0$ for all x.

$\therefore$ $(f(g(x)))' = f'(g(x))g'(x) \le 0$ (as $f'(g(x)) \le 0$)

$\therefore$ $f(g(x))$ is decreasing function.

All the possibilities of the nature of the functions $f(g(x))$ and $g(f(x))$ are listed below.

$f(x)$	$g(x)$	$f(g(x)), g(f(x))$
Increasing	Increasing	Increasing
Decreasing	Decreasing	Increasing
Increasing	Decreasing	Decreasing
Decreasing	Increasing	Decreasing

ILLUSTRATION 6.10

If fogoh(x) is increasing function then which of the following is not possible

(i) $f(x)$, $g(x)$ and $h(x)$ are increasing

(ii) $f(x)$ and $g(x)$ are decreasing and $h(x)$ is increasing

(iii) $f(x)$, $g(x)$ and $h(x)$ are decreasing

Sol.

(i) $f(x)$, $g(x)$ and $h(x)$ are increasing.

$\therefore$ $x_2 > x_1$ then $h(x_2) \geq h(x_1)$

$g(x)$ is increasing

$\therefore$ $g(h(x_2)) \geq g(h(x_1))$

Also, $f(x)$ is increasing

$\therefore$ $f(g(h(x_2))) \geq f(g(h(x_1)))$

Thus $f(g(h(x)))$ is increasing.

(ii) $f(x)$ and $g(x)$ are decreasing and $h(x)$ is increasing h(x) is increasing

$\therefore$ $x_2 > x_1$, then $h(x_2) \geq h(x_1)$

$g(x)$ is decreasing, then

$g(h(x_2)) \leq g(h(x_1))$

Now $f(x)$ is decreasing, then

$f(g(h(x_2))) \geq f(g(h(x_1)))$

Thus $f(g(h(x))$ is increasing.

(iii) obviously when $f(x)$, $g(x)$ are $h(x)$ are decreasing then $f(g(h(x)))$ is decreasing.

Note:

In general $f(x) = f_1(f_2(f_3(\ldots f_n(x))$ is increasing if all $f_i(x)$ are increasing. If odd number of functions among $f_i(x)$ are decreasing then $f(x)$ is decreasing.

ILLUSTRATION 6.11

Let $f(x)$ and $g(x)$ be two continuous functions defined from $R \to R$, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$, $\forall\, x_1 > x_2$, then find the solution set of $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$.

Sol. From the given information, f is increasing and g is decreasing in R.

Now $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$

Since f is increasing, we have

$g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$

Since g is decreasing, we have

$\alpha^2 - 2\alpha < 3\alpha - 4$

$\Rightarrow$ $\alpha^2 - 5\alpha + 4 < 0$

$\Rightarrow$ $(\alpha - 1)(\alpha - 4) < 0$

$\Rightarrow$ $a \in (1, 4)$

ILLUSTRATION 6.12

Let $f: [0, \infty) \to [0, \infty)$ and $g: [0, \infty) \to [0, \infty)$ be non-increasing and non-decreasing functions respectively, and $h(x) = g(f(x))$. If $h(0) = 0$. Then show $h(x)$ is always identically zero.

Sol. Since $f(x) \in [0, \infty)$, $g(f(x)) \in [0, \infty)$ ...(1)

Now $f(x)$ is non–increasing,

$\therefore$ $x_2 > x_1$ then $f(x_2) \leq f(x_1)$

$g(x)$ is non–decreasing,

$\therefore$ $g(f(x_2)) \leq g(f(x_1))$

$\therefore$ For $x > 0$, $g(f(x)) \leq g(f(0))$

$\therefore$ $g(f(x)) \leq 0$ ...(2)

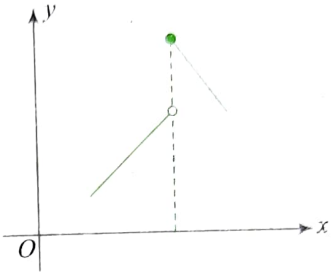
Therefore, from (1) and (2), $g(f(x)) = 0$.

MONOTONIC FUNCTION

A function which is increasing, strictly increasing, decreasing or strictly decreasing in the interval (a, b) is called monotonic function in that interval.

Note:

- If monotonic function is continuous then it is one–one and hence invertible.
- Not all the one–one or invertible functions are monotonic. See the graph of the following function.



- Discontinuous functions like $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$, $\{x\}$ (fractional part function) are increasing in their domain but are many–one.

ILLUSTRATION 6.13

Find the values of p if $f(x) = \cos x - 2px$ is invertible.

Sol. For $f(x) = \cos x - 2px$ to be invertible, it must be monotonic. i.e., either always increasing or always decreasing.

If $f(x)$ is monotonically decreasing then $f'(x) \leq 0$, $\forall\, x \in R$.

$\therefore$ $f'(x) = -\sin x - 2p \leq 0\ \forall\, x \in R$.

$\Rightarrow$ $p \geq -\dfrac{1}{2}\sin x\ \forall\, x \in R$.

$\Rightarrow$ $p \geq \dfrac{1}{2}$ ...(1)

If $f(x)$ is monotonically increasing then $f'(x) \geq 0$

$\therefore$ $f'(x) = -\sin x - 2p \geq 0\ \forall\, x \in R$.

$\Rightarrow$ $p \leq -\dfrac{1}{2}\sin x\ \forall\, x \in R$

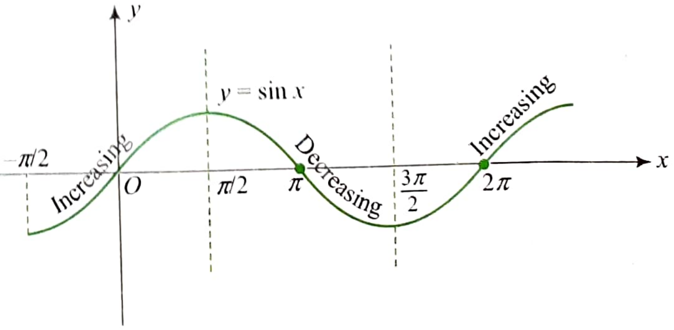
$\Rightarrow$ $p \leq -\dfrac{1}{2}$ ...(2)

From (1) and (2), $|p| \geq \dfrac{1}{2}$

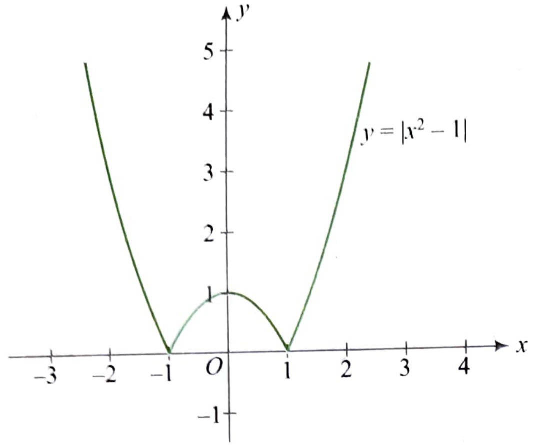
NON-MONOTONIC FUNCTION

A non-monotonic function is neither always decreasing nor always increasing.

e.g. $f(x) = \sin x$, which increases in first and fourth quadrant and decreases in second and third quadrants.



Consider function $f(x) = |x^2 - 1|$. The graph of the function is as shown in the following figure.



Clearly $f(x)$ is non-monotonic. $f(x)$ increases in $(-1, 0)$ and $(1, \infty)$ and decreases in $(-\infty, -1)$ and $(0, 1)$.

Critical Point

Critical point of the function is a point where either derivative is zero or derivative does not exist.

For differentiable function critical point is stationary point, where tangent to the curve is parallel to x-axis.

For $\sin x$, $x = n\pi$, $n \in Z$ are critical points.

For $|x^2 - 1|$, critical points are $x = 0$, where derivative is zero and $x = \pm 1$, where derivative do not exist.

Critical point is also called bifurcation point as it divides the domain of the function in the intervals of increase and decrease of the function.

In non-monotonic function we are interested in separating the intervals where it is increasing and decreasing. If function is differentiable then using differentiation it is easy to find these intervals.

ILLUSTRATION 6.14

Find the critical point(s) and stationary point(s) of the function $f(x) = (x - 2)^{2/3} (2x + 1)$

Sol. We have $f(x) = (x - 2)^{2/3}(2x + 1)$

$$f'(x) = 2(x - 2)^{2/3} + \frac{2}{3}\frac{2x+1}{(x-2)^{1/3}}$$

$$= \left[2(x - 2) + \frac{2}{3}(2x + 1)\right]\frac{1}{(x-2)^{1/3}}$$

$$= \frac{2(5x - 5)}{3(x - 2)^{1/3}}$$

$f'(x)$ does not exist at $x = 2$ and $f'(x) = 0$ at $x = 1$.

$\therefore$ $x = 1, 2$ are critical points and $x = 1$ is stationary point.

ILLUSTRATION 6.15

Separate the intervals of monotonocity of the following functions:

(i) $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 7$

(ii) $f(x) = -\sin^3 x + 3\sin^2 x + 5, x \in (-\pi/2, \pi/2)$

(iii) $f(x) = (2^x - 1)(2^x - 2)^2$

Sol.

(i) $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 7$

$f'(x) = 12x^3 - 24x^2 - 12x + 24$

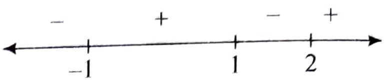
$= 12(x^3 - 2x^2 - x + 2)$

$= 12(x - 1)(x - 2)(x + 1)$

Now, $f'(x) = 0$ when $x = -1, 1$, and 2.

Hence, critical points are $-1, 1$, and 2.

The sign scheme of the derivative is given in the following figure.



Hence, the function increases in the interval $(-1, 1) \cup (2, \infty)$ and decreases in the interval $(-\infty, -1) \cup (1, 2)$.

(ii) $f(x) = -\sin^3 x + 3\sin^2 x + 5, x \in (-\pi/2, \pi/2)$

$\therefore$ $+f'(x) = -3\sin^2 x \cos x + 6\sin x \cos x$

$= 3\sin x \cos x (2 - \sin x)$

As $\cos x > 0$ and $2 - \sin x > 0\ \forall\ x \in (-\pi/2, \pi/2)$, $\sin x > 0$ $\forall\ x \in (0, \pi/2)$, and $\sin x < 0\ \forall\ x \in (-\pi/2, 0)$, $f'(x) > 0$, for $x \in (0, \pi/2)$, and $f'(x) < 0$, for $x \in (-\pi/2, 0)$.

Therefore, $f(x)$ is increasing in $(0, \pi/2)$ and decreasing in $(-\pi/2, 0)$.

(iii) $f(x) = (2^x - 1)(2^x - 2)^2$

$\therefore$ $f'(x) = 2^x \log 2\,(2^x - 2)^2 + 2(2^x - 2)\log 2(2^x - 1)$

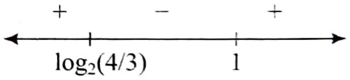
$= 2^x \log 2\,(2^x - 2)[(2^x - 2) + 2(2^x - 1)]$

$= 2^x \log 2\,(2^x - 2)[3 \times 2^x - 4]$

$2^x - 2 = 0$ or $x = 1$

$3 \times 2^x - 4 = 0$ or $x = \log_2(4/3)$

The sign scheme of $f'(x)$ is given in the following figure.



Thus, $f(x)$ is increasing in $(-\infty, \log_2(4/3)) \cup (1, \infty)$ and decreasing in $(\log_2(4/3), 1)$.

ILLUSTRATION 6.16

Find the intervals in which the function f given by

$f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is (a) increasing (b) decreasing.

$x \in (0, 2\pi)$

Sol. $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$

$$= \frac{4\sin x}{2 + \cos x} - \frac{2x + x\cos x}{2 + \cos x} = \frac{4\sin x}{2 + \cos x} - x$$

$$\therefore \quad f'(x) = \frac{4\cos x(2 + \cos x) - (-\sin x)(4\sin x)}{(2 + \cos x)^2} - 1$$

$$= \frac{4 + 8\cos x}{(2 + \cos x)^2} - 1$$

$$= \frac{4 + 8\cos x - (2 + \cos x)^2}{(2 + \cos x)^2}$$

$$= \frac{\cos x(4 - \cos x)}{(2 + \cos x)^2}$$

Now, $f'(x) = 0$

or $\cos x = 0$ (As $\cos x \neq 4$)

or $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\cos x > 0$ for $x \in (0, \pi/2) \cup (3\pi/2, 2\pi)$

and $\cos x < 0$ for $x \in (\pi/2, \pi) \cup (\pi, 3\pi/2)$

(a) Thus, $f(x)$ is increasing for $x \in (0, \pi/2) \cup (3\pi/2, 2\pi)$.

(b) Thus, $f(x)$ is decreasing for $x \in (\pi/2, \pi) \cup (\pi, 3\pi/2)$.

ILLUSTRATION 6.17

Find the interval of monotonocity of the function $f(x) = |x - 1|/x^2$.

Sol. $f(x) = \frac{|x-1|}{x^2} = \begin{cases} \frac{1-x}{x^2}, & x < 1, x \neq 0 \\ \frac{x-1}{x^2}, & x > 1 \end{cases}$

Clearly, $f(x)$ is continuous for all $x \in R$ except at $x = 0$.

$$f'(x) = \begin{cases} \frac{x-2}{x^3}, & x < 1, x \neq 0 \\ \frac{2-x}{x^3}, & x > 1 \end{cases}$$

$f'(x) > 0 \Rightarrow x < 0$ or $1 < x < 2$

$f'(x) < 0 \Rightarrow 0 < x < 1$ or $x > 2$

Hence, $f(x)$ is increasing in $(-\infty, 0) \cup (1, 2)$ and decreasing in $(0, 1) \cup (2, \infty)$.

ILLUSTRATION 6.18

Find the intervals of decrease and increase for the function $f(x) = \cos\left(\frac{\pi}{x}\right)$.

Sol. $f(x) = \cos\left(\frac{\pi}{x}\right)$. The function is defined for all x, where $x \neq 0$.

$$\therefore \quad f'(x) = -\sin\left(\frac{\pi}{x}\right)\pi\left(-\frac{1}{x^2}\right) = \frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right) \quad (1)$$

Therefore, f is differentiable for all x, $(x \neq 0)$.

Here, sign of $f'(x)$ is same as that of $\sin\left(\frac{\pi}{x}\right)$.

Thus, $f'(x)$ is positive if $\sin\left(\frac{\pi}{x}\right) > 0$ and $f'(x)$ is negative if $\sin\left(\frac{\pi}{x}\right) < 0$

or $\sin\left(\frac{\pi}{x}\right) > 0$ if $2k\pi < \frac{\pi}{x} < (2k+1)\pi,\ k \in Z$

and $\sin\left(\frac{\pi}{x}\right) < 0$, if $(2k+1)\pi < \frac{\pi}{x} < (2k+2)\pi,\ k \in Z$

Hence, the function f is increasing in the interval $\left(\frac{1}{2k+1}, \frac{1}{2k}\right)$ and decreasing in the interval $\left(\frac{1}{2k+2}, \frac{1}{2k+1}\right)$ (k being a non-negative integer).

ILLUSTRATION 6.19

Let $g(x) = f(x) + f(1 - x)$ and $f''(x) > 0$, $\forall x \in (0, 1)$. Find the intervals of increase and decrease of $g(x)$. It is given that $f(x)$ and $g(x)$ are differentiable functions.

Sol. We have, $g(x) = f(x) + f(1 - x)$ ...(1)

$\therefore \quad g'(x) = f'(x) - f'(1 - x)$

Given that $f''(x) > 0$, $\forall x \in (0, 1)$

It means that $f'(x)$ is increasing on $(0, 1)$.

Now $g(x)$ is increasing,

$\therefore \quad f'(x) - f'(1 - x) > 0$

$\Rightarrow \quad f'(x) > f'(1 - x)$

$\Rightarrow \quad x > 1 - x$ (as $f'(x)$ is increasing)

$\Rightarrow \quad \frac{1}{2} < x < 1$

So, $g(x)$ is increasing in $\left(\frac{1}{2}, 1\right)$

If $g(x)$ is decreasing,

$\therefore \quad f'(x) - f'(1 - x) < 0$

$\Rightarrow \quad f'(x) < f'(1 - x)$

$\Rightarrow \quad x < 1 - x$ (as $f'(x)$ is increasing)

$\Rightarrow \quad 0 < x < \frac{1}{2}$

So, $g(x)$ is decreasing in $\left(0, \frac{1}{2}\right)$

CONCEPT APPLICATION EXERCISE 6.1

1. Prove that the following functions are strictly increasing:
   (a) $f(x) = \cot^{-1} x + x$ **(b)** $f(x) = \log(1 + x) - \frac{2x}{2+x}$
2. Separate the intervals of monotonocity for the following functions:
   (a) $f(x) = -2x^3 - 9x^2 - 12x + 1$
   (b) $f(x) = x^2 e^{-x}$
   (c) $f(x) = \sin x + \cos x, x \in (0, 2\pi)$
   (d) $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3, x \in [0, \pi]$
   (e) $f(x) = (\log_e x)^2 + (\log_e x)$
3. Discuss monotonocity of $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \leq 1$.
4. Discuss monotonocity of $y = f(x)$ which is given by $x = \frac{1}{1+t^2}$ and $y = \frac{1}{t(1+t^2)}$, $t > 0$.
5. Find the value of a for which the function $(a + 2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all real x.
6. Find the value of a in order that $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ decreases for all real values of x.
7. Discuss the monotonocity of function $f(x) = 2\log|x - 1| - x^2 + 2x + 3$.
8. If $f(x) = \sin x + \log_e|\sec x + \tan x| - 2x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then check the monotonicity of $f(x)$.

9. Find the interval of the monotonicity of the function $f(x) = \log_e\left(\frac{\log_e x}{x}\right)$.

10. Let $g(x) = f(\log x) + f(2 - \log x)$ and $f''(x) < 0\ \forall\, x \in (0, 3)$. Then find the interval in which $g(x)$ increases.

ANSWERS

2. (a) Increasing: $(-2, -1)$
   Decreasing: $(-\infty, -2) \cup (-1, \infty)$
   (b) Increasing: $(0, 2)$
   Decreasing: $(-\infty, 0) \cup (2, \infty)$
   (c) Increasing: $(0, \pi/4) \cup (5\pi/4, 2\pi)$
   Decreasing: $(\pi/4, 5\pi/4)$
   (d) Increasing: $(\pi/2, 2\pi/3)$
   Decreasing: $(0, \pi/2) \cup (2\pi/3, \pi)$
   (e) Increasing: $(1/\sqrt{e}, \infty)$
   Decreasing: $(0, 1/\sqrt{e})$
3. $f(x)$ is increasing and $g(x)$ decreasing
4. Increasing
5. $(-\infty, -3]$
6. $[1, \infty)$
7. Increasing: $(-\infty, 0) \cup (1, 2)$
   Decreasing: $(0, 1) \cup (2, \infty)$
8. Strictly increasing
9. Increasing: $(1, e)$
   Decreasing: (e, ∞)
10. $(0, e)$

APPLICATIONS OF MONOTONICITY

FINDING RANGE OF THE FUNCTION AND NUMBER OF ROOTS OF THE EQUATION USING MONOTONICITY

One of the most important use of concept of monotonicity is finding the range of the function. Once it is known that function is monotonic in the given interval it is easy to find the range of the function by finding the value of the function at the end points of interval.

ILLUSTRATION 6.20

Find the range of the function $f(x) = x \sin x - \frac{1}{2}\sin^2 x$ for $x \in \left(0, \frac{\pi}{2}\right)$

Sol. We have $f(x) = x \sin x - \frac{1}{2}\sin^2 x$

$\therefore \quad f'(x) = x\cos x + \sin x - \sin x \cdot \cos x$

$= \sin x(1 - \cos x) + x\cos x$

Clearly $f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$.

$\therefore \quad f(x)$ is strictly increasing function.

$\therefore \quad$ Range of $f(x)$ is $\left(\lim_{x\to 0} f(x), \lim_{x\to\pi/2} f(x)\right) \equiv \left(0, \frac{\pi - 1}{2}\right)$

ILLUSTRATION 6.21

Find the range of $f(x) = \frac{1}{\pi}(\sin^{-1} x + \tan^{-1} x) + \frac{x+1}{x^2+2x+5}$

Sol. Clearly domain of the function is $[-1, 1]$.
Both $\sin^{-1}x$ and $\tan^{-1}x$ are increasing functions.

Also $\frac{d}{dx}\left(\frac{x+1}{x^2+2x+5}\right) = \frac{x^2+2x+5-(2x+2)(x+1)}{(x^2+2x+5)^2}$

$= \frac{-x^2-2x+3}{(x^2+2x+5)^2}$

$= \frac{-(x-1)(x+3)}{(x^2+2x+5)^2} \geq 0\ \forall\, x \in [-1, 1]$

$\therefore \quad f(x)$ is an increasing function.

Therefore range of $f(x)$ is $[f(-1), f(1)]$

$f(-1) = \frac{1}{\pi}\left(-\frac{\pi}{2} - \frac{\pi}{4}\right) + 0 = -\frac{3}{4}$

$f(1) = \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{4}\right) + \frac{1}{4} = 1$

Hence range of $f(x)$ is $\left[-\frac{3}{4}, 1\right]$.

ILLUSTRATION 6.22

Find the number of roots of the equation

$\log_e(1 + x) - \frac{\tan^{-1} x}{1+x} = 0.$

Sol. Consider $f(x) = (1 + x)\log_e(1 + x) - \tan^{-1}x, x > -1$

$\therefore \quad f'(x) = \log_e(1 + x) + 1 - \frac{1}{1+x^2}$

$\Rightarrow \quad f''(x) = \frac{1}{1+x} + \frac{2x}{(1+x^2)^2}$

$= \frac{(1+x)^2 + 3x^2 + x^4}{(1+x)(1+x^2)^2} > 0$

So, $f'(x)$ is increasing

$\therefore \quad$ For $x < 0, f'(x) < f'(0)$

$\Rightarrow \quad f'(x) < 0 \qquad (\text{as } f'(0) = 0)$

$\therefore \quad f(x)$ is decreasing function.

$\Rightarrow \quad f(x) > f(0) \qquad (\text{as } f(0) = 0)$

$\Rightarrow \quad f(x) > 0$

$\Rightarrow \quad (1 + x)\log_e(1 + x) - \tan^{-1}x > 0$

$\Rightarrow \quad \log_e(1 + x) > \frac{\tan^{-1}x}{x+1}$

For $\quad x > 0, f'(x) > f'(0)$

$\Rightarrow \quad f'(x) > 0$

$\Rightarrow \quad f(x)$ is increasing

$\Rightarrow \quad f(x) > f(0)$

$\Rightarrow \quad f(x) > 0$

Thus $\log_e(1+x) > \dfrac{\tan^{-1} x}{x+1}$

Since $f(x)$ is continuous and increasing, it crosses x-axis only once at $x = 0$.

Therefore, equation has only one real root $x = 0$

ILLUSTRATION 6.23

Find the number of roots of the function $f(x) = \dfrac{1}{(x+1)^3} - 3x + \sin x$.

Sol. $f(x) = \dfrac{1}{(x+1)^3} - 3x + \sin x,\ x \neq -1.$

$\therefore \quad f'(x) = \dfrac{-3}{(x+1)^4} - 3 + \cos x = \cos x - 3\left(1 + \dfrac{1}{(x+1)^4}\right)$

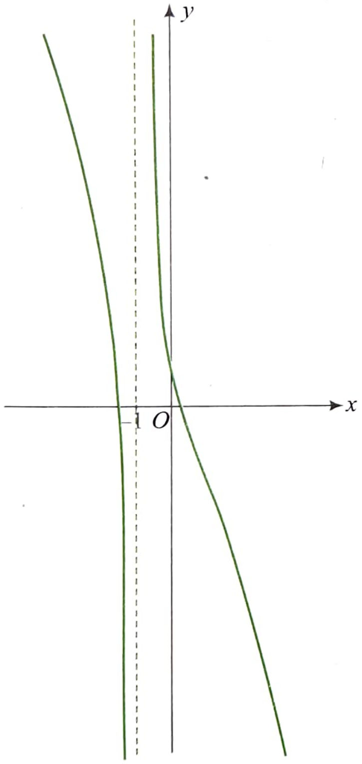
Now, $\cos x \in [-1, 1]$ and $3\left(1+\dfrac{1}{(x+1)^4}\right) > 3$

$\therefore \quad f'(x) < 0\ \forall\ x \in R$

So, $f(x)$ is decreasing function.

Also, $f(x)$ is discontinuous at $x = -1$

Graph of function is as shown in the following figure.



$\therefore \quad f(x) = 0$ has two roots one in each of the intervals $(-\infty, -1)$ and $(-1, \infty)$.

Proving Inequalities Using Monotonicity

One interesting application of monotonicity lies in proving some inequalities. To prove inequality, say $f(x) < g(x)$ in some interval, we consider function $h(x) = f(x) - g(x)$ or $g(x) - f(x)$ and study the nature of $h(x)$ w.r.t. monotonicity of $h(x)$ in that interval.

ILLUSTRATION 6.24

Prove that $\log_e(1+x) < x$ for $x > 0$.

Sol. Let $f(x) = \log_e(1+x) - x$

$\therefore \quad f'(x) = \dfrac{1}{1+x} - 1 = -\dfrac{x}{1+x}$

For $\quad x > 0, f'(x) < 0.$

So, $f(x)$ is strictly decreasing function for $x > 0$.

$\therefore \quad f(x) < f(0)$ for $x > 0$

or $\quad \log_e(1+x) - x < 0 \quad$ (as $f(0) = 0$)

$\therefore \quad \log_e(1+x) < x$ for $x > 0$.

ILLUSTRATION 6.25

Let f and g be differentiable on R and suppose $f(0) = g(0)$ and $f'(x) \le g'(x)$ for all $x \ge 0$. Then show that $f(x) \le g(x)$ for all $x \ge 0$.

Sol. Let $h(x) = f(x) - g(x)$

$h(0) = f(0) - g(0) = 0$

Now, $h'(x) = f'(x) - g'(x) \le 0$ for $x \ge 0$ (Given)

Thus, $h(x)$ is decreasing for $x \ge 0$. Now,

$x \ge 0$

$\therefore \quad h(x) \le h(0)$

$\therefore \quad h(x) \le 0$

$\therefore \quad f(x) - g(x) \le 0$

$\therefore \quad f(x) \le g(x)$

ILLUSTRATION 6.26

Show that $1 + x\log_e(x + \sqrt{x^2+1}) \ge \sqrt{1+x^2}$ for all $x \ge 0$.

Sol. Let $f(x) = 1 + x\log_e(x + \sqrt{x^2+1}) - \sqrt{1+x^2}$

$$\therefore \quad f'(x) = \log_e(x + \sqrt{x^2+1}) + x\frac{1 + \dfrac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} - \frac{x}{\sqrt{1+x^2}}$$

$$= \log_e(x + \sqrt{x^2+1}) \ge 0 \text{ for } x \ge 1$$

$\therefore \quad f(x)$ is an increasing function.

So, for $x \ge 0, f(x) \ge f(0)$

$\Rightarrow \quad 1 + x\log_e(x + \sqrt{x^2+1}) - \sqrt{1+x^2} \ge 0$

$\Rightarrow \quad 1 + x\log_e(x + \sqrt{x^2+1}) \ge \sqrt{1+x^2}$

ILLUSTRATION 6.27

If $a, b > 0$ and $0 < p < 1$ then prove that $(a+b)^p < a^p + b^p$.

Sol. Let $f(x) = (1+x)^p - 1 - x^p, x > 0$

$\therefore \quad f'(x) = p(1+x)^{p-1} - px^{p-1}$

$= p[(1+x)^{p-1} - x^{p-1}] \quad \ldots(1)$

Now $1 + x > x$

$\therefore \quad (1+x)^{1-p} > x^{1-p}$ $\quad (\because 1-p>0)$

$\Rightarrow \quad \dfrac{1}{(1+x)^{p-1}} > \dfrac{1}{x^{p-1}}$

$\Rightarrow (1+x)^{p-1} < x^{p-1}$

$\Rightarrow \quad (1+x)^{p-1} - x^{p-1} < 0$ $\quad \ldots(2)$

From (1) and (2) we get $f'(x) < 0$

So, $f(x)$ is a decreasing function.

Now $f(0) = 0$

$\therefore \quad x > 0$

$\Rightarrow \quad f(x) < f(0)$

$\Rightarrow \quad (1+x)^p - 1 - x^p < 0$

$\Rightarrow \quad (1+x)^p < 1 + x^p$

Putting $x = \dfrac{a}{b}$, we get $(a+b)^p < a^p + b^p$

ILLUSTRATION 6.28

Prove that $|\cos\alpha - \cos\beta| \le |\alpha - \beta|$.

Sol. Let $\alpha < \beta$

We have to prove that

$$\alpha - \beta \le \cos\alpha - \cos\beta \le \beta - \alpha$$

Consider $\alpha - \beta \le \cos\alpha - \cos\beta$

$\therefore \quad \cos\beta - \beta \le \cos\alpha - \alpha$

$\Rightarrow \quad f(\beta) \le f(\alpha)$, where $f(x) = \cos x - x$

Now $\quad f'(x) = -\sin x - 1 \le 0, \forall x \in R$.

$\therefore \quad f(x)$ is decreasing function.

$\Rightarrow \quad f(\beta) \le f(\alpha)$

$\Rightarrow \quad \cos\beta - \beta \le \cos\alpha - \alpha$

$\Rightarrow \quad \alpha - \beta \le \cos\alpha - \cos\beta$ $\quad \ldots(1)$

Now consider $\cos\alpha - \cos\beta \le \beta - \alpha$

$\therefore \quad \alpha + \cos\alpha \le \beta + \cos\beta$

$\Rightarrow \quad g(\alpha) \le g(\beta)$, where $g(x) = x + \cos x$

$\quad g'(x) = 1 - \sin x \ge 0, \forall x \in R$.

$\therefore \quad g(x)$ is an increasing function.

$\Rightarrow \quad g(\alpha) \le g(\beta)$

$\Rightarrow \quad \alpha + \cos\alpha \le \beta + \cos\beta$

$\Rightarrow \quad \cos\alpha - \cos\beta \le \beta - \alpha$ $\quad \ldots(2)$

From (1) and (2), we get $|\cos\alpha - \cos\beta| \le |\alpha - \beta|$

ILLUSTRATION 6.29

If $P(1) = 0$ and $\dfrac{dP(x)}{dx} > P(x)$ for all $x \ge 1$ then prove $P(x) > 0$ for all $x > 1$.

Sol. We are given that, $\dfrac{dP(x)}{dx} > P(x)$, $\forall x \ge 1$ and $P(1) = 0$

$\Rightarrow \quad \dfrac{dP(x)}{dx} - P(x) > 0$

Multiplying both sides by e^{-x}, we get

$$e^{-x}\dfrac{dP(x)}{dx} - e^{-x}P(x) > 0$$

$\Rightarrow \quad \dfrac{d}{dx}\left[e^{-x}P(x)\right] > 0$

$\Rightarrow \quad e^{-x}P(x)$ is an increasing function.

$\therefore \quad \forall x > 1, e^{-x}P(x) > e^{-1}P(1) = 0$ $\quad (\because P(1) = 0)$

$\Rightarrow \quad e^{-x}P(x) > 0, \forall x > 1$

$\Rightarrow \quad P(x) > 0, \forall x > 1$ $\quad (\because e^{-x} > 0)$

ILLUSTRATION 6.30

For $0 < x < \dfrac{\pi}{2}$, prove that $x > \sin x$ and hence $\cos(\sin x) > \sin(\cos x)$

Sol. Let $f(x) = x - \sin x$

$\therefore \quad f'(x) = 1 - \cos x > 0, \forall \in (0, \pi/2)$.

So, $f(x)$ is an increasing function.

$\therefore \quad$ for $x \in (0, \pi/2), f(x) > 0$

or $\quad x - \sin x > 0$

$\therefore \quad x > \sin x$ $\quad \ldots(1)$

Now $\quad \cos x \in (0, 1)$ for $x \in (0, \pi/2)$

So, from (1), we have (replacing x by $\cos x$)

$\quad \cos x > \sin(\cos x)$ $\quad \ldots(2)$

Also, $\cos x$ is decreasing in $(0, \pi/2)$

Therefore from (1), we have (applying cos both sides)

$\quad \cos x < \cos(\sin x)$ $\quad \ldots(3)$

From (2) and (3), we get

$\quad \sin(\cos x) < \cos x < \cos(\sin x)$

or $\quad \cos(\sin x) > \sin(\cos x)$

ILLUSTRATION 6.31

Prove that $e^x \ge 1 + x$ and hence $e^x + \sqrt{1+e^{2x}} \ge (1+x) + \sqrt{2+2x+x^2}$ $\forall x \in R$

Sol. Consider function $f(x) = e^x - 1 - x$

$\quad f'(x) = e^x - 1$

For $\quad x \ge 0, f'(x) \ge 0$.

$\therefore \quad f(x)$ is increasing function for $x \ge 0$.

$\therefore \quad$ for $x \ge 0, f(x) \ge f(0)$

or $\quad e^x - 1 - x \ge 0$

or $\quad e^x \ge 1 + x$

For $\quad x < 0, f'(x) < 0$

$\therefore \quad f(x)$ is decreasing function for $x < 0$.

$\therefore \quad$ for $x < 0, f(x) \ge f(0)$

or $\quad e^x - 1 - x \ge 0$

or $\quad e^x \ge 1 + x$

Thus, $e^x \ge 1 + x \ \forall x \in R$.

Now we have to prove that

$$e^x + \sqrt{1+e^{2x}} \ge (1+x) + \sqrt{1+(1+x)^2}$$

or $\quad f(e^x) \ge f(x+1)$, where $f(x) = x + \sqrt{1+x^2}$

Now $\quad f'(x) = 1 + \dfrac{x}{\sqrt{1+x^2}} = \dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}$

For $x > 0, f'(x) > 0$.

For $x < 0$, $|x| = -x$.

$$\therefore \quad \sqrt{1+x^2} + x = \sqrt{1+x^2} - |x|$$

$$= \sqrt{1+x^2} - \sqrt{x^2} > 0$$

Thus, for $x < 0, f'(x) > 0$

Therefore, $f(x)$ is an increasing function.

Since $e^x \ge x + 1$, we have

$$f(e^x) \ge f(x+1)$$

or $\quad e^x + \sqrt{1+e^{2x}} \ge (1+x) + \sqrt{1+(1+x)^2}$, $\forall x \in R$.

Concept Application Exercise 6.2

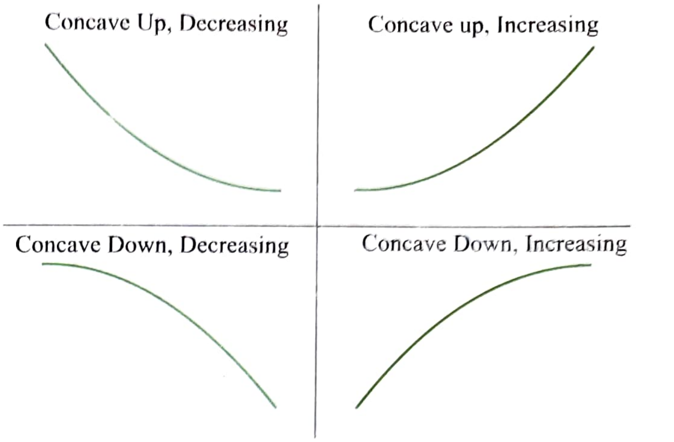
1. Find the range of $f(x) = \tan^{-1} x - \dfrac{1}{2}\log_e x$ in $\left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$
2. Find the range of $f(x) = \dfrac{\sin x}{x} + \dfrac{x}{\tan x}$ in $\left(0, \dfrac{\pi}{2}\right)$
3. Find the range of $f(x) = \sin^{-1}x - \cot^{-1}x + x^2 + 2x + 6$
4. Find the number of solutions of the equation $x^3 + 2x + \cos x + \tan x = 0$ in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
5. Show that $\dfrac{x}{(1+x)} < \ln(1+x)$ for $x > 0$.
6. For $0 < x \le \dfrac{\pi}{2}$, show that $x - \dfrac{x^3}{6} < \sin x < x$.
7. Show that $\tan^{-1} x > \dfrac{x}{1+\dfrac{x^2}{3}}$ if $x \in (0, \infty)$.
8. Show that $2\sin x + \tan x \ge 3x$ where $0 \le x < \dfrac{\pi}{2}$.
9. Prove that $f(x) = \dfrac{\sin x}{x}$ is monotonically decreasing in $[0, \pi/2]$. Hence, prove that $\dfrac{2x}{\pi} < \sin x < x$ for $x \in (0, \pi/2)$.
10. For $0 < x_1 < x_2 < \pi/2$, prove that $\dfrac{x_2}{x_1} < \dfrac{\tan x_2}{\tan x_1}$.

Answers

1. $\left[\dfrac{\pi}{3} - \log_e \sqrt{3}, \dfrac{\pi}{6} - \log_e \dfrac{1}{\sqrt{3}}\right]$
2. $\left(\dfrac{2}{\pi}, 2\right)$
3. $\left[5 - \dfrac{5\pi}{4}, 9 + \dfrac{\pi}{4}\right]$
4. one solution

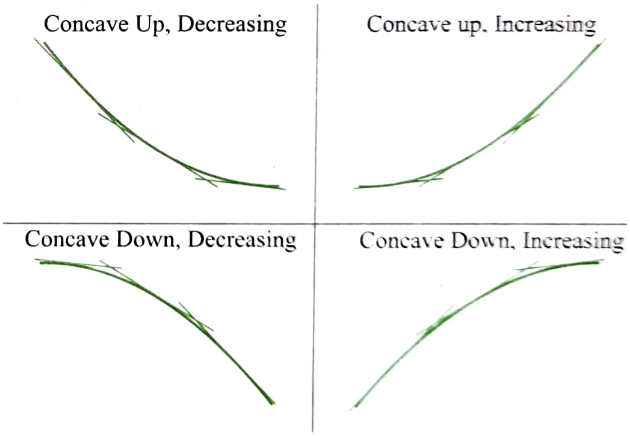
CONCAVITY OF CURVE AND POINT OF INFLECTION

Consider the graphs of some functions as shown in the following figure.



A function is concave up if it "opens" up and the function is concave down if it "opens" down. Also, we observe that concavity has nothing to do with increasing or decreasing. A function can be concave up and either increasing or decreasing. Similarly, a function can be concave down and either increasing or decreasing.

Mathematically, given function $y = f(x)$ is concave up on the interval I if all of the tangents to the curve on I are below the graph and function $y = f(x)$ is concave down on the interval I if all of the tangents to the curve on I are above the graph



We observe when the graph of the double differentiable function is concave upward, slope of tangent $\left(\dfrac{dy}{dx}\right)$ increases from left to right (with the increase in the value of x). So, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2} > 0$.

i.e., for $f(x) = e^x$, $f''(x) = e^x > 0$. So, graph of the function is concave upward.

Also, when graph of the double differentiable function is concave downward, slope of tangent $\left(\dfrac{dy}{dx}\right)$ decreases from left to right (with the increase in the value of x).

So, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d^2y}{dx^2}<0$.

i.e., for $f(x)=-x^2+x-2, f''(x)=-2<0$. So, graph of the function is concave downward. For $f(x)=\sin x, f''(x)=-\sin x<0$ for x lying in first and second quadrants, where graph is concave downward. Also $f''(x)>0$ for x lying in second and third quadrants, where graph is concave upward.

ILLUSTRATION 6.32

Separate the intervals of concavity of the following functions.
(i) $f(x)=\sin^{-1}x$, **(ii)** $f(x)=x+\sin x$

Sol.

(i) $f(x)=\sin^{-1}x$

$$\therefore \quad f'(x)=\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad f''(x)=\frac{x}{(1-x^2)^{3/2}}$$

$f''(x)>0$ for $x>0$ and $f''(x)<0$ for $x<0$.

So, graph is concave upward for $x>0$ and concave downward for $x<0$.

(ii) $f(x)=x+\sin x$

$\therefore \quad f'(x)=1+\cos x$

$\Rightarrow \quad f''(x)=-\sin x$

$f''(x)>0$ for $x\in((2n-1)\pi, 2n\pi)$, $n\in Z$, where graph is concave upward.

$f''(x)<0$ for $x\in(2n\pi, (2n+1)\pi)$, $n\in Z$, where graph is concave downward.

ILLUSTRATION 6.33

If graph of the function $f(x)=3x^4+2x^3+ax^2-x+2$ is concave upward for all real x, then find values of a.

Sol. We have $f'(x)=12x^3+6x^2+2ax-1$

and $\quad f''(x)=36x^2+12x+2a$

Now graph of concave upward for all real x.

$\therefore \quad f''(x)=36x^2+12x+2a>0$ for all real x.

$\therefore \quad$ Discriminant, $12^2-4(36)(2a)<0$

$\Rightarrow \quad 1-2a<0$

$\Rightarrow \quad a>1/2$

INEQUALITIES USING CONCAVITY

We have following important results, which will be useful in proving some inequalities.

(i) If graph of $y=f(x)$ is concave upward,

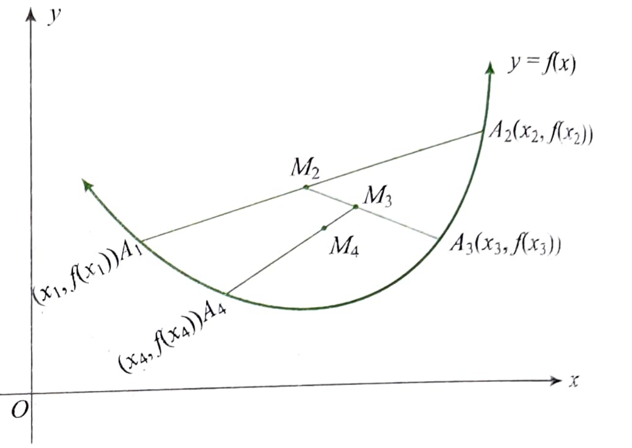
$$\frac{\sum_{i=1}^{n} f(x_i)}{n}\ge f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$$

(ii) If graph of $y=f(x)$ is concave downward

$$\frac{\sum_{i=1}^{n} f(x_i)}{n}\le f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$$

Proof:

Consider the following graph of $y=f(x)$, which is concave upward.



Consider points $A_1(x_1, f(x_1))$ and $A_2(x_2, f(x_2))$ on the graph.

Midpoint of A_1A_2 is $M_2\left(\dfrac{x_1+x_2}{2}, \dfrac{f(x_1)+f(x_2)}{2}\right)$

Now join M_2 and $A_3(x_3, f(x_3))$.

Let point M_3 on M_2A_3 be such that $M_2M_3 : M_3A_3=1:2$

So, coordinates of M_3 are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{f(x_1)+f(x_2)+f(x_3)}{3}\right)$$

Further on M_3A_4, consider point M_4 such that

$M_3M_4 : M_4A_4=1:3$

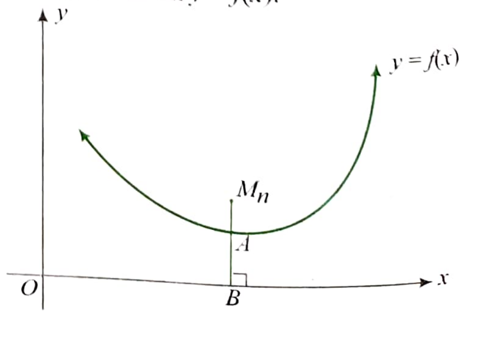
So, coordinates of M_4 are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{f(x_1)+f(x_2)+f(x_3)+f(x_4)}{4}\right)$$

Similarly, we get point M_n whose coordinates are

$$\left(\frac{\sum_{i=1}^{n} x_i}{n}, \frac{\sum_{i=1}^{n} f(x_i)}{n}\right).$$

All the points $M_2, M_3, \ldots, M_n$ lie the region which is above the graph of the function $y=f(x)$.



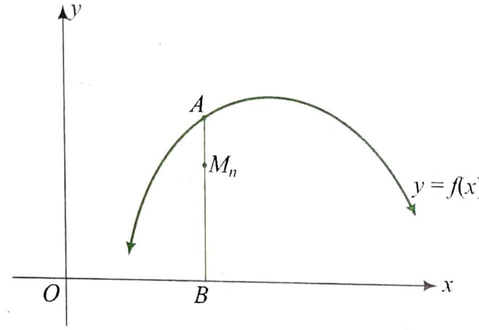
In the above figure from point M_n, draw vertical line, which intersect the graph of $y = f(x)$ at A and x-axis at B.

Coordinates of point A are $\left(\frac{\sum_{i=1}^{n} x_i}{n}, f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right)$.

Clearly $M_nB \geq AB$

$$\therefore \quad \frac{\sum_{i=1}^{n} f(x_i)}{n} \geq f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right).$$

When the graph of the function is concave downward point M_n lies in the region which is below the graph of the function $y = f(x)$.



Here $AB \geq M_nB$

$$\therefore \quad \frac{\sum_{i=1}^{n} f(x_i)}{n} \leq f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$$

ILLUSTRATION 6.34

Prove that for any two numbers x_1 and x_2,

$$\frac{2e^{x_1} + e^{x_2}}{3} > e^{\frac{2x_1 + x_2}{3}}.$$

Sol. Consider function $f(x) = e^x$.

Graph of the function is concave upward.

Now consider two points $A(x_1, e^{x_1})$ and $B(x_2, e^{x_2})$ on the graph of function.

Let C be another point which divides AB internally in ratio 1 : 2.

$\therefore$ coordinate of point C are $\left(\frac{2x_1 + x_2}{3}, \frac{2e^{x_1} + e^{x_2}}{3}\right)$

Since graph is concave upward,

$$f\left(\frac{2x_1 + x_2}{3}\right) < \frac{2e^{x_1} + e^{x_2}}{3}$$

or $$e^{\frac{2x_1 + x_2}{3}} < \frac{e^{2x_1} + e^{x_2}}{3}$$

ILLUSTRATION 6.35

Prove that in triangle ABC, $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$

Sol. Graph of $y = \sin x$ is concave downward.

$$\therefore \quad \frac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\frac{A+B+C}{3}\right)$$

$$\Rightarrow \quad \frac{\sin A + \sin B + \sin C}{3} \leq \sin\frac{\pi}{3}$$

$$\Rightarrow \quad \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

Here equality holds when $A = B = C$.

ILLUSTRATION 6.36

Prove that $\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$, if $m < 0$ or $m > 1$ and $a_i > 0$ for all i.

Sol. Consider function $y = f(x) = x^m$, where $x > 0$ and $m < 0$ or $m > 1$

$$\therefore \quad \frac{dy}{dx} = mx^{m-1} \text{ and } \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

Since $x > 0$ and $m < 0$ or $m > 1$, $\frac{d^2y}{dx^2} > 0$.

So, graph of function is concave upward.

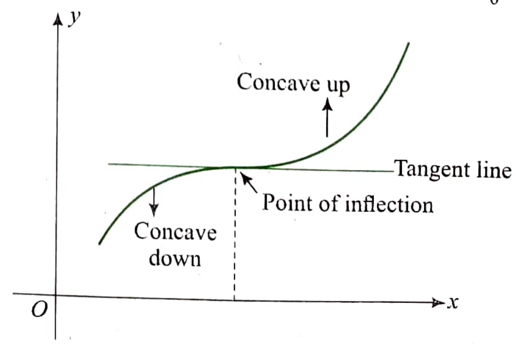
$$\text{Therefore} \quad \frac{\sum_{i=1}^{n} f(x_i)}{n} > f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$$

$$\text{or} \quad \frac{a_1^m + a_2^m + \ldots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \ldots\ldots + a_n}{n}\right)^m$$

POINT OF INFLECTION

A point on the graph where the curve changes its concavity (from up to down or from down to up) is called point of inflection.

For smooth curve, at point of inflection 'x_0' tangent crosses the curve. Also, second derivative $f''(x)$ changes sign in the neighborhood of point of inflection. So, second derivative is zero (if it exists), i.e., $f''(x_0) = 0$ at the point of inflection x_0.

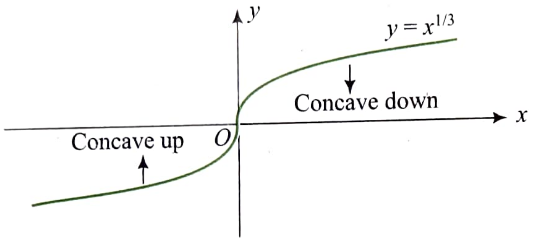


- But the converse of this is not true, i.e., if second derivative of function is zero at some point x_0, it is not necessary that it is point of inflection. A sufficient condition requires for x_0 to be point inflection, $f''(x_0 - \delta)$ and $f''(x_0 + \delta)$ to have opposite signs in the neighborhood of x_0.

For function $f(x) = x^4$, $f''(0) = 0$, but $x = 0$ is not a point of inflection as $f''(x) = 12x^2$ does not change sign in $(0 - \delta, 0 + \delta)$.

Consider function $f(x) = x^{1/3}$.

Here $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. Thus, $f(x)$ is non-differentiable at $x = 0$ as graph has vertical tangent at this point.



But we observe that curve changes its concavity at $x = 0$ as $f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$ changes its sign across $x = 0$. So, $x = 0$ is a point of inflection. Thus, it is not necessary that function is differentiable at the point of inflection.

Also, it is not necessary that at point of inflection function is continuous. Consider the graph of the following function which is discontinuous at $x = x_0$. But it is point of inflection as curve changes its concavity (second derivative changes its sign)

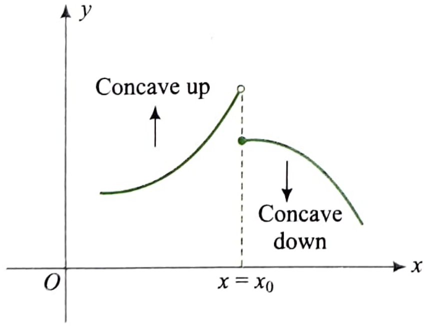


ILLUSTRATION 6.37

Find the points of inflection of the following functions.

(i) $f(x) = \sin x$ **(ii)** $f(x) = 3x^4 - 4x^3$

Sol.

(i) $f(x) = \sin x$

$\therefore \quad f'(x) = \cos x$

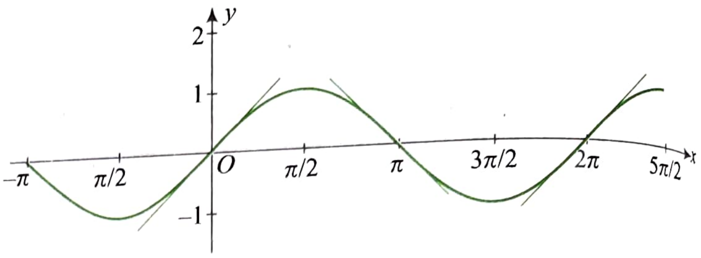
$\Rightarrow \quad f''(x) = -\sin x$

$f''(x) = 0$

$\Rightarrow \quad x = n\pi, n \in Z.$

So, points of inflection are $x = n\pi, n \in Z$.

At all points of inflection, tangent crosses the curve.



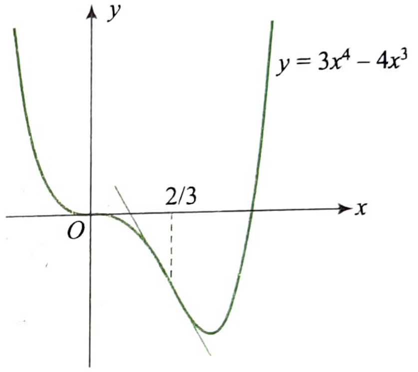
(ii) $f(x) = 3x^4 - 4x^3$

$\therefore \quad f'(x) = 12x^3 - 12x^2$

$\Rightarrow \quad f''(x) = 36x^2 - 24x$

Now $f''(x) = 0$

$\Rightarrow \quad x = 0, 2/3$, which are points of inflection.



We observe that tangents to the curve at $x = 0$ and $x = 2/3$ are crossing the curve.

ILLUSTRATION 6.38

Find the coordinates of the point of inflection of the curve $f(x) = e^{-x^2}$

Sol. We have $f(x) = e^{-x^2}$

$\therefore \quad f'(x) = -2xe^{-x^2}$

$\Rightarrow \quad f''(x) = -2(e^{-x^2}.1 + x.e^{-x^2}(-2x))$

If $f''(x) = 0$, then $2e^{-x^2}(2x^2 - 1) = 0$

$\Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$

When $x = \pm\frac{1}{\sqrt{2}}$, $y = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

Thus, the points of inflection are $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$.

CONCEPT APPLICATION EXERCISE 6.3

1. Show that graph of the function $(x) = \log_e(x - 2) - \frac{1}{x}$ always concave downwards.
2. Separate the interval of concavity of $y = x\log_e x - \frac{x^2}{2} + \frac{1}{2}$.

3. Consider, $f(x) = \cos 2x + 2x\lambda^2 + (2\lambda + 1)(\lambda - 1)x^2$, $\lambda \in R$. If $\alpha \neq \beta$ and $f\left(\frac{\alpha+\beta}{2}\right) < \frac{f(\alpha)+f(\beta)}{2}$ for all α and β, then find the values of λ.
4. Find the values of x where function $f(x) = (\sin x + \cos x)\, e^x$ in $(0, 2\pi)$ has point of inflection.
5. Prove that $\frac{a_1^m + a_2^m + ... + a_n^m}{n} < \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m$, if $0 < m < 1$ and $a_i > 0$ for all i.
6. In acute angled triangle ABC prove that $\sec A + \sec B + \sec C \geq 6$.

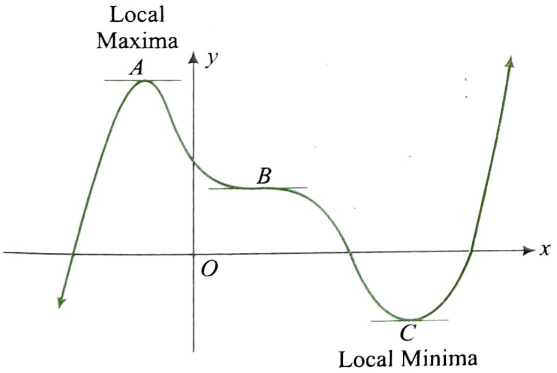
ANSWERS

2. Concave upward in $(0, 1)$
Concave downward in $(1, \infty)$
3. $(-\infty, -1] \cup [3/2, \infty)$
4. $x = \pi/4, 5\pi/4$

MAXIMA AND MINIMA OF FUNCTION

In calculus maxima/minima collectively called extrema of the function is one of the most powerful tools to study the functional behavior. Here maxima/minima of the function should not be confused with greatest and least value of the function in the given interval. In study of the function we are interested in finding the points of extrema. Often function has more than one points of extrema. For differentiable function these points are turning points on the graph of the function.

Consider the graph of one differentiable function as shown in the following figure.



We observe that points A, B and C are stationary points where tangent to the curve is parallel to x-axis, i.e., derivative is zero.

At points A and B the curve actually turns. These two stationary points are referred to as turning points.

Point C is not a turning point because, although the graph is flat for a short time, the curve continues to go down as we look from left to right. So, all turning points are stationary points. But not all stationary points are turning points (e.g. point C).

In other words, there are points for which derivative is zero but not turning points.

We have already learned that such point is called point of inflection where curve changes its concavity.

Point A in figure is called a point of local maxima because in its immediate area(neighborhood) it is the highest point, and so represents the greatest or maximum value of the function in that area.

Point B is called a local minimum because in its immediate area it is the lowest point, and so represents the least or minimum value of the function in that area.

Loosely speaking, we refer to a point of local maxima as simply a point of maxima.

Similarly, a point of local minimum is often just called a point of minima.

Consider following graph of the function which has several points of maxima and minima.

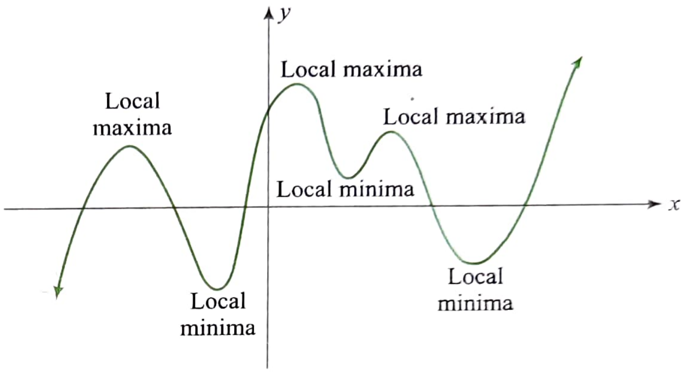


ILLUSTRATION 6.39

The function $y = \frac{ax+b}{(x-1)(x-4)}$ has turning point at $P(2, -1)$.

Then find the values of a and b.

Sol. $y = \frac{ax+b}{(x-1)(x-4)} = \frac{ax+b}{x^2-5x+4}$ has turning point at $P(2,-1)$.

Thus, $P(2, -1)$ lies on the curve. Therefore, $2a + b = 2$ (1)

Also, $\frac{dy}{dx} = 0$ at $P(2, -1)$.

Now, $\frac{dy}{dx} = \frac{a(x^2-5x+4)-(2x-5)(ax+b)}{(x^2-5x+4)^2}$.

At $P(2, -1)$, $\frac{dy}{dx} = \frac{-2a+2a+b}{4} = 0$

or $b = 0$ or $a = 1$ [From equation (1)]

ILLUSTRATION 6.40

How many critical points function $f(x) = (x-1)\,|x-3| - 4x + 12$ has? How many of these are points of local extrema?

Sol. $f(x) = (x-1)\,|x-3| - 4x + 12$

$$= \begin{cases} 9 - x^2, & x < 3 \\ x^2 - 8x + 15, & x \geq 3 \end{cases}$$

$$\therefore \quad f'(x) = \begin{cases} -2x & x < 3 \\ 2x - 8, & x > 3 \end{cases}$$

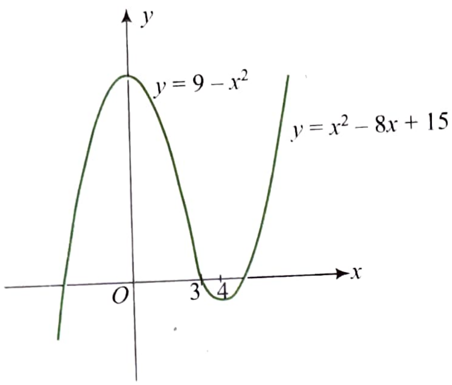
$$f'(3^-) = -6, f'(3^+) = -2.$$

So, $f(x)$ is non–differentiable at $x = 3$.

Also, $f'(x) = 0$ at $x = 0$ and $x = 4$

So, function has three critical points.

Graph of the function is as shown in the following figure.



At $x = 3$ derivative is not changing sign.

So, $x = 3$ is not a point of extrema.

At $x = 0$, derivative changes sign from positive to negative.

So, $x = 0$ is point of local maxima.

At $x = 3$, derivative changes sign from negative to positive.

So, $x = 3$ is point of local minima.

ILLUSTRATION 6.41

Discuss the extremum of the function $f(x) = (x - 1)|(x - 2)(x - 3)|$. How many critical points $f(x)$ has? Draw the graph of the function.

Sol. We have $f(x) = (x - 1)|x - 2||x - 3|$

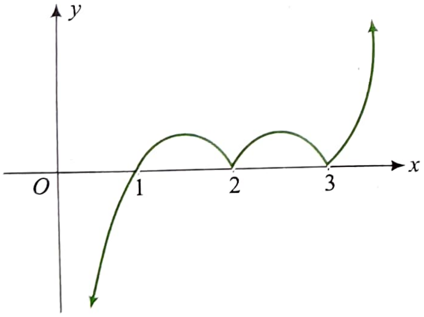
For such function we first draw the rough sketch of the graph.

Here sign scheme of $f(x)$ will help.

–	1	+	2	+	3	+

Sign of $f(x)$ is not changing across $x = 2$ and $x = 3$ as we have corresponding factors $|x - 2|$ and $|x - 3|$.

So, rough sketch of the graph is as follows.



Clearly $f(x)$ is non–differentiable at $x = 2$ and $x = 3$, which are critical points.

Also, $f(x)$ has one turning point in (1, 2) and one in (2, 3).

To, get these turning points, we consider

$$g(x) = (x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6 .$$

$$\therefore \quad g'(x) = 3x^2 - 12x + 11 = 0$$

$$\Rightarrow \quad x = \frac{12 \pm \sqrt{12}}{6} = 2 \pm \frac{1}{\sqrt{3}}$$

Clearly, both these values give point of maxima.

ILLUSTRATION 6.42

Find the points of extrema of the function $f(x) = 2 \sec x + 3 \operatorname{cosec} x$ in

(i) $(0, \pi/2)$ **(ii)** $(\pi, 3\pi/2)$

Sol. $f(x) = 2 \sec x + 3 \operatorname{cosec} x$

$$f'(x) = 2 \sec x \times \tan x - 3 \operatorname{cosec} x \times \cot x$$

Let $f'(x) = 0$,

$$\therefore \quad 2\frac{\sin x}{\cos^2 x} = 3\frac{\cos x}{\sin^2 x}$$

$$\Rightarrow \quad \tan^3 x = 3/2$$

$$\Rightarrow \quad \tan x = \left(\frac{3}{2}\right)^{1/3}$$

$$\Rightarrow \quad x = n\pi + \tan^{-1}\left(\frac{3}{2}\right)^{1/3}, n \in Z.$$

(i) if $x \in (0, \pi/2)$, then $x = \tan^{-1}\left(\frac{3}{2}\right)^{1/3}$

Now $\lim\limits_{x \to 0^+} (2\sec x + 3\operatorname{cosec} x) \to \infty$ and

$$\lim_{x \to \frac{\pi}{2}^-} (2\sec x + 3\operatorname{cosec} x) \to \infty$$

$$\therefore \quad x = \tan^{-1}\left(\frac{3}{2}\right)^{1/3} \text{ is point of minima.}$$

(ii) if $x \in (\pi, 3\pi/2)$, then $x = \pi + \tan^{-1}\left(\frac{3}{2}\right)^{1/3}$

Now $\lim\limits_{x \to \pi^+} (2\sec x + 3\operatorname{cosec} x) \to -\infty$

$$\lim_{x \to \frac{3\pi}{2}^-} (2\sec x + 3\operatorname{cosec} x) \to -\infty$$

$$\therefore \quad x = \pi + \tan^{-1}\left(\frac{3}{2}\right)^{1/3} \text{ is point of maxima.}$$

ILLUSTRATION 6.43

Prove that for all $a, b \in R$ the function $f(x) = 3x^4 - 4x^3 + 6x^2 + ax + b$ has exactly one extremum

Sol. We have $f'(x) = 12(x^3 - x^2 + x) - a$

and $f''(x) = 12(3x^2 - 2x + 1) > 0$

$\Rightarrow$ $f'(x)$ is an increasing function.

Since $f'(x)$ is polynomial of degree three, $f'(x) = 0$ has exactly one real root.

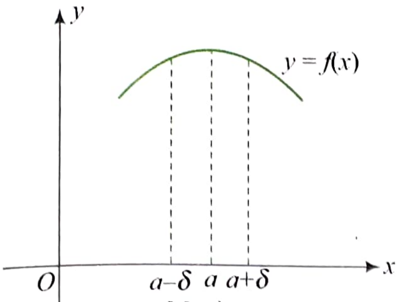
So, function has exactly one extremum for all real values of a and b.

IDENTIFYING MAXIMA AND MINIMA BY CHECKING VALUE OF FUNCTION IN THE NEIGHBOURHOOD OF POINT OF EXTREMA

From the above discussion, we have following findings.

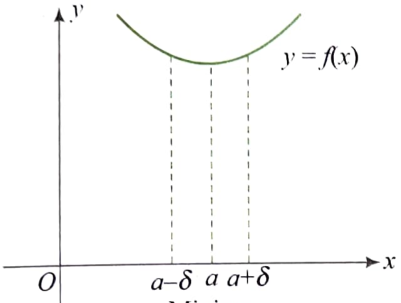
- A function $f(x)$ is said to have point of maxima at $x = a$ if $f(a)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x = a$.

  i.e., $f(a) > f(a - \delta)$ and $f(a) > f(a + \delta)$, where $\delta \to 0^+$



Maxima

- A function $f(x)$ is said to have a point of minima at $x = a$ if $f(a)$ is smaller than every other value assumed by $f(x)$ in the immediate neighbourhood of $x = a$.
i.e., $f(a) < f(a - \delta)$ and $f(a) < f(a + \delta)$, where $\delta \to 0^+$



Minima

The above definition of maxima/minima works well with discontinuous and non–differentiable functions also.

ILLUSTRATION 6.44

The function $f(x) = (x^2 - 4)^n (x^2 - x + 1)$, $n \in N$, assumes a local minimum value at $x = 2$. Then find the possible values of n.

Sol. $f(x) = (x^2 - 4)^n (x^2 - x + 1)$

$f(2) = 0,$

Now, $x^2 - x + 1 > 0 \ \forall \ x$

$$f(2^+) = \lim_{x \to 2^+} (x^2-4)^n (x^2 - x + 1)$$

$$= 3 \lim_{h \to 0} ((h+2)^2 - 4)^n$$

$$= 3 \lim_{h \to 0} (4h + h^2)^n$$

$$> 0$$

$$f(2^-) = \lim_{x \to 2^-} (x^2-4)^n (x^2 - x + 1)$$

$$= 3 \lim_{h \to 0} ((h-2)^2 - 4)^n$$

$$= 3 \lim_{h \to 0} (h^2 - 4h)^n$$

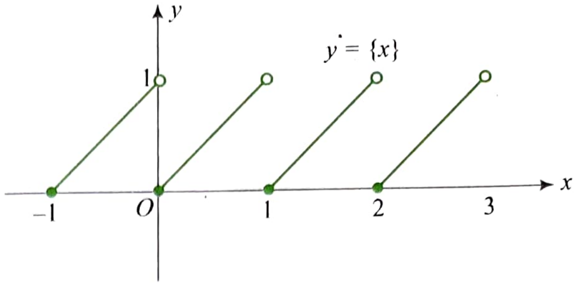
$$= 3 \times (\text{very small negative value})^n$$

For $x = 0$ to be a point of minima, we must have $f(2^-) > 0$ for which n must be an even integer.

ILLUSTRATION 6.45

Test $f(x) = \{x\}$ for the existence of a local maximum and minimum at $x = 1$, where $\{\cdot\}$ represents the fractional part function.

Sol.



We know that $f(x) = \{x\}$ is discontinuous at integral values of x and hence at $x = 1$.

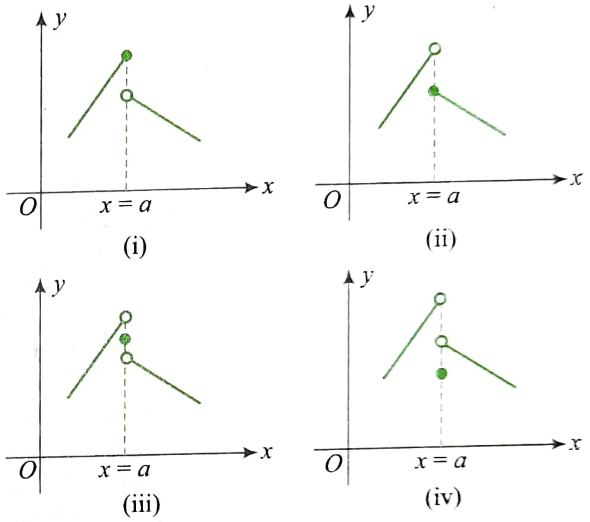
Now $f(1) = 0$ and $f(1^+) = 0^+$ and $c = 1^-$

Thus $f(1) < f(1^+)$ and $f(1) < f(1^-)$

So, $x = 1$ is point of minima.

ILLUSTRATION 6.46

Consider the following graphs of the functions. Check each for the extrema at $x = a$.



Sol.

(i) $f(a) > f(a - \delta)$ and $f(a) > f(a - \delta)$
So, $x =$ a is point of maxima.

(ii) $f(a) < f(a - \delta)$ and $f(a) > f(a + \delta)$
So, $x = a$ is neither point of maxima nor point of minima.

(iii) $f(a) < f(a - \delta)$ and $f(a) > f(a + \delta)$
So, $x = a$ is neither point of maxima nor point of minima.

(iv) $f(a) < f(a - \delta)$ and $f(a) < f(a + \delta)$.
So, $x = a$ is point of minima.

ILLUSTRATION 6.47

Let $f(x) = \begin{cases} x^3 + x^2 + 10x, & x < 0 \\ -3\sin x, & x \geq 0 \end{cases}$. Investigate $x = 0$ for local maxima/minima.

Sol. Here $f(0) = -3 \sin 0 = 0$

$$\lim_{x \to 0^-} [x(x^2 + x + 10)] = 0^-$$

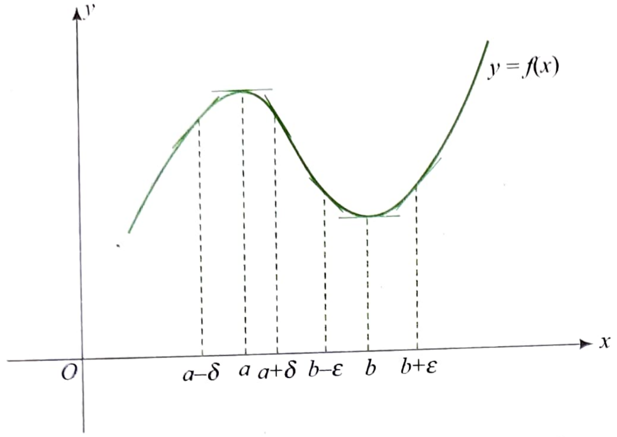
$$\lim_{x \to 0^+} (-3 \sin x) = 0^-$$

Thus $f(0) > f(0 - \delta)$ and $f(0) > f(0 + \delta)$

So, $x = 0$ is point of maxima.

Second Derivative Test for Maxima and Minima of Differentiable Function

Consider the graph of one differentiable function having point of maxima and minima as shown in the following figure.



In the above figure $x = a$ is point of maxima and $x = b$ is point of minima.

We observe that slope of the tangent decreases from positive to negative (with slope zero at $x = a$) if we move left to right in the interval $(a - \delta, a + \delta)$.

In other words, derivative $\frac{dy}{dx}$ of the function decreases in $(a - \delta, a + \delta)$.

i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} < 0$ in $(a - \delta, a + \delta)$.

Thus, at point of maxima $\frac{d^2y}{dx^2} < 0$ or $f''(a) < 0$.

Now slope of the tangent increases from negative to positive (with slope zero at $x = b$) if we move left to right in the interval $(b - \varepsilon, b + \varepsilon)$.

In other words, derivative $\frac{dy}{dx}$ of the function increases in $(b - \varepsilon, b + \varepsilon)$.

i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} > 0$ in $(b - \varepsilon, b + \varepsilon)$.

Thus, at point of minima $\frac{d^2y}{dx^2} > 0$ or $f''(b) > 0$.

It is important to realize that this test for a minima/maxima is not conclusive.

It is possible for a stationary point to be a minima/maxima even if $\frac{d^2y}{dx^2} = 0$.

e.g. for function $f(x) = x^4, f''(x) = 12x^2$.

$\therefore$ $f''(0) = 0$, still function has point of minima at $x = 0$.

In such cases we either use the previous method or refer the graph or use the change of sign of derivative method.

Illustration 6.48

Let $f(x) = \frac{a}{x} + x^2$. If it has minima at $x = -3$ then find the value of a.

Sol. $f'(x) = -\frac{a}{x^2} + 2x$

For $f'(x) = 0,\ x^3 = \frac{a}{2}$

For $x = -3,\ a = -54$

Now $f''(x) = \frac{2a}{x^3} + 2$

$\Rightarrow$ $f''(-3) = \frac{-108}{(-3)^3} + 2 = 6 > 0$

So, $f(x)$ has minima at $x = -3$ if $a = -54$.

Illustration 6.49

Discuss the extremum of $f(x) = \sin x\,(1 + \cos x),\ x \in (0, \pi/2)$.

Sol. Let $f(x) = \sin x\,(1 + \cos x)$

$\therefore$ $f'(x) = \cos 2x + \cos x$

and $f''(x) = -2\sin 2x - \sin x = -(2\sin 2x + \sin x)$

For maximum or minimum value of $f(x)$, $f'(x) = 0$, i.e.,

or $\cos 2x + \cos x = 0$

or $\cos x = -\cos 2x$

or $\cos x = \cos(\pi \pm 2x)$

$\therefore$ $x = \pi \pm 2x$ or $x = \frac{\pi}{3}$

Now, $f''\left(\frac{\pi}{3}\right) = -2\sin\frac{2\pi}{3} - \sin\frac{\pi}{3}$

$= -2\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2} = -\text{ve}$

Hence, $f(x)$ has maxima at $x = \frac{\pi}{3}$.

Illustration 6.50

Discuss the extremum of $f(x) = x^2 + \frac{1}{x^2}$.

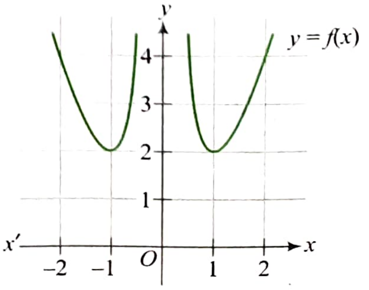
Sol. $f(x) = x^2 + \frac{1}{x^2}$

$f'(x) = 2x - \frac{2}{x^3}$

Let $f'(x) = 0$. Then $x^4 = 1$ or $x = \pm 1$.

Also, $f''(x) = 2 + \frac{6}{x^4} > 0$ for all $x \neq 0$.

Thus, both the points, $x = 1$ and $x = -1$, are the points of minima.



Note:
Here, two consecutive points of extrema are minima. This is because $f(x)$ is discontinuous at $x = 0$. However, dis-continuous function can also have two consecutive points of extrema of which one is maxima and the other minima, e.g., for $f(x) = x + \frac{1}{x}$. For continuous function, consecutive points of extrema are maxima and minima.

First Derivative Test for Maxima/Minima of Differentiable Function

If second derivative is zero at point of extrema or it is difficult to find the second derivative, we use first derivative test to find the point of maxima or minima.

In this method we differentiate the function and find the stationary point.

A stationary point across which derivative changes sign from negative to positive (moving from left to right) is point of minima.

A stationary point across which derivative changes sign from positive to negative (moving from right to left) is point of maxima.

A stationary point across which derivative does not change the sign is called point of inflection.

Sign scheme of derivative helps in finding points of maxima, minima and inflection.

Illustration 6.51

Find the points at which the function f given by $f(x) = (x-2)^4 (x+1)^3$ has

(i) local maxima (ii) local minima
(iii) point of inflection

Sol. The given function is $f(x) = (x-2)^4(x+1)^3$

$$\therefore \quad f'(x) = 4(x-2)^3(x+1)^3 + 3(x+1)^2(x-2)^4$$
$$= (x-2)^3(x+1)^2[4(x+1) + 3(x-2)]$$
$$= (x-2)^3(x+1)^2(7x-2)$$

Now, $f'(x) = 0 \Rightarrow x = -1$ and $x = \frac{2}{7}$ or $x = 2$

Sign scheme of $f'(x)$ is as shown in the following figure.

+   +   −   +
—— -1 —— $\frac{2}{7}$ —— 2 ——

(a) From the figure, $x = 2$ is the point of local minima as derivative changes sign from "–" to "+."

(b) $x = \frac{2}{7}$ is the point of local maxima as derivative changes sign from "+" to "–."

(c) $x = -1$ is point of inflection as derivative does not change sign.

Illustration 6.52

Discuss the extremum of $f(x) = 40/(3x^4 + 8x^3 - 18x^2 + 60)$.

Sol. $f(x) = \dfrac{40}{3x^4 + 8x^3 - 18x^2 + 60}$

$$\therefore \quad f'(x) = -\frac{40(12x^3 + 24x^2 - 36x)}{(3x^4 + 8x^3 - 18x^2 + 60)^2}$$
$$= -\frac{480x(x^2 + 2x - 3)}{(3x^4 + 8x^3 - 18x^2 + 60)^2}$$
$$= \frac{-480x(x-1)(x+3)}{(3x^4 + 8x^3 - 18x^2 + 60)^2}$$

The sign scheme of $f'(x)$ is given in the following figure.

+   −   +   −
—— -3 —— 0 —— 1 ——

Hence, $x = -3$ and $x = 1$ are the points of maxima and $x = 0$ is the point of minima.

Illustration 6.53

Discuss extrema of the function

$$f(x) = \int_1^x (2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2)\,dx.$$

Sol. We have

$$f'(x) = 2(x-1)(x-2)3 + 3(x-1)2(x-2)2$$
$$= (x-1)(x-2)^2(5x-7)$$

Signe scheme of $f'(x)$ is as shown in the following figure.

+   −   +   +
—— -1 —— $7/5$ —— 2 ——

Clearly, $f(x)$ has point of maxima at $x = 1$ and point of minima at $x = 7/5$.

Also, sign of derivative is not changing at $x = 2$, so it is not a point of extrema.

Illustration 6.54

Let $f(x) = \sin^3 x + \lambda \sin^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lies in order that $f(x)$ has exactly one minimum and exactly one maximum.

Sol. $f(x) = \sin^3 x + \lambda \sin^2 x$

$$\therefore \quad f'(x) = 3\sin^2 x \times \cos x + \lambda(2\sin x \times \cos x)$$
$$= \sin x \cos x (3\sin x + 2\lambda)$$

For extrema, $f'(x) = 0$

$\therefore \quad \sin x = 0$, $\cos x = 0$, $\sin x = -2\lambda/3$

Since $-\pi/2 < x < \pi/2$, $\cos x \neq 0$

$\therefore \quad \sin x = 0 \Rightarrow x = 0$ and $\sin x = \frac{-2\lambda}{3}$

So, we must have $-1 < \frac{-2\lambda}{3} < 1$

$\Rightarrow \quad -3 < 2\lambda < 3$

$\Rightarrow \quad -3/2 < \lambda < 3/2$

But if $\lambda = 0$ then $\sin x = 0$ has only one solution.

$\therefore$ for two points of extrema, $\lambda \in (-3/2, 3/2) - \{0\}$

For these values of λ there are two points of extrema.

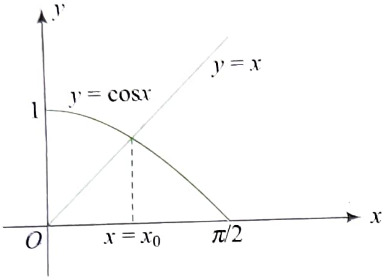
ILLUSTRATION 6.55

Discuss extrema of $f(x)=\dfrac{x}{1+x\tan x},\ x\in\left(0,\dfrac{\pi}{2}\right)$.

Sol. $f'(x)=\dfrac{1-x^2\sec^2 x}{(1+x\tan x)^2}=\dfrac{\sec^2 x(\cos x+x)(\cos x-x)}{(1+x\tan x)^2}$

If $f'(x)=0$, then $\cos x=x$.

Draw the graphs of $y=x$ and $y=\cos x$ for $x\in\left(0,\dfrac{\pi}{2}\right)$.



From the figure, graph intersect at $x=x_0$.

For $x<x_0$, $\cos x>x$ or $\cos x-x>0$.

For $x>x_0$, $\cos x<x$ or $\cos x-x<0$.

Thus, derivative changes sign from positive to negative across $x=x_0$.

So, $x=x_0$ is point of maxima.

ILLUSTRATION 6.56

Find the point of extrema of the function $f(x)=\left(\dfrac{1}{x}\right)^x$. Draw the graph of the function and hence find the range of the function. Also determine which is bigger, $\left(\dfrac{1}{\pi}\right)^{\frac{1}{e}}$ or $\left(\dfrac{1}{e}\right)^{\frac{1}{\pi}}$?

Sol. $f(x)=\left(\dfrac{1}{x}\right)^x$

$\therefore\quad f'(x)=\left(\dfrac{1}{x}\right)^x\left(\log_e\dfrac{1}{x}-1\right)$

$f'(x)=0\Rightarrow\quad \log_e\dfrac{1}{x}=1\ \Rightarrow x=\dfrac{1}{e}$

Sign scheme of $f'(x)$ is as follows:

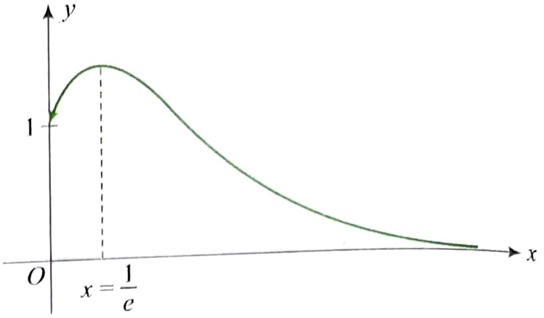
$$\begin{array}{ccc} + & & - \\ \hline & 1/e & \end{array}$$

From the sign scheme $x=\dfrac{1}{e}$ is point of maxima.

Also $\lim\limits_{x\to0}\left(\dfrac{1}{x}\right)^x=e^{\lim\limits_{x\to0}x\log\left(\frac{1}{x}\right)}=e^{-\lim\limits_{x\to0}x\log x}=e^0=1$

and $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)^x=0$

So, graph of the function is as shown in the following figure.



From the graph range of the function is $(0, f(1/e)]$ or $(0, e^{1/e})$

Now $\pi>e$

$\therefore\quad f(\pi)<f(e)$

$\Rightarrow\quad \left(\dfrac{1}{\pi}\right)^{\pi}<\left(\dfrac{1}{e}\right)^{e}$

$\Rightarrow\quad \left(\dfrac{1}{\pi}\right)^{\frac{1}{e}}<\left(\dfrac{1}{e}\right)^{\frac{1}{\pi}}$

TESTS FOR MAXIMA AND MINIMA OF DISCONTINUOUS AND NON–DIFFERENTIABLE FUNCTIONS

Case I: When function is discontinuous at $x=a$ with $f(a)$ exists finitely.

Here we use most elementary definition of maxima and minima i.e., if $f(a)>f(a-\delta)$ and $f(a)>f(a+\delta)$, $x=a$ is point of maxima. If $f(a)<f(a-\delta)$ and $f(a)<f(a+\delta)$, $x=a$ is point of minima.

Case II: When function is continuous and non–differentiable at $x=a$.

Here we use either method of above case or we check change in sign of derivative across $x=a$.

i.e., if $f'(a-\delta)>0$ and $f'(a+\delta)<0$, then $x=a$ is point maxima and if $f'(a-\delta)<0$ and $f'(a+\delta)>0$, then $x=a$ is point minima.

In some of the cases graph of the function also helps.

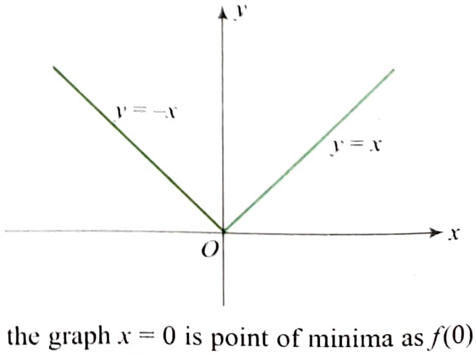
ILLUSTRATION 6.57

Discuss the extrema of the following functions

(i) $f(x)=|x|$ **(ii)** $f(x)=e^{-|x|}$ **(iii)** $f(x)=x^{2/3}$

Sol.

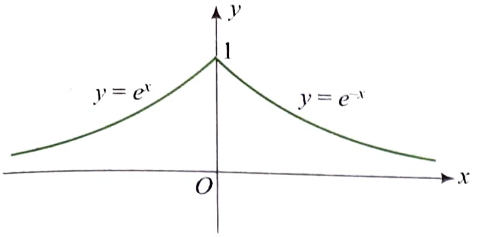
(i) $f(x)=|x|$



From the graph $x=0$ is point of minima as $f(0)<f(0-\delta)$ and $f(0)<f(0+\delta)$.

(ii) $f(x) = e^{-|x|}$

$$f(x) = \begin{cases} e^{-x}, & x \ge 0 \\ e^{x}, & x < 0 \end{cases}$$



From the graph $x = 0$ is point of maxima

(iii) $f(x) = x^{2/3}$.

$$f'(x) = \frac{2}{3x^{\frac{1}{3}}}.$$

Clearly $f'(0)$ does not exist as function has vertical tangent at $x = 0$.

$f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.

$\therefore$ $f(x)$ has point of minima at $x = 0$.

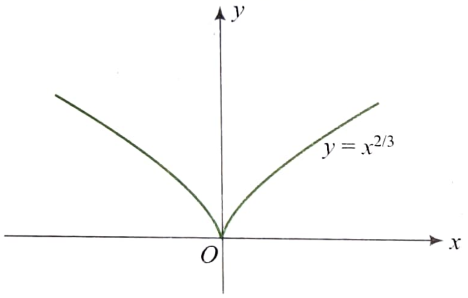


ILLUSTRATION 6.58

If $f(x) = \begin{cases} x^2 & x \le 0 \\ 2\sin x & x > 0 \end{cases}$. Investigate the function at $x = 0$ for maxima/minima.

Sol. Clearly $f(0) = 0, f(0^+) = \lim\limits_{x \to 0^+} (2\sin x) = 0^+$

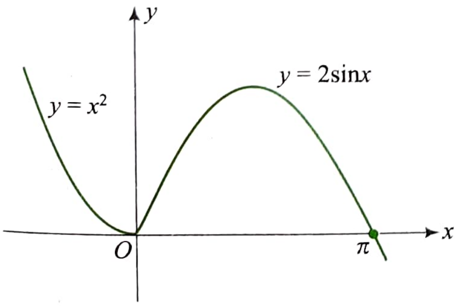
Also $f(0^-) = \lim\limits_{x \to 0^-} x^2 = 0^+$

Thus, $f(x)$ is continuous at $x = 0$.

$f(0) < f(0 + \delta)$ and $f(0) < f(0 - \delta)$.

Thus $x = 0$ is point of minima.

We can draw the graph of the function to verify this.



Also, $f'(x) = \begin{cases} 2x & x \le 0 \\ 2\cos x & x > 0 \end{cases}$

$\therefore$ $f'(0 - \delta) < 0$ and $f'(0 + \delta) > 0$.

Thus derivative changes sign from negative to positive with $f(0) = 0$.

So, $x = 0$ is point of minima.

Note:

Change in sign of derivative method does not always work with discontinuous functions.

$$\text{Consider function } f(x) = \begin{cases} x^2, & x < 0 \\ 5, & x = 0 \\ 2\sin x, & x > 0 \end{cases}$$

Here derivative changes sign from negative to positive across $x = 0$.

But $x = 0$ is not a point of minima as $f(0) > f(0 - \delta)$ and $f(0) > f(0 + \delta)$.

Thus $x = 0$ is point maxima.

ILLUSTRATION 6.59

Discuss extremum of $f(x) = 2x + 3x^{2/3}$.

Sol. $f(x) = 2x + 3x^{2/3}$

$$f'(x) = 2 + 3 \times \frac{2}{3} x^{-1/3} = 2(1 + x^{-1/3})$$

If $f'(x) = 0$, then $x^{-1/3} + 1 = 0$ or $x = -1$

$$f''(x) = -\frac{2}{3} x^{-\frac{4}{3}}$$

$$\therefore \quad f''(-1) = -\frac{2}{3}(-1)^{-4/3} = -\frac{2}{3} < 0$$

So, $x = -1$ is point of maxima.

Also, $f(x)$ is non–differentiable at $x = 0$.

$f(x) = x^{2/3}(3 + x^{1/3})$

$f(0) = 0$ and $f(0^+) > 0$ and $f(0^-) > 0$

$\therefore$ $x = 0$ is point of minima.

We can draw the graph of the function to verify this.

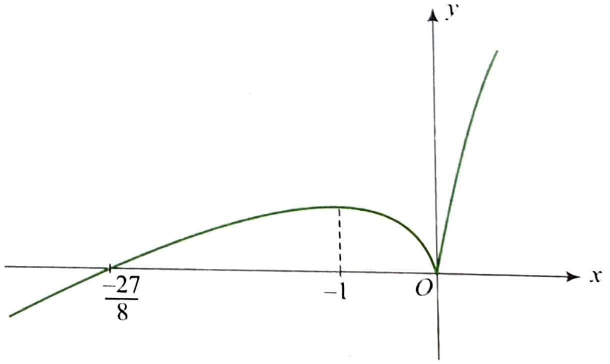


ILLUSTRATION 6.60

$f(x) = \begin{cases} \cos \dfrac{\pi x}{2}, & x > 0 \\ x + a, & x \le 0 \end{cases}$. Find the values of a if $x = 0$ is point of maxima.

Sol. For $x < 0$, $f(x) = x + a$, which is increasing.

For $0 < x < 2$, $f(x) = \cos\dfrac{\pi x}{2}$, which is decreasing.

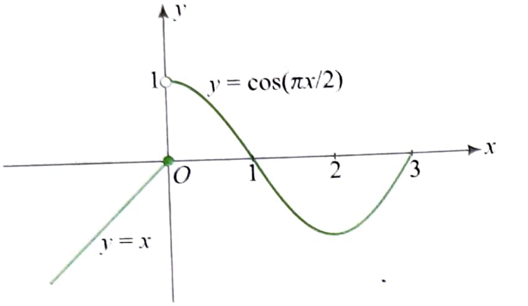
$\therefore$ $x = 0$ is point of maxima if $f(0) \geq \lim\limits_{x\to 0+} f(x)$

$\Rightarrow$ $f(0) \geq \lim\limits_{x\to 0+} \cos\dfrac{\pi x}{2}$

$\Rightarrow$ $a \geq 1$

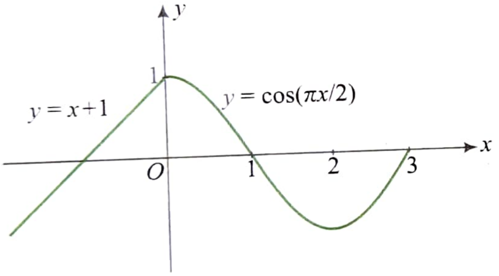
Graphical Method:

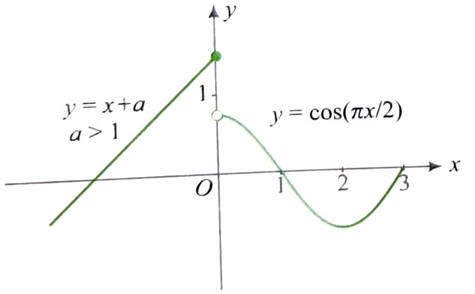
First set $a = 0$. Then graph of the function is as follows.



From the figure, $x = 0$ is not a point of maxima.

For $x = 0$ to be point of maxima graph of $y = x$, must be shifted upward at least by '1' unit.





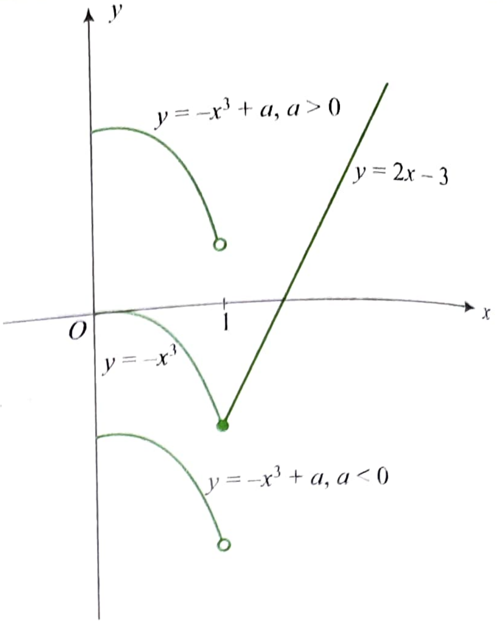
So, we must have $a \geq 1$

ILLUSTRATION 6.61

Let $f(x) = \begin{cases} -x^3 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}, & 0 \leq x < 1 \\ 2x - 3, & 1 \leq x \leq 3 \end{cases}$. Find all possible real value of b such that $f(x)$ has the smallest value at $x = 1$.

Sol. Let $\dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} = a$

If $a = 0$ then $f(x) = \begin{cases} -x^3, & 0 \leq x < 1 \\ 2x - 3, & 1 \leq x \leq 3 \end{cases}$



Clearly when $a = 0$, $x = 1$ is point of minima.

If $a > 0$, then graph of $y = -x^3 + a$ shifts upward, then also $x =$ 1 remains point of minima.

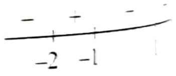
If $a < 0$, then graph of $y = -x^3 + a$ shifts downward. In this case $x = 1$ is not a point of minima.

So, for point of minima at $x = 1$, $a \geq 0$

$$\Rightarrow \quad \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \geq 0$$

$$\Rightarrow \quad \frac{(b^2 + 1)(b - 1)}{(b + 2)(b + 1)} \geq 0$$

Sign scheme of $\dfrac{(b^2 + 1)(b - 1)}{(b + 2)(b + 1)}$



From the sign scheme, $b \in (-2, -1) \cup [1, \infty)$

ILLUSTRATION 6.62

The function $f(x) = |ax - b| + c\,|x|\ \forall\ x \in (-\infty, \infty)$, where $a > 0$, $b > 0$, $c > 0$. Find the condition if $f(x)$ attains minimum value only at one point.

Sol. We have $f(x) = |ax - b| + c|x|$

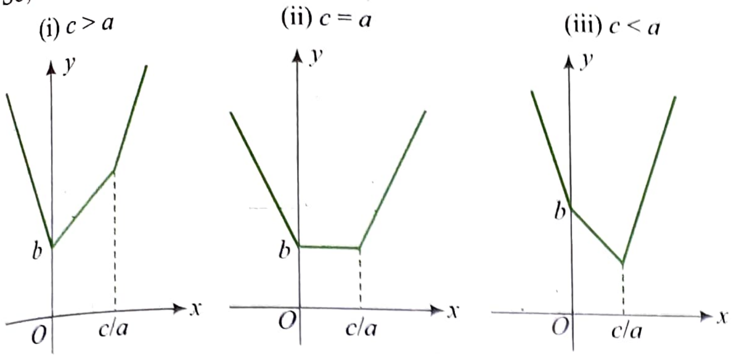
$$= \begin{cases} b - (a + c)x, & x < 0 \\ b + (c - a)x, & 0 \leq x < \dfrac{b}{a} \\ (a + c)x + b, & x \geq \dfrac{b}{a} \end{cases}$$

Slope of $y = b - (a + c)x$ is negative.

Slope of $y = (a + c)x + b$ is positive.

Slope of $y = b + (c - a)x$ is negative, zero or positive if $c < a$, $c = a$ and $c > a$ respectively.

So, we have following possible graphs of given function.



In fig. (i), function has one point of minima at $x = 0$.

In fig. (ii), function has infinite points of minima from $x = 0$ to $x = c/a$.

In fig. (iii), function has one point of minima at $x = c/a$.

Hence function attains minimum value at only one point of $c \neq a$.

Global (Absolute) Maximum and Minimum

Let $y = f(x)$ be a given function with domain D. Let $[a, b] \subseteq D$. Global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest / least value of $f(x)$ in $[a, b]$.

Global maximum and minimum in $[a, b]$ would occur at critical point of $f(x)$ within $[a, b]$ or at the end points of the interval.

Global Maximum and Minimum in $[a, b]$

In order to find the global maximum and minimum of $f(x)$ in $[a, b]$, first we find out all the critical points of $f(x)$ in (a, b).

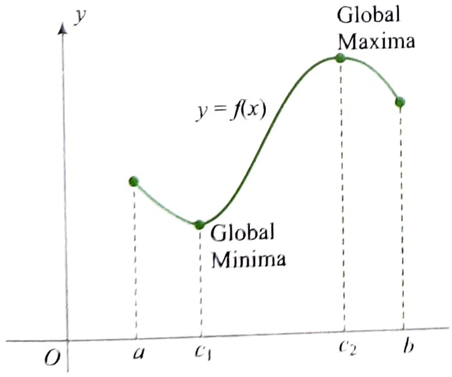
Let $c_1, c_2, \ldots, c_n$ be the different critical points.

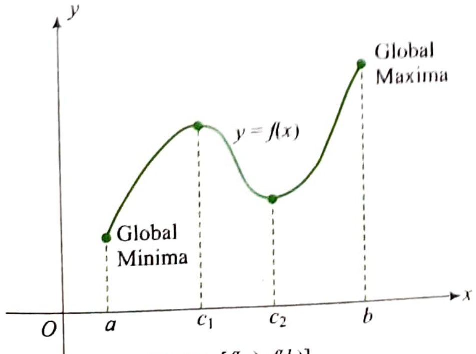
Now we find the values of the function at these critical points, i.e., $f(c_1), f(c_2), \ldots, f(c_n)$

Say, $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

and $M_2 = \min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

Then M_1 is the greatest value of $f(x)$ in $[a, b]$ and M_2 is the least value of $f(x)$ in $[a, b]$





Range: $[f(a), f(b)]$

Global Maximum and Minimum in (a, b)

Method for obtaining the greatest and least values of $f(x)$ in (a, b) is almost same as the method used for obtaining the greatest and least values in $[a, b]$, however with a caution.

Let $y = f(x)$ be a function and $c_1, c_2, \ldots, c_n$ be the different critical points of the function in (a, b)

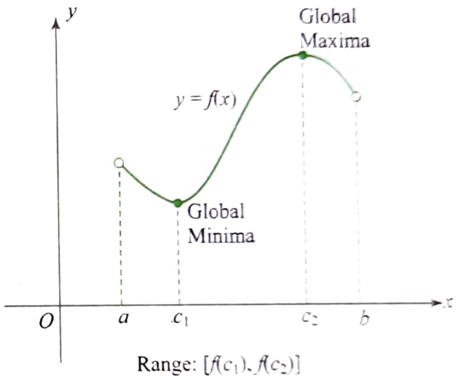
Let $M_1 = \max.\{f(c_1), f(c_2), f(c_3), \ldots, f(c_n)\}$

and $M_2 = \min.\{f(c_1), f(c_2), f(c_3), \ldots, f(c_n)\}$

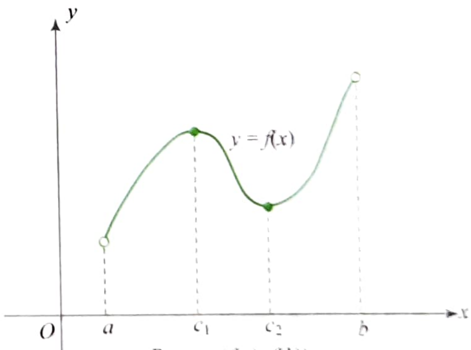
Now if $\lim\limits_{\substack{x \to a^+ \\ or\, x \to b^-}} f(x) > M_1$ or $< M_2$, $f(x)$ would have not global maximum (or global minimum) in (a, b)

This means that if the limiting values at the end points are greater than M_1 or less than M_2, then $f(x)$ would not have global maximum/minimum in (a, b).

On the other hand if $M_1 > \lim\limits_{\substack{x \to a^+ \\ or\, x \to b^-}} f(x)$ and $M_2 < \lim\limits_{\substack{x \to a^+ \\ or\, x \to b^-}} f(x)$, then M_1 and M_2 would respectively be the global maximum and global minimum of $f(x)$ in (a, b).



Range: $[f(c_1), f(c_2)]$



Range: $(f(a), f(b))$
Function has no global minima and global maxima

ILLUSTRATION 6.63

Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. Discuss the global maxima and minima of $f(x)$ in $[0, 2]$ and $(1, 3)$ and, hence, find the range of $f(x)$ for corresponding intervals.

Sol. $f(x) = 2x^3 - 9x^2 + 12x + 6$

$\therefore \quad f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$

Clearly, the critical point of $f(x)$ in $[0, 2]$ is $x = 1$.

Now, $f(0) = 6$, $f(1) = 11$, $f(2) = 10$.

Thus, $x = 0$ is the point of global minimum of $f(x)$ in $[0, 2]$ and $x = 1$ is the point of global maximum.

Hence, range is $[6, 11]$.

For $x \in (1, 3)$, clearly, $x = 2$ is the only critical point in $(1, 3)$.

$$f(2) = 10, \lim_{x \to 1^+} f(x) = 11, \text{ and } \lim_{x \to 3^-} f(x) = 15.$$

Thus, $x = 2$ is the point of global minimum in $(1, 3)$ and the global maximum in $(1, 3)$ does not exist.

Hence, range is $[10, 15)$.

ILLUSTRATION 6.64

Find the range of the function $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$.

Sol. $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$.

$$\therefore \quad f'(x) = 12x^3 - 24x^2 + 24x - 48$$
$$= 12(x^3 - 2x^2 + 2x - 4)$$
$$= 12[x^2(x-2) + 2(x-2)]$$
$$= 12(x-2)(x^2+2)$$

Now, $f'(x) = 0$ gives $x = 2$, which is point of minima.

Now $f(2) = 3(16) - 8(8) + 12(4) - 48(2) + 25 = -39$

Also $\lim_{x \to \pm\infty} f(x) = \infty$

Therefore, range of the function is $[-39, \infty)$.

ILLUSTRATION 6.65

Find the range of the function $f(x) = 4\cos^3 x - 8\cos^2 x + 1$.

Sol. $f(x) = 4\cos^3 x - 8\cos^2 x + 1$

Let $g(x) = 4x^3 - 8x^2 + 1$, where $x \in [-1, 1]$

$$\therefore \quad g'(x) = 12x^2 - 16x$$
$$g'(x) = 0$$
$$\Rightarrow \quad x = 0 \text{ or } x = \frac{4}{3} \notin [-1, 1]$$

Now $g(-1) = -11$, $g(0) = 1$, $g(1) = -3$

Therefore, the range of $f(x)$ is $[-11, 1]$

ILLUSTRATION 6.66

Find the range of the function $f(x) = 2\sqrt{x-2} + \sqrt{4-x}$.

Sol. Clearly, domain of the function is $[2, 4]$. Now,

$$f'(x) = \frac{1}{\sqrt{x-2}} - \frac{1}{2\sqrt{4-x}}$$
$$f'(x) = 0$$

or $\sqrt{x-2} = 2\sqrt{4-x}$

or $x - 2 = 16 - 4x$

or $x = \frac{18}{5}$

Now, $f(2) = \sqrt{2}$, $f\left(\frac{18}{5}\right) = 2\sqrt{\frac{18}{5} - 2} + \sqrt{4 - \frac{18}{5}} = \sqrt{10}$,

$f(4) = 2\sqrt{2}$

Hence, range of the function is $[\sqrt{2}, \sqrt{10}]$.

Also, here $x = (18/5)$ is the point of global maxima.

ILLUSTRATION 6.67

A function $y = f(x)$ is represented parametrically as follows:

$$x = \phi(t) = t^5 - 5t^3 - 20t + 7$$
$$y = \psi(t) = 4t^3 - 3t^2 - 18t + 3$$

where $t \in [-2, 2]$.

Find the intervals of monotonicity and also find the points of extrema. Also find the range of function.

Sol. We have $x = \phi(t) = t^5 - 5t^3 - 20t + 7$

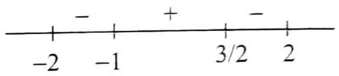
$$y = \psi(t) = 4t^3 - 3t^2 - 18t + 3$$

$$\therefore \quad \frac{dx}{dt} = \phi'(t) = 5t^4 - 15t^2 - 20 = 5(t^2 - 4)(t^2 + 1)$$

$$\frac{dy}{dt} = \psi'(t) = 12t^2 - 6t - 18 = 6(t+1)(2t-3)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6(t+1)(2t-3)}{(t-2)(t+2)(t^2+1)}$$

Sign scheme of $\frac{dy}{dx}$ w.r.t. t is as follows:



$\frac{dy}{dx} > 0$ for $t \in (-1, 3/2)$, where $y = f(x)$ increases.

$\frac{dy}{dx} < 0$ for $t \in [-2, -1) \cup (3/2, 2]$, where $y = f(x)$ decreases.

Also, $t = -1$ is point minima and $t = 3/2$ is point of maxima.

Now $\psi(-2) = 4(-2)^3 - 3(-2)^2 - 18(-2) + 3 = -5$

$\psi(2) = 4(2)^3 - 3(2)^2 - 18(2) + 3 = -13$

$\psi(-1) = 4(-1)^3 - 3(-1)^2 - 18(-1) + 3 = 14$

CONCEPT APPLICATION EXERCISE 6.4

1. Find the critical (stationary) points of the function $f(x) = \frac{x^5}{20} - \frac{x^4}{12} + 5$. Name these points. Also find the point of inflection.
2. The curve $f(x) = \frac{x^2 + ax + b}{x - 10}$ has a stationary point at $(4, 1)$. Find the values of a and b. Also, show that $f(x)$ has point of maxima at this point.
3. If the function $f(x) = axe^{bx^2}$ has maximum value at $x = 2$ such that $f(2) = 1$, then find the values of a and b.
4. Discuss the extremum of $f(x) = \frac{1}{3}\left(x + \frac{1}{x}\right)$.
5. Discuss the extremum of $f(x) = 1 + 2\sin x + 3\cos^2 x$, $0 \le x \le 2\pi/3$.
6. Discuss the extremum of $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$, $0 \le x \le \pi$.

7. Let $f(x) = -\sin^3 x + 3\sin^2 x + 5$ on $[0, \pi/2]$. Find the local maximum and local minimum of $f(x)$.

$\psi(3/2) = 4(3/2)^3 - 3(3/2)^2 - 18(3/2) + 3 = -17.25$

So, range of the function is $[-17.25, 14]$

8. Discuss the extremum of $f(\theta) = \sin^p \theta \cos^q \theta$, $p, q > 0$, $0 < \theta < \pi/2$.

9. Find the maximum and minimum values of the function $y = \log_e(3x^4 - 2x^3 - 6x^2 + 6x + 1)$, $x \in (0, 2)$. Given that $(3x^4 - 2x^3 - 6x^2 + 6x + 1) > 0\ \forall\ x \in (0, 2)$.

10. Discuss the extremum of $f(x) = x(x^2 - 4)^{-1/3}$.

11. Discuss the maxima and minima of the function $f(x) = x^{2/3} - x^{4/3}$. Draw the graph of $y = f(x)$ and find the range of $f(x)$.

12. Discuss the extremum of $f(x) = \begin{cases} |x^2 - 2|, & -1 \le x < \sqrt{3} \\ \dfrac{x}{\sqrt{3}}, & \sqrt{3} \le x < 2\sqrt{3} \\ 3 - x, & 2\sqrt{3} \le x \le 4 \end{cases}$.

13. Discuss the extremum of $f(x) = \begin{cases} 1 + \sin x, & x < 0 \\ x^2 - x + 1, & x \ge 0 \end{cases}$ at $x = 0$.

14. Find the points of minima and minimum value of

$|x| + \left|x + \dfrac{1}{2}\right| + |x - 3| + \left|x - \dfrac{5}{2}\right|$.

15. Let $f(x)$ be defined as $f(x) = \begin{cases} \tan^{-1}\alpha - 5x^2, & 0 < x < 1 \\ -6x, & x \ge 1 \end{cases}$.

If $f(x)$ has a maximum at $x = 1$, then find the values of α.

16. Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \le 1 \\ -2x + \log_2(b^2 - 2), & x > 1 \end{cases}$. Find the values of b for which $f(x)$ has the greatest value at $x = 1$.

ANSWERS

1. Point of maxima: $x = 0$
   Point of minima: $x = 4/3$
   Point of inflection: $x = 1$
2. $a = 7$, $b = -6$
3. $a = \dfrac{\sqrt{e}}{2}$, $b = -\dfrac{1}{8}$
4. Point of local maxima: $x = -1$
   Point of local minima: $x = 1$
5. Point of global minima: $x = \pi/3$
   Point of global maxima: $x = \sin^{-1}(1/3)$
6. Point of global maxima: $x = \pi/4$
   Point of local minima: $x = 2\pi/3$
   Point of local maxima: $x = 3\pi/4$
7. Strictly increasing
8. Point of maxima: $x = \tan^{-1}\sqrt{\dfrac{p}{q}}$
9. Point of local maxima: $x = 1/2$
   Point of local minima: $x = 1$
10. Point of local maxima: $x = -2\sqrt{3}$
    Point of local minima: $x = 2\sqrt{3}$
11. Points of global maxima: $x = \pm\dfrac{1}{2\sqrt{2}}$
    Point of local minima: $x = 0$
12. Point of global maxima: $x = 0$
    Point of local minima: $x = \sqrt{2}$
13. Point of local maxima
14. Points of minima: $[0, 5/2]$
    Minimum value: 6
15. $\alpha < -\tan 1$
16. $b \in [-\sqrt{130}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{130}]$

APPLICATIONS OF EXTREMA IN CURVE TRACING AND ANALYSING ROOTS OF EQUATION

There are not any set rules to draw the graph of the function. Depending upon the type of function we start the arguments and draw the graph. But having learned about different aspects of function like continuity, differentiability, monotonicity and extrema we can somehow draw a sequence of operations to be performed.

1. Check the domain of the function.
2. Examine the point of intersection of $y = f(x)$ with x-axis and y-axis.
3. If function is discontinuous at some point say $x = a$, find its left hand limit and right hand limit at $x = a$.
4. Check the value of the function at the end points of the given interval.
5. Differentiate the function and separate the intervals of monotonicity.
6. Check for the points of maxima, minima, inflection etc.

ILLUSTRATION 6.68

How many roots of the equation $x^4 + 2x^2 - 8x + 3 = 0$ are real? Also find the integral values of the real roots.

Sol. Let $f(x) = x^4 + 2x^2 - 8x + 3$

$\Rightarrow \quad f'(x) = 4x^3 + 4x - 8 = 4(x - 1)(x^2 + x + 2)$

Now $\quad f'(x) = 0 \Rightarrow x = 1$

Hence graph of $y = f(x)$ has only one turn(maxima minima).

Now $\quad f(1) = 1 + 2 - 8 + 3 < 0$

Also, $\quad \lim_{x \to \pm\infty} f(x) = \infty$

The graph of the function is as shown in the following figure.
From the graph equation has two real roots.
Also, $\alpha \in (0, 1)$. $\therefore [\alpha] = 0$

$f(2) > 0$.

$\therefore \quad \beta \in (1, 2)$. $\therefore [\beta] = 1$

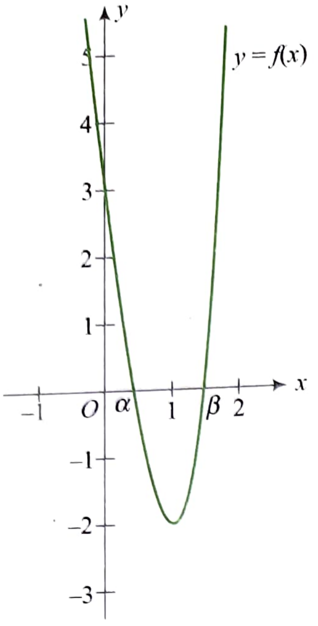


ILLUSTRATION 6.69

Draw the graph of $y = \dfrac{x^2}{\sqrt{x+1}}$

Sol. We have $y = f(x) = \dfrac{x^2}{\sqrt{x+1}}$

Clearly domain of function is $[-1, \infty)$

$$\frac{x^2}{\sqrt{x+1}} \geq 0, \forall\, x \in R.$$

$\therefore$ Graph entirely lies above x-axis.

$$\lim_{x \to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty$$

$$f(0) = 0$$

$$f'(x) = \frac{2x\sqrt{x+1} - \dfrac{x^2}{2\sqrt{x+1}}}{x+1}$$

$$= \frac{4x(x+1) - x^2}{2(x+1)\sqrt{x+1}}$$

$$= \frac{x(3x+4)}{2(x+1)\sqrt{x+1}},\ x > -1$$

$f'(x) < 0$ for $-1 < x < 0$, where $f(x)$ decreases

and $f'(x) > 0$ for $x > 0$, where $f(x)$ increases.

Hence $x = 0$ is point of minima.

$$\lim_{x \to \infty} \frac{x^2}{\sqrt{x+1}} = \infty$$

From the above information the graph of function is as shown in the figure.

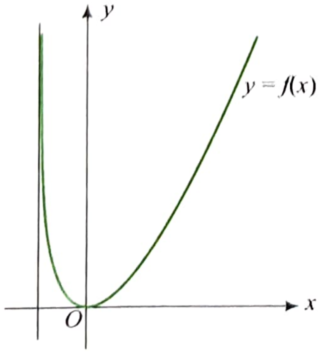


ILLUSTRATION 6.70

Draw the graph of $y = xe^x$. Also,

(i) find the range of the function

(ii) find the point of inflection.

(iii) find the values of k if $xe^x = k$ has two distinct real roots.

Sol. We have $y = f(x) = x\,e^x$

Clearly domain of the function is R.

$f(0) = 0$.

$\therefore$ Curve passes through origin.

Now $f'(x) = e^x + xe^x = e^x(1 + x)$

$f'(x) = 0,\ \therefore x = -1$

$f'(x) > 0$ for $x > -1$, where $f(x)$ increases.

$f'(x) < 0$ for $x < -1$, where $f(x)$ decreases.

$\therefore$ $x = -1$ is point of minima.

$f(-1) = -e^{-1} = -1/e$

$$\lim_{x \to \infty} xe^x = \infty$$

and $$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0$$

Thus in $(-\infty, -1)$, f(x) decreases from 0^- to $-1/e$

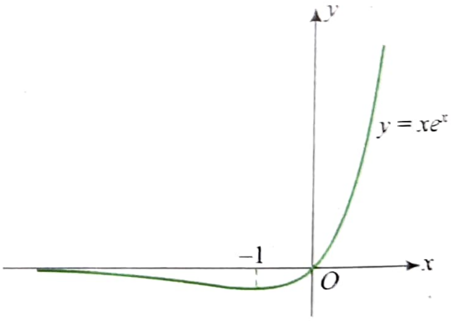
In $(-1, \infty)$, f(x) increases, interesting x-axis at x = 0.

Also $f''(x) = e^x(2 + x)$

$f''(x) = 0,\ \therefore x = -2.$

So, $x = -2$ is point of inflection.

From the above discussion the graph of the function is as shown in the following figure.



Range of the function is $[f(-1), \infty)$ or $[-e^{-1}, \infty)$

If $xe^x = k$ has two distinct real roots, line $y = k$ must intersect the graph between $f(-1)$ and 0, i.e., $-1/e$ and 0.

So, $k \in (-1/e, 0)$

ILLUSTRATION 6.71

Find the minimum integral value of k for which the equation $e^x = kx^2$ has exactly three real distinct solutions.

Sol. Given equation is $e^x = kx^2$ ...(1)

$\Rightarrow$ $k = \dfrac{e^x}{x^2} = f(x)$ (say)

$\therefore$ $f'(x) = \dfrac{(x-2)\,e^x}{x^3}$

$f'(x) = 0,$

$\therefore$ $x = 2$, which is point of minima.

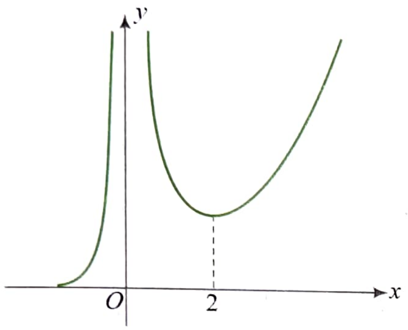
Also $f(2) = \frac{e^2}{4}$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \infty \quad \text{(using L'Hospital Rule twice)}$$

$$\lim_{x\to-\infty} \frac{e^x}{x^2} = 0 \text{ and } \lim_{x\to 0} \frac{e^x}{x^2} = \infty$$

Further $f(x) > 0, \forall x \in R$.

From this information graph of $f(x)$ is as shown below



For three real distinct solutions of equation (1), line $y = k$ must intersect the graph above $\frac{e^2}{4}$.

So, $k > \frac{e^2}{4}$

ILLUSTRATION 6.72

Draw the graph of $f(x) = \log_e(\sqrt{1-x^2} - x)$. Find the range of the function. Also find the values of k if $(x) = k$ has two distinct real roots.

Sol. We have $y = f(x) = \log_e(\sqrt{1-x^2} - x)$

$f(x)$ is defined if $\sqrt{1-x^2} - x > 0$ and $1 - x^2 \geq 0$

For $1 - x^2 \geq 0, -1 \leq x \leq 1$

For $[-1, 0]$,

$$\sqrt{1-x^2} - x > 0$$

For $(0, 1)$, we have $1 - x^2 > x^2$

or $x^2 < \frac{1}{2}$

or $0 < x < \frac{1}{\sqrt{2}}$

Thus, domain of the function is $\left[-1, \frac{1}{\sqrt{2}}\right)$.

$f(0) = 0$.

So, curve passes through origin.

$$\text{Now} \quad f'(x) = \frac{-\frac{x}{\sqrt{1-x^2}} - 1}{\sqrt{1-x^2} - x}$$

$$= \frac{-x - \sqrt{1-x^2}}{(\sqrt{1-x^2})\sqrt{1-x^2} - x}$$

$$f'(x) = 0$$

$$\therefore \quad -x = \sqrt{1-x^2}$$

$$\therefore \quad x = -\frac{1}{\sqrt{2}}$$

$$f'(x) > 0 \text{ for } x \in \left(-1, -\frac{1}{\sqrt{2}}\right)$$

$$f'(x) < 0 \text{ for } x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

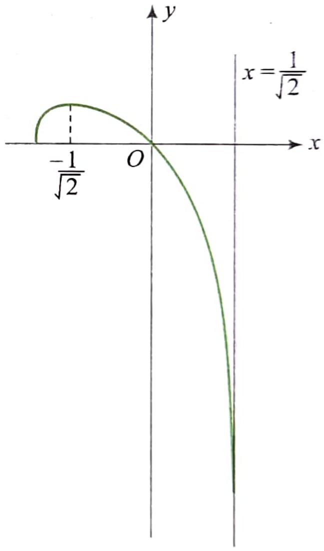
$\therefore$ $x = -\frac{1}{\sqrt{2}}$ is point of maxima.

$$f(-1) = 0$$

and $\lim_{x\to\frac{1}{\sqrt{2}}} \log_e(\sqrt{1-x^2} - x) = -\infty \quad \left(\because \lim_{x\to\frac{1}{\sqrt{2}}} (\sqrt{1-x^2} - x) = 0\right)$

Thus $x = \frac{1}{\sqrt{2}}$ is an asymptote.

From the above discussion the graph of the function is as shown in the following figure.



Clearly range of $f(x)$ is $\left(-\infty, f\left(-\frac{1}{\sqrt{2}}\right)\right]$ or $\left(-\infty, \log_e \sqrt{2}\right]$

For two distinct real roots of $f(x) = k$, $k \in \left(0, \log_e \sqrt{2}\right)$

ILLUSTRATION 6.73

Draw the graph of $f(x) = \frac{x^2 - 5x + 6}{x^2 - x}$

Sol. $f(x) = \frac{(x-2)(x-3)}{x(x-1)}$

$f(x)$ is discontinuous at $x = 0$ and $x = 1$

$$\lim_{x\to 1^+} f(x) = +\infty \text{ and } \lim_{x\to 1^-} f(x) = -\infty$$

$$\lim_{x\to 0^+} = -\infty \text{ and } \lim_{x\to 0^-} f(x) = +\infty$$

$f(x)$ intersect the x-axis at $x = 2$ and $x = 3$.

$$\lim_{x\to\pm\infty}\frac{x^2-5x+6}{x^2-x}=1$$

Now $f'(x)=\dfrac{(2x-5)(x^2-x)-(2x-1)(x^2-5x+6)}{(x^2-x)^2}$

$=2\dfrac{2x^2-6x+3}{(x^2-x)^2}$

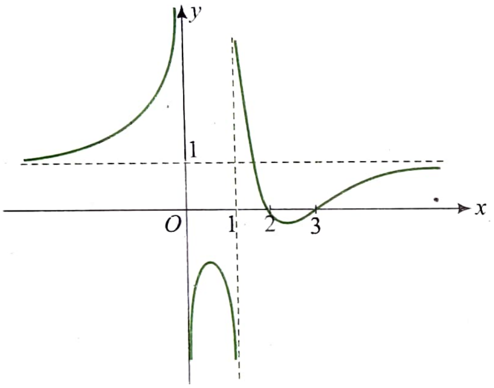
$f'(x)=0 \quad\Rightarrow 2x^2-6x+3=0$ or $x=\dfrac{3\pm\sqrt{3}}{2}$

Sign scheme of $f'(x)$ is as follows.

$+ \quad \dfrac{3-\sqrt{3}}{2} \quad - \quad \dfrac{3+\sqrt{3}}{2} \quad +$

Clearly $x=\dfrac{3+\sqrt{3}}{2}$ is point of minima and $x=\dfrac{3-\sqrt{3}}{2}$ is point of maxima.

From the above information graph of $y=f(x)$ is as shown in the following figure.



Nature of the Roots of the Cubic Equation

Let $f(x)=x^3+ax^2+bx+c$ be the given cubic polynomial, and $f(x)=0$ be the corresponding cubic equation, where $a, b, c\in R$.

$\therefore \quad f'(x)=3x^2+2ax+b$

Discriminant of $f'(x)=0$ is $D=4a^2-12b=4(a^2-3b)$

Now we have following cases.

Case I: $D<0$

Here $f'(x)>0\ \forall\ x\in R$.

So, $f(x)$ in increasing.

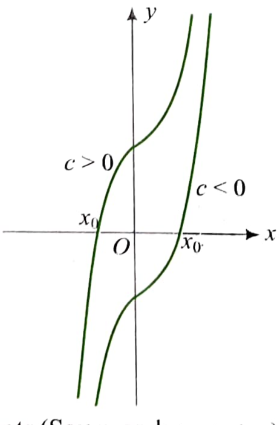
Also $\lim_{x\to-\infty} f(x)=-\infty$

and $\lim_{x\to+\infty} f(x)=\infty$.

Therefore, graph intersect x-axis exactly once.

So, equation $f(x)=0$ has one root x_0.

x_0 is positive if $c<0$ and x_0 is negative if $c>0$



Case II: $D>0$

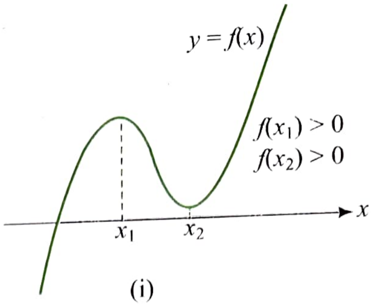
If $D>0, f'(x)=0$ would have two real roots (Say x_1 and $x_2, x_1<x_2$)

$\therefore \quad f'(x)=3(x-x_1)(x-x_2)$

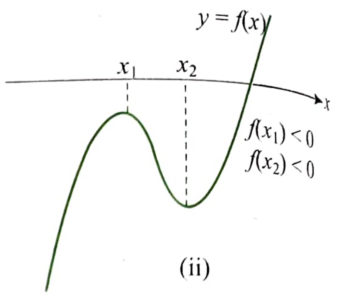
$f'(x)=0$ if $x=x_1$ of $x=x_2$.

Clearly x_2 is point of minima and x_1 is point of maxima.

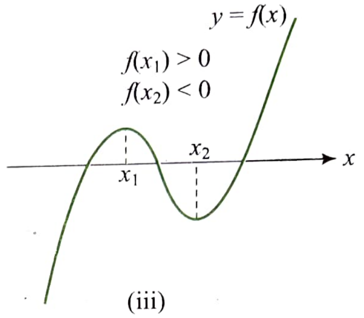
We have following possible graphs w.r.t. intersection with x-axis.



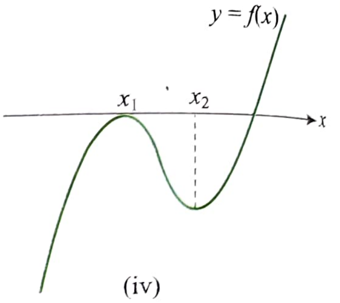
(i)



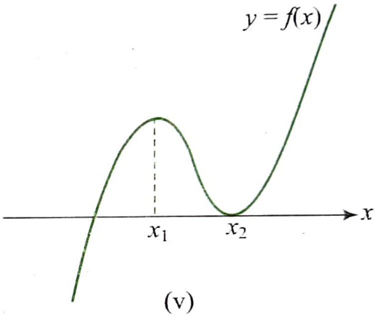
(ii)



(iii)



(iv)



(v)

From the above graphs we have following findings.

(i) If $f(x_1)f(x_2)>0$ (for fig. (i) and (ii)), $f(x)=0$ has just one real root.

(ii) If one of the $f(x_1)$ and $f(x_2)$ is zero, i.e., $f(x_1)f(x_2)=0$. $f(x)=0$ has three real roots but one of the roots is repeated.

(iii) If $f(x_1)>0$ and $f(x_2)<0$, i.e., $f(x_1)f(x_2)<0$ (considering the case when coefficient of x^3 is negative), $f(x)=0$ has three real and distinct roots.

Case III: If $D=0$

Here $f'(x)=3(x-x_1)^2$

$\therefore \quad f(x)=(x-x_1)^3+k$.

Now if $k=0$, then $f(x)=0$ has three equal real roots and if $k\neq 0$ then $f(x)=0$ has one real root.

ILLUSTRATION 6.74

Find the value of a if $x^3-3x+a=0$ has three real distinct roots.

Sol. Let $f(x)=x^3-3x+a$

If $f'(x)=0$, then $3x^2-3=0\Rightarrow x=\pm 1$

For three distinct roots, $f(1)\times f(-1)<0$

$\Rightarrow \quad (1-3+a)(-1+3+a)<0$

$\Rightarrow \quad (a+2)(a-2)<0$

$\Rightarrow \quad -2<a<2$

Illustration 6.75

Prove that there exist exactly two non-similar isosceles triangles ABC such that $\tan A + \tan B + \tan C = 100$.

Sol. Let $A = B$.

$\therefore \quad 2A + C = 180°$

$\Rightarrow \quad 2A = 180° - C$

$\Rightarrow \quad \tan 2A = -\tan C \quad \ldots(1)$

Also, we have $\tan A + \tan B + \tan C = 100$

$\therefore \quad 2\tan A + \tan C = 100 \quad \ldots(2)$

From (1) and (2), we have

$$2\tan A - \tan 2A = 100$$

$$\Rightarrow \quad 2\tan A - \frac{2\tan A}{1-\tan^2 A} - 100 = 0$$

Let $\tan A = x$.

So we have $x^3 - 50x^2 + 50 = 0$

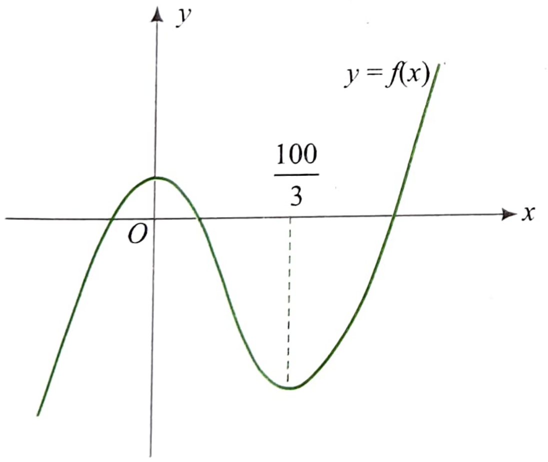
Let $f(x) = x^3 - 50x^2 + 50$.

$\therefore \quad f'(x) = 3x^2 - 100x = 0$

$$\Rightarrow \quad x = 0, \frac{100}{3}$$

Also $f(0) = 50$.

So, graph of function is as shown in the following figure.



Clearly $f(x) = 0$ has exactly three distinct real roots.

Therefore, tan A and hence A has three distinct values but one of them will be obtuse angle as one of the roots of the equation $f(x) = 0$ is negative.

Hence, there exist exactly two non-similar isosceles triangles.

Illustration 6.76

If t be a real number satisfying the equation $2t^3 - 9t^2 + 30 - a = 0$, then find the values of the parameter 'a' for which the equation $x + \frac{1}{x} = t$ gives six real and distinct values of x. $[a \in R]$

Sol. We have, $2t^3 - 9t^2 + 30 - a = 0 \quad \ldots(1)$

Now $\quad t = x + \frac{1}{x}$

Since $\left|x + \frac{1}{x}\right| \ge 2$, $|t| \ge 2$.

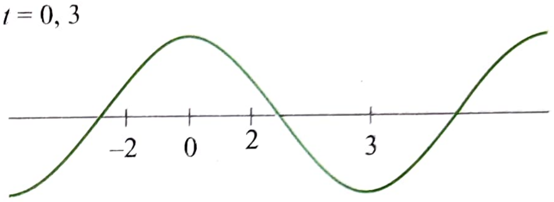
So, any real root t_0 of equation (1), gives two real and distinct values of x if $|t_0| > 2$.

Thus, we need to find the condition for the equation in 't' to have three real and distinct roots none of which lies in $[-2, 2]$

Let $\quad f(t) = 2t^3 - 9t^2 + 30 - a$

$\therefore \quad f'(t) = 6t^2 - 18t = 0$

$\Rightarrow \quad t = 0, 3$



So the equation $f(t) = 0$ has three real and distinct roots if $f(0) \cdot f(3) < 0$

$\Rightarrow \quad (30 - a)(54 - 81 + 30 - a) < 0$

$\Rightarrow \quad (30 - a)(3 - a) < 0$

$\Rightarrow \quad a \in (3, 30) \quad \ldots(1)$

Also, none of the roots lies in $[-2, 2]$ if $f(-2) > 0$ and $f(2) > 0$

$-16 - 36 + 30 - a > 0$ and $16 - 36 + 30 - a > 0$

$-22 - a > 0$ and $10 - a > 0$

$\Rightarrow \quad a + 22 < 0$ and $a - 10 < 0$

$\Rightarrow \quad a < -22$ and $a < 10$

$\Rightarrow \quad a < -22 \quad \ldots(2)$

From (1) and (2) no real value of a exists.

Concept Application Exercise 6.5

1. Draw the graph of $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$.
2. Discuss the number of roots of the equation $e(k - x\log_e x) = 1$, for different values of k.
3. Draw the graph of $y = (x+1)^{\frac{2}{3}} + (x-1)^{\frac{2}{3}}$
4. Draw the graph of $f(x) = \log_e(1 - \log_e x)$. Find the point of inflection.
5. Draw the graph of $f(x) = \frac{4\log_e x}{x^2}$. Also find the range.

Answers

1. Two distinct roots for $k \in (0, 1/e)$
   No real roots for $k < 0$
   One real root for $k = 0$ or $k \ge 1/e$
4. Point of inflection: $x = 1$
5. Range: $(-\infty, 2/e]$

APPLICATIONS OF EXTREMA IN OPTIMIZATION, PLANE GEOMETRY AND COORDINATE GEOMETRY

There are some interesting applications of extrema in above said fields. Let us study these applications through some illustrations.

ILLUSTRATION 6.77

Find two positive numbers x and y such that $x + y = 60$ and x^3y is maximum.

Sol. $x + y = 60$

or $y = 60 - x$

or $x^3y = (60 - x)x^3$

Let $f(x) = (60 - x)\,x^3,\ x \in (0, 60)$

Let $f'(x) = 3x^2(60 - x) - x^3 = 0$

or $x = 45$ $\quad (\because x \neq 0)$

$f'(45^+) < 0$ and $f'(45^-) > 0$

Hence, local maxima is at $x = 45$.

So, $x = 45$ and $y = 15$.

ILLUSTRATION 6.78

Assuming the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against the current of c miles per hour is $(3c/2)$ miles per hour.

Sol. Let the speed of the motor boat be v m/h. Then

Velocity of the boat relative to the current = $(v - c)$ m/h

If s miles is the distance covered, then the time taken to cover this distance is $t = s/(v - c)$ hours.

Since the petrol burnt = kv^3 per hour, where k is a constant, the total amount of petrol burnt for a distance of s miles,

$$z = kv^3 s/(v - c)$$

$$\therefore \quad \frac{dz}{dv} = \frac{2ksv^2(v - 3c/2)}{(v - c)^2}$$

For maximum or minimum of z, $dz/dv = 0$ or $v = 3c/2$.

If v is little less or little greater than $3c/2$, then the sign of dz/dv changes from –ve to + ve. Hence, z is minimum when $v = 3c/2$ m/h.

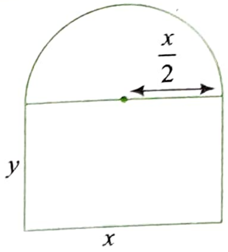
Since minima is the only extreme value, z is least at $v = 3c/2$, i.e., the most economical speed is $3c/2$ m/h.

ILLUSTRATION 6.79

A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Sol. Let x and y be the length and breadth of the rectangular window. Therefore,

Radius of the semicircular opening $= \dfrac{x}{2}$



It is given that the perimeter of the window is 10 m. Therefore,

$$x + 2y + \frac{\pi x}{2} = 10$$

or $$y = 5 - \frac{x}{2}\left(\frac{\pi}{2} + 1\right)$$

Therefore, area of the window is given by

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - \frac{x}{2}\left(\frac{\pi}{2} + 1\right)\right] + \frac{\pi}{8}x^2$$

$$= 5x - x^2\left(\frac{\pi}{4} + \frac{1}{2}\right) + \frac{\pi}{8}x^2$$

$$\therefore \quad \frac{dA}{dx} = 5 - 2x\left(\frac{\pi}{4} + \frac{1}{2}\right) + \frac{\pi}{4}x$$

Now, $$\frac{dA}{dx} = 0$$

or $$5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$

or $$x = \frac{20}{\pi + 4}$$

Also, A is a quadratic function having coefficient of x^2 negative.

Therefore, $x = \dfrac{20}{\pi + 4}$ is a point of maxima. Thus,

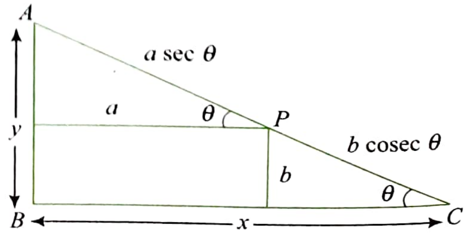
$$y = 5 - \frac{20}{\pi + 4}\left(\frac{2 + \pi}{4}\right) = \frac{10}{\pi + 4}\text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given by length $= \dfrac{20}{\pi + 4}$ m and breadth $= \dfrac{10}{\pi + 4}$ m.

ILLUSTRATION 6.80

A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$

Sol.



From the figure,

$$PC = b\operatorname{cosec}\theta$$

and $AP = a\sec\theta$

$$AC = PC + AP$$

or $AC = b\operatorname{cosec}\theta + a\sec\theta$ (1)

$\therefore$ $\dfrac{d(AC)}{d\theta} = -b\operatorname{cosec}\theta\cot\theta + a\sec\theta\tan\theta$

$$\frac{d(AC)}{d\theta} = 0$$

or $a\sec\theta\tan\theta = b\operatorname{cosec}\theta\cot\theta$

or $\tan\theta = \left(\dfrac{b}{a}\right)^{1/3}$

$\therefore$ $\sin\theta = \dfrac{(b)^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$ and $\cos\theta = \dfrac{(a)^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$ (2)

Also, $\theta \in (0, \pi/2)$

$$\lim_{\theta\to 0^+} (a\sec\theta + b\operatorname{cosec}\theta) \to \infty$$

and $\lim_{\theta\to \frac{\pi}{2}^{-}} (a\sec\theta + b\operatorname{cosec}\theta) \to \infty$

Therefore, $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)^{1/3}$ is a point of minima

For this value of θ,

$$AC = \frac{b\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}} + \frac{a\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} \quad \text{[Using (1) and (2)]}$$

$$= \sqrt{a^{2/3}+b^{2/3}}\left(b^{2/3}+a^{2/3}\right)$$

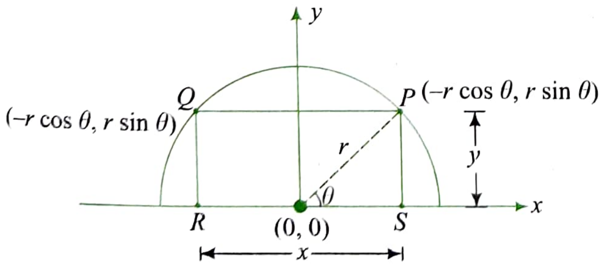
$$= \left(a^{2/3}+b^{2/3}\right)^{3/2}$$

Hence, the minimum length of the hypotenuse is $\left(a^{2/3}+b^{2/3}\right)^{3/2}$.

ILLUSTRATION 6.81

Rectangles are inscribed inside a semi-circle of radius r. Find the rectangle with maximum area.

Sol. Let us choose coordinate system with the origin as the center of circle.



Area of rectangle $PQRS$,

$$A = 2\,(r\cos\theta)\,(r\sin\theta),\ \theta \in \left(0, \frac{\pi}{2}\right)$$

$$A = r^2 \sin 2\theta$$

A is maximum when $\sin 2\theta = 1$ or $2\theta = \pi/2$

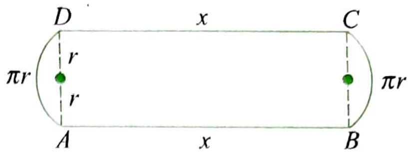
or $\theta = \pi/4$

Therefore, sides of the rectangle are $2r\cos(\pi/4) = \sqrt{2}r$ and $r\sin(\pi/4) = r/\sqrt{2}$.

ILLUSTRATION 6.82

A running track of 440 ft is to be laid out enclosing a football field, the shape of which is a rectangle with a semi-circle at each end. If the area of the rectangular portion is to be maximum, then find the lengths of its sides.

Sol.



Perimeter = 440 ft.

$\therefore$ $2x + \pi r + \pi r = 440$ or $2x + 2\pi r = 440$

A = Area of the rectangular portion = $x\, 2r$

$$= x\,\frac{(440-2x)}{\pi} = \frac{1}{\pi}\,(440x - 2x^2)$$

Let $\dfrac{dA}{dx} = \dfrac{1}{\pi}\,(440 - 4x) = 0.$

Then, $x = 110$ for which $\dfrac{d^2A}{dx^2} < 0.$

Thus, A is maximum when $x = 110$. Therefore,

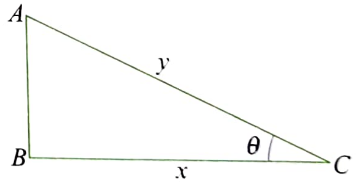
$$2r = \frac{440-2x}{\pi} = \frac{440-220}{22/7} = 70$$

or $r = 35$ ft and $x = 110$ ft

ILLUSTRATION 6.83

If the sum of the lengths of the hypotenuse and another side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between these sides is $\pi/3$.

Sol.



Let ABC be a right-angled triangle in which side $BC = x$ (say) and hypotenuse $AC = y$ (say). Given $x + y = k$ (constant)

or $y = k - x.$

Now, the area of triangle ABC is given by

$$A = \frac{1}{2}BC\,AB = \frac{1}{2}x\sqrt{(y^2 - x^2)} = \frac{1}{2}x\sqrt{[(k-x)^2 - x^2]}$$

Let $u = A^2 = \dfrac{1}{4}x^2(k^2 - 2kx)$

$\therefore$ $du/dx = \dfrac{1}{2}k(kx - 3x^2)$ and $d^2u/dx^2 = \dfrac{1}{2}k(k - 6x)$

For maximum or minimum of u,

$$du/dx = 0$$

or $x = k/3$ $(\because x \neq 0)$

When $x = k/3$, $d^2u/dx^2 = \dfrac{1}{2}k(k - 6 \times \dfrac{1}{3}k) = -\dfrac{1}{2}k^2$ (–ve).

Therefore, u, i.e., A is maximum when $x = k/3$ and when $y = k - x = 2k/3$.

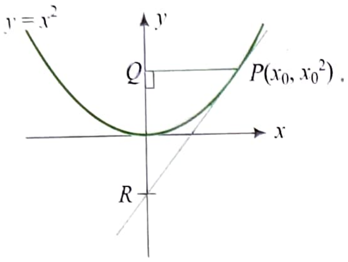
Now, $\cos\theta = BC/AC = x/y = 1/2$ or $\theta = \pi/3$.

Hence, the required angle is $\pi/3$.

ILLUSTRATION 6.84

The tangent to the parabola $y = x^2$ has been drawn so that the abscissa x_0 of the point of tangency belongs to the interval $[1, 2]$. Find x_0 for which the triangle bounded by the tangent, the axis of ordinates, and the straight line $y = x_0^2$ has the greatest area.

Sol.



$y = x^2$ or $dy/dx = 2x$

Therefore, equation of the tangent at (x_0, x_0^2) is $y - x_0^2 = 2x_0(x - x_0)$.

It meets y-axis in $R(0, -x_0^2)$. Q is $(0, x_0^2)$. Thus,

Z = area of triangle PQR

$$= \frac{1}{2} 2x_0^2\, x_0 = x_0^3,\ 1 \le x_0 \le 2$$

$$\therefore \quad \frac{dZ}{dx_0} = 3x_0^2 > 0 \text{ in } 1 \le x_0 \le 2$$

Thus, Z is an increasing function in $[1, 2]$.

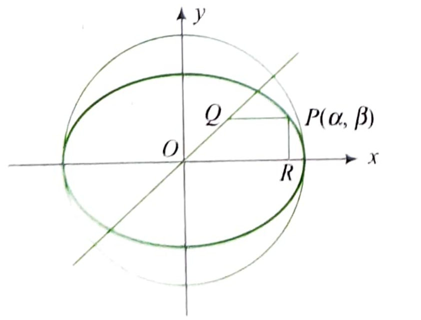
Hence, Z, i.e., the area of ΔPQR, is greatest at $x_0 = 2$.

ILLUSTRATION 6.85

Find the point (α, β) on the ellipse $4x^2 + 3y^2 = 12$, in the first quadrant, so that the area enclosed by the lines $y = x, y = \beta, x = \alpha$, and the x-axis is maximum.

Sol. Equation of the ellipse is $x^2/3 + y^2/4 = 1$.

Let point P be $(\sqrt{3}\cos\theta, 2\sin\theta)$, $\theta \in (0, \pi/2)$.



Clearly, line PQ is $y = 2\sin\theta$, line PR is $x = \sqrt{3}\cos\theta$, line OQ is $y = x$, and Q is $(2\sin\theta, 2\sin\theta)$.

Z = Area of the region $PQORP$ (trapezium)

$$= \frac{1}{2}(OR + PQ)PR$$

$$= \frac{1}{2}(\sqrt{3}\cos\theta + (\sqrt{3}\cos\theta - 2\sin\theta))2\sin\theta$$

$$= \frac{1}{2}(2\sqrt{3}\cos\theta\sin\theta - 2\sin^2\theta)$$

$$= \frac{1}{2}(\sqrt{3}\sin 2\theta + \cos 2\theta - 1)$$

$$= \cos\left(2\theta - \frac{\pi}{3}\right) - \frac{1}{2}$$

which is maximum when

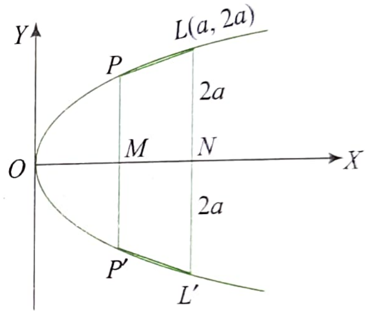
$$\cos\left(2\theta - \frac{\pi}{3}\right) = 1 \text{ or } 2\theta - \frac{\pi}{3} = 0 \text{ or } \theta = \frac{\pi}{6}$$

Hence, point P is $(3/2, 1)$.

ILLUSTRATION 6.86

LL' is the latus rectum of the parabola $y^2 = 4ax$ and PP' is a double ordinate drawn between the vertex and the latus rectum. Show that the area of the trapezium $PP'LL'$ is maximum when the distance PP' from the vertex is $a/9$.

Sol. Let $LL' = 4a$ be the latus rectum of the parabola $y^2 = 4ax$ and let $(at^2, 2at)$ be the coordinates of the point P.



Here, PP' is the double ordinate of the parabola. Then.

$OM = at^2$ or $MN = ON - OM = a - at^2$

and $PP' = 2PM = 4at$

Now, area of trapezium $PP'L'L$

$$A = \frac{1}{2}(PP' + LL') \times MN$$

$$= \frac{1}{2}(4at + 4a)(a - at^2)$$

$$= 2a^2(-t^3 - t^2 + t + 1)$$

or $dA/dt = 2a^2(-3t^2 - 2t + 1)$ and $d^2A/dt^2 = 2a^2(-6t - 2)$

For maximum or minimum of A, $dA/dt = 0$

or $-2a^2(3t - 1)(t + 1) = 0$

or $t = -1, 1/3$

When $t = -1$, $d^2A/dt^2 = 8a^2$ (+ve)

Thus, A is minimum when $t = -1$.

When $t = 1/3$, $d^2A/dt^2 = -8a^2$ (−ve)

Thus, A is maximum when $t = 1/3$ (only point of maxima).

Therefore, for the area of trapezium $PP'LL'$ to be maximum distance of PP' from vertex O is $OM = at^2 = a(1/3)^2 = a/9$.

ILLUSTRATION 6.87

Find the points on the curve $5x^2 - 8xy + 5y^2 = 4$ whose distance from the origin is maximum or minimum.

Sol. Let (r, θ) be the polar coordinates of any point P on the curve where r is the distance of the point from the origin. Then,

$$r^2 [5 (\cos^2 \theta + \sin^2 \theta) - 8 \sin \theta \cos \theta] = 4$$

or $$r^2 = \frac{4}{5 - 4 \sin 2\theta}$$

r^2 is maximum when $5 - 4 \sin 2\theta$ is minimum, i.e., $5 - 4 = 1$ (when $\sin 2\theta = 1$) or

$$2\theta = 90° \text{ or } \theta = 45° \text{ or } r = \pm 2, \theta = 45° \quad (1)$$

Again, r^2 is minimum when $5 - 4 \sin 2\theta$ is maximum, i.e., $5 + 4 = 9$ when

$$\sin 2\theta = -1 \text{ or } 2\theta = \frac{3\pi}{2} \text{ or } \theta = \frac{3\pi}{4}$$

or $$r = \pm\frac{2}{3}, \theta = \frac{3\pi}{4} \quad (2)$$

Hence, the points are $(r \cos \theta, r \sin \theta)$ where r and θ are given by equations (1) and (2).

Thus, we get four points $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$,

$$\left(\frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}\right), \text{ and } \left(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right).$$

CONCEPT APPLICATION EXERCISE 6.6

1. A private telephone company serving a small community makes a profit of ₹12.00 per subscriber, if it has 725 subscribers. It decides to reduce the rate by a fixed sum for each subscriber over 725, thereby reducing the profit by 1 paise per subscriber. Thus, there will be profit of ₹ 11.99 on each of the 726 subscribers, ₹ 11.98 on each of the 727 subscribers, etc. What is the number of subscribers which will give the company the maximum profit?
2. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.
3. A figure is bounded by the curves $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$. At what point (a, b) should a tangent be drawn to curve $y = x^2 + 1$ for it to cut off a trapezium of greatest area from the figure?
4. Find the minimum length of radius vector of the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$.
5. Find the point at which the slope of the tangent of the function $f(x) = e^x \cos x$ attains minima, when $x \in [0, 2\pi]$.
6. An electric light is placed directly over the center of a circular plot of lawn 100 m in diameter. Assuming that the intensity of light varies directly as the sine of the angle at which it strikes an illuminated surface and inversely as the square of its distance from its surface, how should the light be hung in order that the intensity may be as great as possible at the circumference of the plot?
7. A swimmer S is in the sea at a distance d km from the closest point A on a straight shore. The house of the swimmer is on the shore at distance L km from A. He can swim at a speed of u km/h per hour and walk at a speed of v km/h, $u < v$. At what point on the shore should he land so that he reaches his house in the shortest possible time?
8. Let (h, k) be a fixed point, where $h > 0$, $k > 0$. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. Find the minimum area of triangle OPQ, O being the origin.
9. A point P is given on the circumference of a circle of radius r. Chords QR are parallel to the tangent at P. Determine the maximum possible area of triangle PQR.

ANSWERS TO CAE 6.6

1. 962 or 963	3. (1/2, 5/4)
4. $\lvert a + b\rvert$	5. π

6. $25\sqrt{2}$ m above the ground

7. $L - \frac{ud}{\sqrt{v^2 - u^2}}$ km from his house

8. $2hk$

9. $\frac{3\sqrt{3}}{4} r^2$ sq. units.

APPLICATIONS OF EXTREMA IN SOLID GEOMETRY

Consider a simple optimization problem in a day to day life. Suppose we want to make a cylindrical tank of fixed capacity.

Here volume $V = \pi r^2 h$ is fixed, but radius 'r' and height 'h' are variables. Now what dimensions (r and h) we choose?

Obviously, we consider dimensions for which material used is least, i.e., surface area of the cylinder is least.

If we consider a tank which is closed on both sides, then surface area is $S = 2\pi rh + 2\pi r^2$. This is function of two variables.

Using constant volume, we replace h by $V/\pi r^2$. So, function $S = \frac{2V}{r} + 2\pi r^2$, which can be optimized.

In most of the problems we have more or less similar type of algorithm.

Let us recall some useful formulas of mensuration.

- **Cuboid**

  For cuboid having length 'l', breadth 'b' and height 'h'

  Volume $= l \times b \times h$

  Surface area $= 2 (l \times b + b \times h + h \times l)$.

- **Cone**

  For cone having radius of the base 'r' and slant height 'l' and vertical height 'h'

  Curved surface area $= \pi rl$

  Total surface area including base $= \pi rl + \pi r^2$

  Volume $= \frac{1}{3} \pi r^2 h$

- **Cylinder**

  For closed cylinder having radius of the base 'r' and height 'h'.

  Curved surface area = $2\pi rh$

  Total surface area = $2\pi rh + 2\pi r^2$

  Volume = $\pi r^2 h$

- **Sphere**

  For sphere of radius 'r'

  Surface area = $4\pi r^2$

  Volume = $\frac{4}{3}\pi r^3$

- Volume of a prism = area of the base × height
- Volume of a pyramid = $\frac{1}{3}$(area of the base) × (height).

ILLUSTRATION 6.88

Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Sol. Let r and h be the radius and height of the cylinder respectively.

Then, the surface area (S) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh$$

$$\Rightarrow \quad h = \frac{S - 2\pi r^2}{2\pi r} = \frac{S}{2\pi}\left(\frac{1}{r}\right) - r$$

Let V be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2\left[\frac{S}{2\pi}\left(\frac{1}{r}\right) - r\right] = \frac{Sr}{2} - \pi r^3$$

Then, $\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2, \frac{d^2V}{dr^2} = -6\pi r$

Now, $\frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} = 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$

When $r^2 = \frac{S}{6\pi}$, then $\frac{d^2V}{dr^2} = -6\pi\left(\sqrt{\frac{S}{6\pi}}\right) < 0.$

∴ By second derivative test, the volume is the maximum when $r^2 = \frac{S}{6\pi}$.

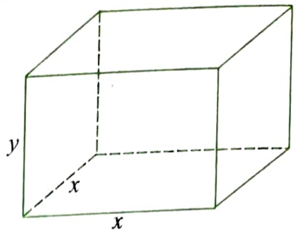
Now, when $r^2 = \frac{S}{6\pi}$, then $h = \frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right) - r = 3r - r = 2r.$

Hence, the volume is the maximum when the height is twice the radius, i.e., when the height is equal to the diameter.

ILLUSTRATION 6.89

A sheet of area 40 m^2 is used to make an open tank with square base. Find the dimensions of the base such that the volume of this tank is maximum.

Sol. Let the length of base be x meters and height be y meters.



Volume $V = x^2 y$

Again, x and y are related to the surface area of this tank which is equal to 40 m^2. Thus,

$$x^2 + 4xy = 40$$

or $$y = \frac{40 - x^2}{4x}, x \in (0, \sqrt{40})$$

∴ $$V(x) = x^2\left(\frac{40 - x^2}{4x}\right) = \frac{40x - x^3}{4}$$

Maximizing volume,

Let $V'(x) = \frac{40 - 3x^2}{4} = 0$ or $x = \sqrt{\frac{40}{3}}$ m

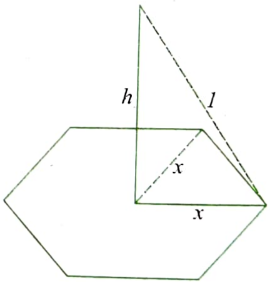
and $V''(x) = -\frac{3x}{2}$ or $V''\left(\sqrt{\frac{40}{3}}\right) < 0$

Therefore, volume is maximum at $x = \sqrt{\frac{40}{3}}$ m.

ILLUSTRATION 6.90

The lateral edge of a regular hexagonal pyramid is 1 cm. If the volume is maximum, then find its height.

Sol.



$$x^2 + h^2 = 1;$$

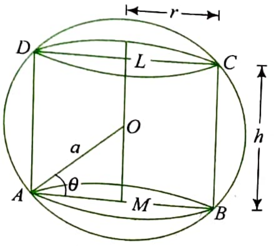
Volume, $V = \frac{1}{3} \times 6 \times \frac{\sqrt{3}}{4}x^2 h = \frac{\sqrt{3}}{2}h(1 - h^2)$

For $V'(h) = 0, h = \frac{1}{\sqrt{3}}$ or $V_{max} = 1/3$

ILLUSTRATION 6.91

Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a.

Sol.



If r is the radius and h is the height of cylinder, then from figure

$$r^2 + (h^2/4) = a^2 \text{ or } r^2 = a^2 - \frac{h^2}{4}$$

Now, $V = \pi r^2 h = \pi\left(a^2 - \frac{1}{4}h^2\right)h = \pi\left(a^2 h - \frac{1}{4}h^3\right)$

$\Rightarrow \quad \frac{dV}{dh} = \pi\left(a^2 - \frac{3}{4}h^2\right) = 0$ for maximum or minimum

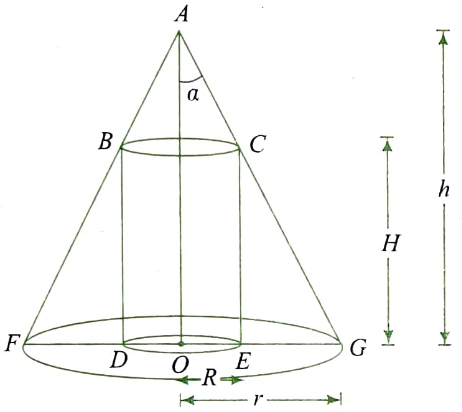
This gives $h = (2/\sqrt{3})\, a$ for which $d^2V/dh^2 = -6h/4 < 0$.

Hence, V is maximum when $h = 2a/\sqrt{3}$.

ILLUSTRATION 6.92

Show that the greatest volume of the cylinder which can be inscribed in a right circular cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^2 \tan^2\alpha$.

Sol. The given right circular cone of fixed height (h) and semi-vertical angle (α) can be drawn as follows.



Here, a cylinder of radius R and height H is inscribed in the cone.

Then, $\angle GAO = \alpha$, $OG = r$, $OA = h$, $OE = R$, and $CE = H$.

We have

$$r = h\tan\alpha$$

Now, since ΔAOG is similar to ΔCEG, we have

$$\frac{AO}{OG} = \frac{CE}{EG}$$

or $\quad \frac{h}{r} = \frac{H}{r-R} \quad [EG = OG - OE]$

or $\quad H = \frac{h}{r}(r-R) = \frac{h}{h\tan\alpha}(h\tan\alpha - R) = \frac{1}{\tan\alpha}(h\tan\alpha - R)$

Now, the volume (V) of the cylinder is given by

$$V = \pi R^2 H = \frac{\pi R^2}{\tan\alpha}(h\tan\alpha - R) = \pi R^2 h - \frac{\pi R^3}{\tan\alpha}$$

$\therefore \quad \frac{dV}{dR} = 2\pi R h - \frac{3\pi R^2}{\tan\alpha}$

Now, $\frac{dV}{dR} = 0$

or $\quad 2\pi R h = \frac{3\pi R^2}{\tan\alpha}$

or $\quad R = \frac{2h}{3}\tan\alpha$

Also, for this value of R,

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan\alpha}\,\frac{2h}{3}\tan\alpha$$
$$= 2\pi h - 4\pi h$$
$$= -2\pi h < 0$$

Therefore, $R = \frac{2h}{3}\tan\alpha$ is point of maxima.

Now, the maximum volume of the cylinder is

$$\pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(\frac{h}{3}\right) = \pi\left(\frac{4h^2}{9}\tan^2\alpha\right)\left(\frac{h}{3}\right)$$
$$= \frac{4}{27}\pi h^3 \tan^2\alpha$$

CONCEPT APPLICATION EXERCISE 6.7

1. A metal box with a square base and vertical sides is to contain 1024 cm^3. The material for the top and bottom costs ₹ 5/cm^2 and the material for the sides costs ₹ 2.50/cm^2. Find the least cost of the box.
2. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area ?
3. The lateral edge of a regular rectangular pyramid is a cm long. The lateral edge makes an angle α with the plane of the base. Find the value of α for which the volume of the pyramid is greatest.
4. Prove that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to 2/3rd the diameter of the sphere.
5. A regular square based pyramid is inscribed in a sphere of given radius r so that all vertices of the pyramid belong to the sphere. Find the greatest value of the volume of the pyramid.

ANSWERS

1. ₹ 1920
2. Radius = $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$; Height = $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$
3. $\alpha = \tan^{-1}\frac{1}{\sqrt{2}}$
5. $\frac{64}{81}r^3$

Solved Examples

EXAMPLE 6.1

Find the possible values of a such that $f(x) = e^{2x} - (a+1)\,e^x + 2x$ is monotonically increasing for $x \in R$.

Sol. $f(x) = e^{2x} - (a+1)\,e^x + 2x$

$\therefore \quad f'(x) = 2e^{2x} - (a+1)\,e^x + 2$

Now, $2e^{2x} - (a+1)e^x + 2 \geq 0$ for all $x \in R$

or $\quad 2\left(e^x + \frac{1}{e^x}\right) - (a+1) \geq 0$ for all $x \in R$

or $\quad (a+1) \leq 2\left(e^x + \frac{1}{e^x}\right)$ for all $x \in R$

or $\quad \frac{a+1}{2} \leq 2 \qquad \left(\because e^x + \frac{1}{e^x} \text{ has minimum value } 2\right)$

or $\quad a \leq 3$

EXAMPLE 6.2

If $f(x) = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \log(x^2+x+1) + (\lambda^2 - 5\lambda + 3)x + 10$ is a decreasing function for all $x \in R$, find the permissible values of λ.

Sol. $f'(x) = \frac{2}{\sqrt{3}} \frac{1}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2} \cdot \frac{2}{\sqrt{3}} - \frac{2x+1}{x^2+x+1} + (\lambda^2 - 5\lambda + 3) \leq 0$

$= \frac{-2x}{x^2+x+1} + (\lambda^2 - 5\lambda + 3) \leq 0, \forall x \in R$

or $\quad (\lambda^2 - 5\lambda + 3) \leq \frac{2x}{x^2+x+1}, \forall x \in R \qquad (1)$

Now, let $y = \frac{2x}{x^2+x+1} = \frac{2}{x+1+\frac{1}{x}}$

Putting $x = 1$ and $x = -1$, $y = \frac{2}{3}$, $y = -2$, respectively.

So, range of $y \in \left[-2, \frac{2}{3}\right]$.

From (1), $\lambda^2 - 5\lambda + 3 < -2$

or $\quad \lambda^2 - 5\lambda + 5 < 0$

or $\quad \left(\lambda - \frac{5-\sqrt{5}}{2}\right)\left(\lambda - \frac{5+\sqrt{5}}{2}\right) \leq 0$

or $\quad \lambda \in \left[\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right]$

EXAMPLE 6.3

If $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ be an increasing function on the set R, then prove that $a^2 - 3b + 15 < 0$

Sol. $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ is increasing on R

$\therefore \quad f'(x) > 0 \ \forall x \in R$

$\Rightarrow \quad 3x^2 + 2ax + b + 5\sin 2x > 0 \ \forall x \in R$

$\Rightarrow \quad 3x^2 + 2ax + b > -5\sin 2x \ \forall x \in R$

$\Rightarrow \quad 3x^2 + 2ax + b > 5 \ \forall x \in R$

$\Rightarrow \quad 3x^2 + 2ax + (b-5) > 0 \ \forall x \in R$

$\Rightarrow \quad (2a)^2 - 4 \times 3 \times (b-5) < 0 \qquad$ (discriminant < 0)

$\Rightarrow \quad a^2 - 3b + 15 < 0$

EXAMPLE 6.4

Prove that $\left[\lim_{x\to 0^+} \frac{\sin x . \tan x}{x^2}\right] = 1$, where $[\cdot]$ represents the greatest integer function.

Sol. Let $f(x) = \sin x \tan x - x^2$

$\therefore \quad f'(x) = \sin x \sec^2 x + \sin x - 2x$

Since the sign of $f'(x)$ cannot be determined easily, differentiate $f'(x)$.

$\therefore \quad f''(x) = 2\sin x \tan x \sec^2 x + (\cos x + \sec x - 2)$

$> 0 \ \forall x \in \left(0, \frac{\pi}{2}\right)$

So, $f'(x)$ is an increasing function.

$\therefore \quad f'(x) > f'(0)$

$\Rightarrow \quad \sin x \sec^2 x + \sin x - 2x > 0$

$\Rightarrow \quad f(x)$ is an increasing function

$\Rightarrow \quad f(x) > f(0)$

$\Rightarrow \quad \sin x \tan x - x^2 > 0$

$\Rightarrow \quad \sin x \tan x > x^2$

Now $\quad \lim_{x\to 0+} \frac{\sin x \cdot \tan x}{x^2} = \lim_{x\to 0+} \frac{\sin x}{x} \cdot \frac{\tan x}{x} = 1$

Since $\sin x \tan x > x^2$, $\lim_{x\to 0+} \frac{\sin x \cdot \tan x}{x^2} = 1^+$

$\therefore \quad \left[\lim_{x\to 0+} \frac{\sin x . \tan x}{x^2}\right] = 1$

EXAMPLE 6.5

Using the relation $2(1 - \cos x) < x^2$, $x \neq 0$ or otherwise, prove that $\sin(\tan x) \geq x$, $\forall x \in \left[0, \frac{\pi}{4}\right]$.

Sol. Given that $2(1 - \cos x) < x^2$, $x \neq 0$.

To prove $\sin(\tan x) \geq x$, $x \in [0, \pi/4]$, let us consider

$f(x) = \sin(\tan x) - x$

or $\quad f'(x) = \cos(\tan x)\sec^2 x - 1$

$= \frac{\cos(\tan x) - \cos^2 x}{\cos^2 x}$

As given $2(1 - \cos x) < x^2$, $x \neq 0$

or $\quad \cos x > 1 - \frac{x^2}{2}$

Similarly, $\cos(\tan x) > 1 - \frac{\tan^2 x}{2}$

$\therefore \quad f'(x) > \frac{1 - \frac{1}{2}\tan^2 x - \cos^2 x}{\cos^2 x}$

$= \frac{\sin^2 x\left[1 - \frac{1}{2\cos^2 x}\right]}{\cos^2 x}$

$$= \frac{\sin^2 x(\cos 2x)}{2\cos^4 x} > 0 \ \forall x \in [0, \pi/4]$$

Therefore, $f'(x) > 0$. Thus, $f(x)$ is an increasing function.
So, for $x \in [0, \pi/4]$, we have

$x \geq 0$ or $f(x) \geq f(0)$

or $\sin(\tan x) - x \geq \sin(\tan 0) - 0$

or $\sin(\tan x) - x \geq 0$

or $\sin(\tan x) \geq x$.

EXAMPLE 6.6

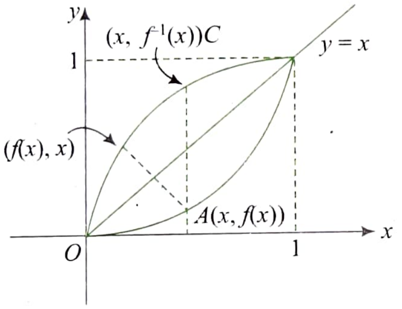
If $f'(x) > 0, f''(x) > 0 \ \forall x \in (0, 1)$ and $f(0) = 0, f(1) = 1$, then prove that $f(x) f^{-1}(x) < x^2$.

Sol. Given that $f'(x) > 0, f''(x) > 0 \ \forall x \in (0, 1)$

$\therefore$ $f(x)$ is increasing function and concave upward for $x \in (0, 1)$.

$\Rightarrow$ $f^{-1}(x)$ is increasing function and concave downward for $x \in (0, 1)$

Graphs of $y = f(x)$ and $y = f^{-1}(x)$ are as shown in the following figure.



From the figure

Slope of $OB >$ slope of OC

$$\Rightarrow \quad \frac{x}{f(x)} > \frac{f^{-1}(x)}{x}$$

$$\Rightarrow \quad f(x) \times f^{-1}(x) < x^2$$

EXAMPLE 6.7

Discuss the monotonicity of $g(x)$, where $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2) \ \forall x \in R$. It is given that $f''(x) > 0 \ \forall x \in R$. Find also the points of maxima and minima of $g(x)$.

Sol. Given $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$$\therefore \quad g'(x) = 2xf'\left(\frac{x^2}{2}\right) - 2xf'(6 - x^2)$$

$$= 2x\left\{f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right\}$$

But given that $f''(x) > 0$.

So, $f'(x)$ is increasing for all $x \in R$.

Let $g'(x) = 0$

$$\therefore \quad 2x\left\{f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right\} = 0$$

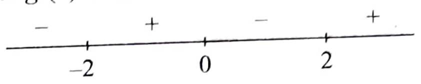
$$\Rightarrow \quad x = 0 \text{ or } f'\left(\frac{x^2}{2}\right) = f'(6 - x^2)$$

$$\Rightarrow \quad x = 0 \text{ or } \frac{x^2}{2} = 6 - x^2 \qquad \text{(as } f'(x) \text{ is increasing)}$$

$$\Rightarrow \quad x = 0 \text{ or } x^2 = 4$$

$$\Rightarrow \quad x = 0 \text{ or } x = \pm 2$$

Sign scheme of $g'(x)$ is as follows.



From the sign scheme of $g'(x)$, $g(x)$ increases in $(-2, 0) \cup (2, \infty)$ and decreases in $(-\infty, -2) \cup (0, 2)$

Also, $x = -2$ and $x = 2$ are points of minima and $x = 0$ is point of maxima.

EXAMPLE 6.8

Prove that $\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{e^2+1}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{e^2+1}}$.

Sol. We have to prove

$$\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{e^2+1}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{e^2+1}}$$

$$\text{or} \quad \left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2}{\sqrt{\left(\frac{1}{e}\right)^2 + 1}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{e^2+1}}$$

$$\text{or} \quad f\left(\frac{1}{e}\right) < f(e), \text{ where } f(x) = (\tan^{-1} x)^2 + \frac{2}{\sqrt{x^2+1}} \qquad \ldots(1)$$

$$\text{Now} \quad f'(x) = \frac{2\tan^{-1} x}{1+x^2} - \frac{2x}{(x^2+1)^{3/2}}$$

$$= \frac{2}{1+x^2}\left\{\tan^{-1} x - \frac{x}{\sqrt{x^2+1}}\right\} \qquad \ldots(2)$$

It is difficult to study the sign of $f'(x)$.

So, we let $g(x) = \tan^{-1} x - \frac{x}{\sqrt{x^2+1}}$

$$\therefore \quad g'(x) = \frac{1}{1+x^2}\left\{1 - \frac{1}{\sqrt{x^2+1}}\right\} > 0$$

$\therefore$ $g(x)$ is an increasing function

$\therefore$ for $x > 0$, $g(x) > g(0)$

$\Rightarrow$ $g(x) > 0$ (as $g(0) = 0$)

$\Rightarrow$ $f'(x) > 0$ {From (2)}

$\Rightarrow$ $f(x)$ is increasing function

Since $\frac{1}{e} < e$, we have $f\left(\frac{1}{e}\right) < f(e)$

Example 6.9

Prove that $\sin^2 \theta < \theta \sin(\sin \theta)$ for $0 < \theta < \dfrac{\pi}{2}$.

Sol. We have to prove that $\sin^2 \theta < \theta \sin(\sin \theta)$

or $$\frac{\sin(\sin\theta)}{\sin\theta} > \frac{\sin\theta}{\theta}$$

or $f(\sin \theta) > f(\theta)$, where $f(x) = \dfrac{\sin x}{x}$

Now $$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

$$= \frac{\cos x(x - \tan x)}{x^2} < 0 \text{ (as } x < \tan x \text{ for } x \in (0, \pi/2))$$

$\therefore$ $f(x)$ is decreasing function

Also, we know that $\sin \theta < \theta$, for $0 < \theta < \dfrac{\pi}{2}$

$\Rightarrow$ $f(\sin \theta) > f(\theta)$ (as $f(x)$ is decreasing)

$\Rightarrow$ $$\frac{\sin(\sin\theta)}{\sin\theta} > \frac{\sin\theta}{\theta}$$

$\Rightarrow$ $\sin^2 \theta < \theta \sin(\sin \theta)$ for $0 < \theta < \dfrac{\pi}{2}$

Example 6.10

Let $f(x) = x^3 - 3x^2 + 6\ \forall\ x \in R$

and $$g(x) = \begin{cases} \max. f(t);\ x+1 \le t \le x+2, -3 \le x \le 0 \\ 1 - x, \qquad x \ge 0 \end{cases}.$$

Find the function $g(x)$.

Sol. Clearly nature of $g(x)$ depends upon the nature of $f(x)$.

So, we first find the behavior of function $f(x)$.

We have $f(x) = x^3 - 3x^2 + 6$

$\therefore$ $f'(x) = 3x^2 - 6x = 3x(x - 2)$

Sign scheme of $f'(x)$

$$\underset{}{+}\quad \underset{0}{|} \quad - \quad \underset{2}{|} \quad +$$

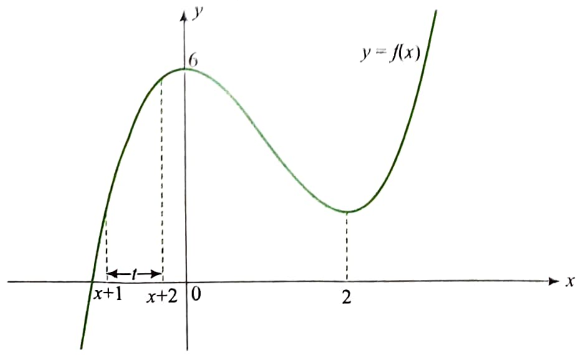
Clearly $x = 0$ is point of maxima and $x = 2$ is point of minima.

$f(0) = 6$ and $f(2) = 2$

Graph of the $f(x)$ is as shown in the following figure.

Now for $g(x) = \max. f(t)$, $x + 1 \le t \le x + 2$, we have following cases.

Case 1: $x + 2 \le 0 \Rightarrow x \le -2$

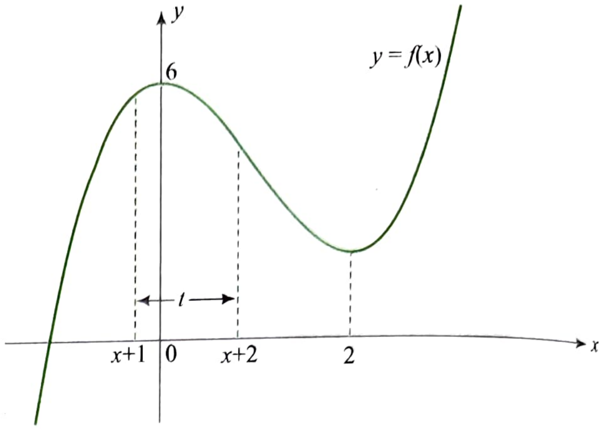


In this case $f(x)$ is increasing function.

So, $\max. f(t) = f(x + 2)$, where $-3 \le x \le -2$

$\therefore$ $g(x) = (x + 2)^3 - 3(x + 2)^2 + 6$

Case 2: $x + 1 < 0$ and $0 < x + 2 < 2$



$\Rightarrow$ $x < -1$ and $-2 < x < 0$

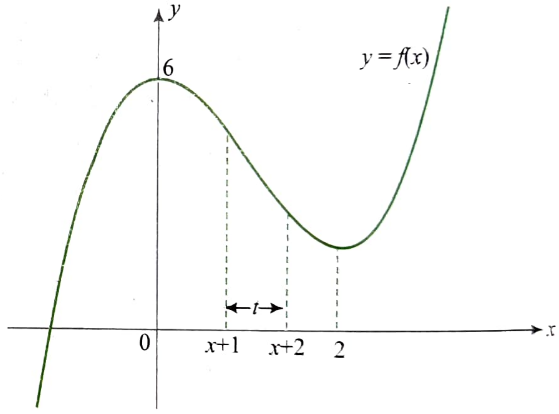
$\Rightarrow$ $-2 < x < -1$

In this case $\max. f(t) = f(0)$

$\therefore$ $g(x) = 6$

Case 3 : $0 \le x + 1, x + 2 \le 2$

$\Rightarrow$ $-1 \le x \le 0$



In this case $f(x)$ is decreasing.

$\therefore$ $\max. f(t) = f(x + 1)$

$\therefore$ $g(x) = (x + 1)^3 - 3(x + 1)^2 + 6$

Thus, $$g(x) = \begin{cases} (x+2)^3 - (x+2)^2 + 6, & -3 \le x \le -2 \\ f(0), & -2 < x < -1 \\ (x+1)^3 - (x+1)^2 + 6, & 1 \le x < 0 \\ 1 - x, & x \ge 0 \end{cases}$$

Example 6.11

Show that $5x \le 8 \sin x - \sin 2x \le 6x$ for $0 \le x \le \dfrac{\pi}{3}$.

Sol. Let $f(x) = 8 \sin x - \sin 2x$

$\therefore$ $f'(x) = 8 \cos x - 2 \cos 2x$

$\Rightarrow$ $f''(x) = -8 \sin x + 4 \sin 2x$

$= -8 \sin x(1 - \cos x)$

$\therefore$ $f''(x) < 0$ in $\left[0, \dfrac{\pi}{3}\right]$

So, $f'(x)$ is decreasing function in $\left[0, \frac{\pi}{3}\right]$

$\therefore \quad f'\left(\frac{\pi}{3}\right) \le f'(x) \le f'(0)$

$f'\left(\frac{\pi}{3}\right) = 5$ and $f'(0) = 6$

$\therefore \quad 5 \le f'(x) \le 6$ in $\left[0, \frac{\pi}{3}\right]$.

Integrating from 0 to x, we get

$\Rightarrow \quad 5x \le f(x) \le 6x$ in $\left[0, \frac{\pi}{3}\right]$

EXAMPLE 6.12

Let $f(x)$, $x \ge 0$, be a non-negative continuous function.

If $f'(x) \cos x \le f(x) \sin x$, $\forall x \ge 0$, then find $f\left(\frac{5\pi}{3}\right)$.

Sol. Given $f(x) \ge 0 \ \forall x \ge 0$ ...(1)

and $\quad f'(x) \cos x - f(x) \sin x \le 0$

$\Rightarrow \quad (f(x) \cos x)' \le 0$ ...(2)

Let $\quad g(x) = f(x) \cos x$

$\therefore \quad g'(x) \le 0$

$\Rightarrow \quad g(x)$ is a decreasing function.

Now $\quad f\left(\frac{5\pi}{3}\right) \ge 0 \quad$ (as $f(x) \ge 0 \ \forall x \ge 0$) ...(2)

Also $\quad g\left(\frac{\pi}{2}\right) \ge g\left(\frac{5\pi}{3}\right) \quad$ (as $g(x)$ is decreasing)

$\Rightarrow \quad g\left(\frac{5\pi}{3}\right) \le 0$

$\Rightarrow \quad f\left(\frac{5\pi}{3}\right) \le 0$ ...(3)

From statement (2) and (3), $f\left(\frac{5\pi}{3}\right) = 0$

EXAMPLE 6.13

If $ax^2 + \frac{b}{x} \ge c$ for all positive x where $a > 0$ and $b > 0$ show that $27ab^2 \ge 4c^3$.

Sol. We have

$$ax^2 + \frac{b}{x} - c \ge 0, \forall x > 0 \quad ...(1)$$

So, least value of $f(x) = ax^2 + \frac{b}{x} - c$ is non–negative.

Now $\quad f'(x) = 2ax - \frac{b}{x^2}$

$f'(x) = 0 \Rightarrow x^3 = b/2a \Rightarrow x = \left(\frac{b}{2a}\right)^{1/3}$

Also $\quad f''(x) = 2a + \frac{2b}{x^3}$

$\Rightarrow \quad f''\left(\left(\frac{b}{2a}\right)^{1/3}\right) = 2a + \frac{2b}{b} \times 2a = 6a > 0$

$\therefore \quad f$ is min at $x = \left(\frac{b}{2a}\right)^{1/3}$

Since (1) is true $\forall x$, we must have

$f\left(\left(\frac{b}{2a}\right)^{1/3}\right) \ge 0$

$\Rightarrow \quad a\left(\frac{b}{2a}\right)^{2/3} + \frac{b}{(b/2a)^{1/3}} - c \ge 0$

$\Rightarrow \quad \frac{a\left(\frac{b}{2a}\right) + b}{(b/2a)^{1/3}} \ge c$

$\Rightarrow \quad \frac{3b}{2}\left(\frac{2a}{b}\right)^{1/3} \ge c$

$\Rightarrow \quad \frac{3b^{2/3}}{2}(2a)^{1/3} \ge c$

$\Rightarrow \quad \frac{27b^2}{8} . 2a \ge c^3$

$\Rightarrow \quad 27ab^2 \ge 4c^3$

EXAMPLE 6.14

Prove that for $x \in \left[0, \frac{\pi}{2}\right]$, $\sin x + 2x \ge \frac{3x(x+1)}{\pi}$.

Sol. Let $f(x) = \sin x + 2x - \frac{3x(x+1)}{\pi}$

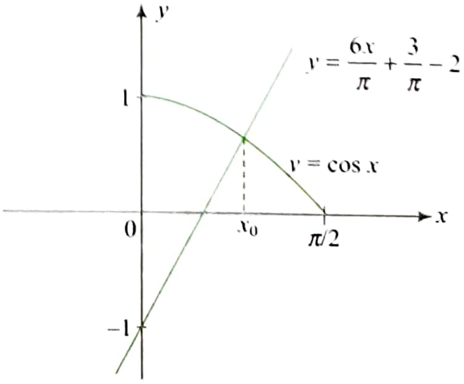
$\therefore \quad f'(x) = \cos x + 2 - \frac{3}{\pi}(2x+1)$

$= \cos x - \left(\frac{6x}{\pi} + \frac{3}{\pi} - 2\right)$

Let $\quad f'(x) = 0$

$\therefore \quad \cos x = \frac{6}{\pi}x + \frac{3}{\pi} - 2$

Graphs of $y = \cos x$ and $y = \frac{6}{\pi}x + \frac{3}{\pi} - 2$ are as shown in the following figure.



There is only one point of intersection for which $x = x_0$.

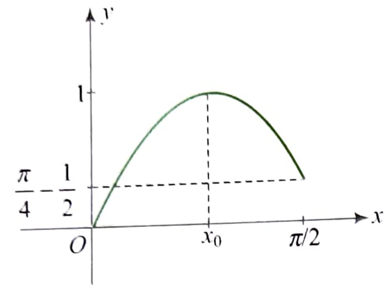
Clearly for $x < x_0$, $\cos x > \frac{6}{\pi}x + \frac{3}{\pi} - 2$. So, $f'(x) > 0$.

For $x > x_0$, $f'(x) < 0$.

So, $x = x_0$ is point of maxima.

Now $f(0) = 0$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{4} - \frac{1}{2}$

So, graph of the function is as shown in the following figure.



From the graph $f(x) \geq 0$ for $x \in \left[0, \frac{\pi}{2}\right]$

$$\Rightarrow \quad \sin x + 2x - \frac{3x}{\pi}(x+1) \geq 0 \text{ for } x \in \left[0, \frac{\pi}{2}\right]$$

Example 6.15

Find the set of values of p for which line $2px - 4y + 2\pi - p = 0$ intersect the curve $y = \cos^{-1}(2x - 1)$ at three distinct points.

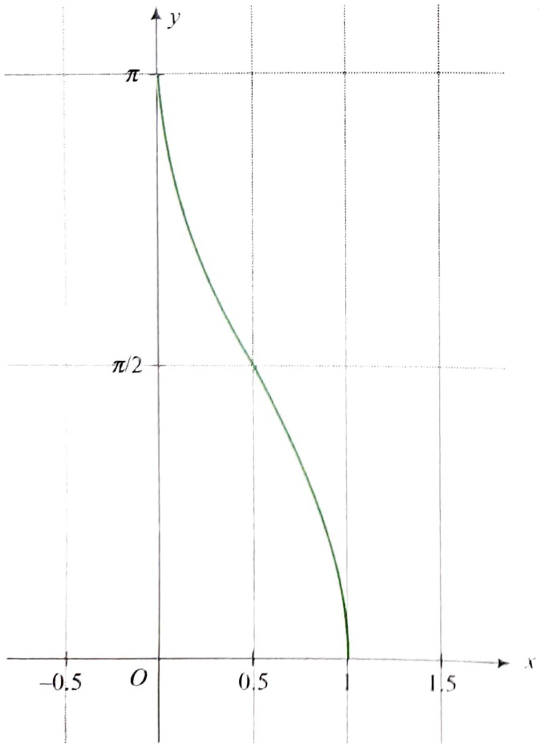
Sol. We have curve $y = f(x) = \cos^{-1}(2x - 1)$ ...(1)

and line $2px - 4y + 2\pi - p = 0$ ...(2)

Domain of $y = f(x) = \cos^{-1}(2x - 1)$ is $[0, 1]$.

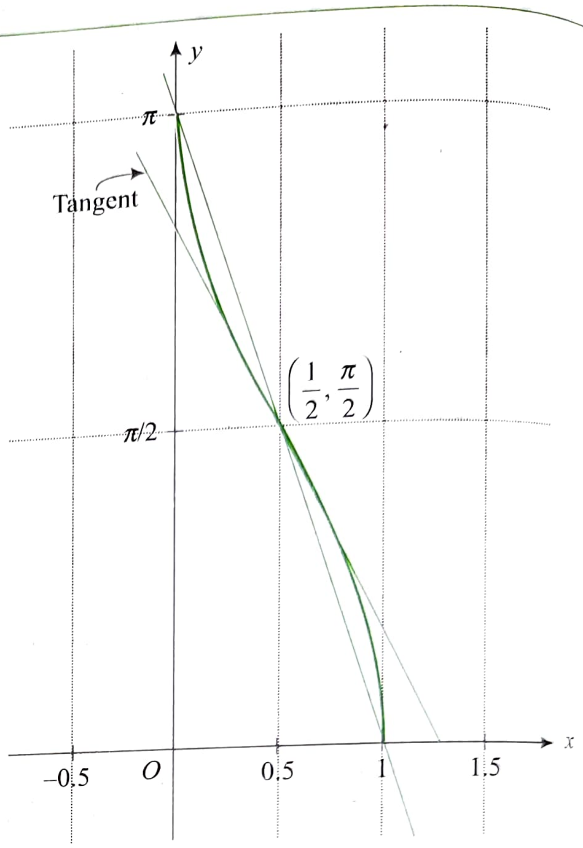
$f(0) = \cos^{-1}(-1) = \pi$ and $f(1) = \cos^{-1}1 = 0$

Graph of the function $y = f(x)$ is as shown in the following figure.



Clearly curve has point of inflection at $(1/2, \pi/2)$.

Examining the line, we find that it passes through $(1/2, \pi/2)$.



This line must intersect the curve at three distinct points.

Now $\frac{dy}{dx} = \frac{-2}{\sqrt{1-(2x-1)^2}} = \frac{-1}{\sqrt{x-x^2}} = -(x-x^2)^{-1/2}$

$\therefore \quad \left(\frac{dy}{dx}\right)_{x=0.5} = -2$

= slope of tangent to the curve at $(1/2, \pi/2)$.

It can be verified that that points $(0, \pi)$ and $(1/2, \pi/2)$ and $(1, 0)$ are collinear.

Slope of line joining these points is $\frac{\pi - \frac{\pi}{2}}{0 - \frac{1}{2}} = -\pi$

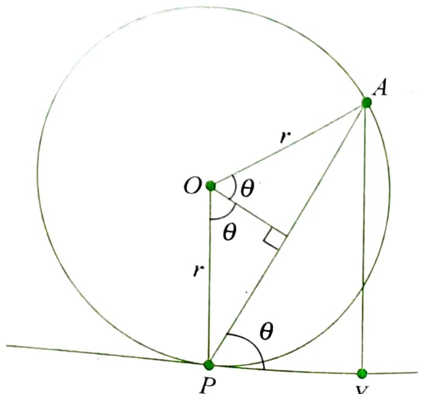
Hence given line intersect the curve at three distinct point if its slope is less than '−2' but more than or equal to $-\pi$.

$\therefore \quad \frac{p}{2} \in [-\pi, -2) \Rightarrow p \in [-2\pi, -4)$

Example 6.16

From a fixed point A on the circumference of a circle of radius r, the perpendicular AY falls on the tangent at P. Find the maximum area of triangle APY.

Sol.



From the figure, $AP = 2r \sin\theta$

$PY = 2r\sin\theta\cos\theta = r\sin 2\theta$

and $AY = 2r\sin\theta\sin\theta$

$\therefore \quad \Delta = \text{Area of } \Delta APY = \frac{1}{2} PY \cdot AY$

$$= r^2 \sin^2\theta \sin 2\theta,\ 0 < \theta < \pi/2 \quad (1)$$

$$d\Delta/d\theta = r^2 [\sin^2 2\theta + 2\cos 2\theta \sin^2\theta]$$
$$= r^2[4\sin^2\theta\cos^2\theta + 2\sin^2\theta\cos 2\theta]$$
$$= 2r^2\sin^2\theta[4\cos^2\theta - 1]$$

$$d\Delta/d\theta = 0 \Rightarrow \sin\theta = 0,\ \cos\theta = \pm 1/2$$

Therefore, the only critical point within $(0, \pi/2)$ is $\pi/3$. Clearly,

$$\left(\frac{d\Delta}{d\theta}\right)_{(\pi/3-h)} > 0 \text{ and } \left(\frac{d\Delta}{d\theta}\right)_{(\pi/3+h)} < 0$$

Thus, Δ is maximum at $\theta = \pi/3$. Therefore, the greatest area of such triangle $= (3\sqrt{3}r^2)/8$.

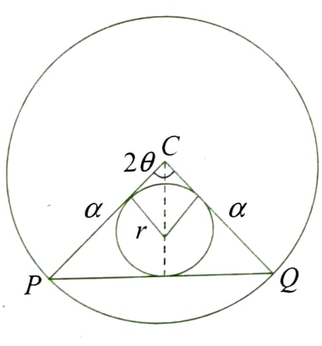
EXAMPLE 6.17

P and Q are two points on a circle of center C and radius α. The angle PCQ being 2θ, find the value of $\sin\theta$ when the radius of the circle inscribed in the triangle CPQ is maximum.

Sol. We know that $r = \frac{\Delta}{s}$, where

Δ = Area of triangle CPQ and s = semiperimeter of ΔCPQ

$$\therefore \quad r = \frac{\alpha^2 \sin 2\theta}{2s} = \frac{\alpha^2\sin 2\theta}{2\alpha + 2\alpha\sin\theta} = \frac{\alpha}{2}\cdot\frac{\sin 2\theta}{1+\sin\theta}$$



Now, for $f(\theta) = \frac{\sin 2\theta}{1+\sin\theta}$,

$$f'(\theta) = \frac{(1+\sin\theta)\,2\cos 2\theta - \sin 2\theta\cdot\cos\theta}{(1+\sin\theta)^2} = 0$$

or $2(1+\sin\theta)(1-2\sin^2\theta) - 2\sin\theta(1-\sin^2\theta) = 0$

or $2(1-2\sin^2\theta) = 2\sin\theta(1-\sin\theta)$

or $1 - 2\sin^2\theta = \sin\theta - \sin^2\theta$

or $\sin^2\theta + \sin\theta - 1 = 0$

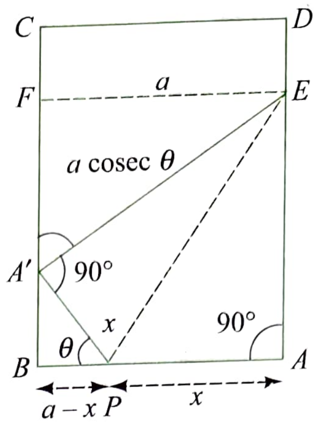
or $\sin\theta = \frac{\sqrt{5}-1}{2}$

EXAMPLE 6.18

The lower corner of a leaf in a book is folded over so as to reach the inner edge of the page. Show that the fraction of the width folded over when the area of the folded part is minimum is 2/3.

Sol. Let the corner A of the leaf $ABCD$ be folded over to A' which is on the inner edge BC of the page.

Let $AP = x$ and $AB = a$. Therefore, $BP = a - x$.



If $\angle A'PB = \theta$, then $\angle EA'F = \theta$. EF is parallel to AB, then $EF = a$.

In $\Delta A'BP$, $\cos\theta = BP/PA' = (a-x)/x$.

In $\Delta A'FE$, $A'E = EF \operatorname{cosec}\theta = a/\sqrt{1-\cos^2\theta}$

$$= a/\sqrt{1-\{(a-x)/x\}^2}$$

$$= \frac{ax}{\sqrt{\left[x^2-(a-x)^2\right]}} = \frac{ax}{\sqrt{(2ax-a^2)}} = AE$$

Triangle APE is folded to triangle $A'PE$. Therefore,

$$\text{Area of folded part} = \text{Area of } \Delta PAE = \frac{1}{2} AP\cdot AE$$

$$= \frac{1}{2}x\frac{ax}{\sqrt{(2ax-a^2)}} = A \text{ (say)}$$

or $$A^2 = \frac{a^2x^4}{4(2ax-a^2)} = \frac{a^2/4}{(2a/x^3 - a^2/x^4)} = \frac{a^2/4}{y}$$

where $y = (2a/x^3) - a^2/x^4$, $0 < x < a$.

Obviously, A^2 (i.e., A) is minimum when y is maximum. Now,

$$y = 2a/x^3 - a^2/x^4$$

or $$\frac{dy}{dx} = -\frac{6a}{x^4} + \frac{4a^2}{x^5} \text{ and } \frac{d^2y}{dx^2} = \frac{24a}{x^5} - \frac{20a^2}{x^6}$$

For maximum or minimum of y,

$$\frac{dy}{dx} = -\frac{6a}{x^4} + \frac{4a^2}{x^5} = 0 \text{ or } x = 2a/3$$

When $x = 2a/3$, $\frac{d^2y}{dx^2} = \frac{4a}{x^5}\left(6 - \frac{5a}{x}\right)$

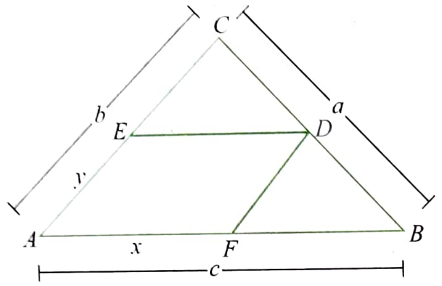
$$= 4a\left(\frac{3}{2a}\right)^5\left(6 - 5a\frac{3}{2a}\right) = -4a\left(\frac{3}{2a}\right)^5\frac{3}{2} < 0$$

Thus, y is maximum, i.e., A is minimum, when $x = 2a/3$ which is the only critical point (least). Hence, the folded area is minimum when 2/3 of the width of the page is folded over.

EXAMPLE 6.19

Let $A(p^2, -p)$, $B(q^2, q)$, $C(r^2, -r)$ be the vertices of triangle ABC. A parallelogram $AFDE$ is drawn with D, E, and F on the line segments BC, CA, and AB, respectively. Using calculus, show that the maximum area of such a parallelogram is $\frac{1}{4}|(p+q)(q+r)(p-r)|$.

Sol.



From similar Δs FBD and ABC, $\frac{c-x}{c} = \frac{y}{b}$

or $\quad y = (b/c)(c-x)$

$\therefore \quad Z = \text{Area of } AFDE = xy \sin A = \frac{b \sin A}{c}(cx - x^2) \qquad (1)$

$0 < x < c$

$\therefore \quad \frac{dZ}{dx} = \frac{b \sin A}{c}(c-2x) = 0 \text{ or } x = c/2$

$$\left(\frac{d^2Z}{dx^2}\right)_{x=c/2} = \frac{-2}{c} b \sin A < 0$$

Thus, Z has maxima at $x = \frac{c}{2}$. So, the greatest area of parallelogram $AFDE$ is

$$(b/c)\sin A\,(c^2/4) = \frac{1}{2}\left(\frac{1}{2}bc \sin A\right)$$

$$= \frac{1}{2}\Delta_{ABC} = \frac{1}{2}\times\frac{1}{2}\left\|\begin{matrix} p^2 & -p & 1 \\ q^2 & q & 1 \\ r^2 & -r & 1 \end{matrix}\right\|$$

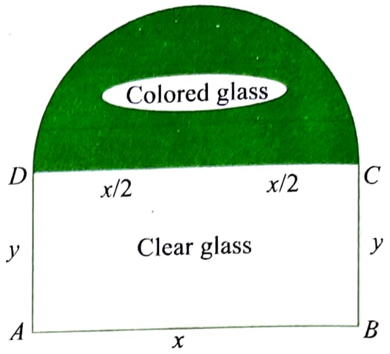
$$= \frac{1}{4}\left\|\begin{matrix} p^2 & -p & 1 \\ q^2-p^2 & q+p & 0 \\ r^2-p^2 & -r+p & 0 \end{matrix}\right\| \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \frac{1}{4}|(q+p)(q+r)(p-r)|$$

EXAMPLE 6.20

A window of perimeter P (including the base of the arch) is in the form of a rectangle surrounded by a semi-circle. The semi-circular portion is fitted with the colored glass while the rectangular part is fitted with the clear glass that transmits three times as much light per square meter as the colored glass does. What is the ratio for the sides of the rectangle so that the window transmits the maximum light?

Sol. Let x and y meters be the lengths of the sides of rectangle $ABCD$ and let there be a semi-circle on side CD of length x.



Therefore, perimeter of the window (including the base of the arch) = perimeter of the rectangle + perimeter of the semi-circle

$$= 2x + 2y + \frac{1}{2}(2\pi x/2)$$

$$= 2x + 2y + \frac{1}{2}\pi x = c \quad \text{(constant)}$$

$$\therefore \quad y = \frac{1}{2}\left(c - 2x - \frac{1}{2}\pi x\right) \qquad (1)$$

Let k be the light per square meter transmitted by colored glass so that transmitted by clear glass will be $3k$ per square meter.

Hence, the total light transmitted by the window is given by

$A = (\text{Area of colored glass})\,k + (\text{Area of clear glass})\,3k$

$$= \frac{1}{2}\pi(x/2)^2 k + xy(3k)$$

$$= \frac{1}{8}\pi k x^2 + 3kx\,\frac{1}{2}\left(c - 2x - \frac{1}{2}\pi x\right)$$

[Substituting for y from (1)]

$$= \frac{1}{8}k(-5\pi x^2 - 24x^2 + 12cx)$$

$$\therefore \quad \frac{dA}{dx} = \frac{1}{8}k(-10\pi x - 48x + 12c)$$

and $\quad \frac{d^2A}{dx^2} = -\frac{1}{4}(5\pi + 24)k = -\text{ve}$

For maximum or minimum of A, $dA/dx = 0$

or $\quad x = 6c/(5\pi + 24)$

Therefore, from (1), $y = (\pi + 6)\,c/(5\pi + 24)$.

Since $\frac{d^2A}{dx^2}$ is −ve, A has maxima

Hence, ratio $x/y = 6(\pi + 6)$.

Exercises

Single Correct Answer Type

1. A function is matched below against an interval where it is supposed to be increasing. Which of the following parts is incorrectly matched?

Interval	Function
(1) $[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(2) $(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(3) $(-\infty, -4]$	$x^3 + 6x^2 + 6$
(4) $\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$

2. On which of the following intervals is the function $x^{100} + \sin x - 1$ decreasing?
(1) $(0, \pi/2)$ (2) $(0, 1)$
(3) $(\pi/2, \pi)$ (4) None of these

3. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in
(1) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$ (2) $\left(0, \frac{\pi}{2}\right)$
(3) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (4) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

4. Consider the following statements in S and R
S: Both $\sin x$ and $\cos x$ are decreasing functions in the interval $\left(\frac{\pi}{2}, \pi\right)$.
R: If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b). Which of the following is true?
(1) Both S and R are wrong.
(2) Both S and R are correct, but R is not the correct explanation of S.
(3) S is correct and R is the correct explanation for S.
(4) S is correct and R is wrong.

5. The length of the longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing is
(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{2}$ (3) $\frac{3\pi}{2}$ (4) π

6. The function x^x decreases in the interval
(1) $(0, e)$ (2) $(0, 1)$
(3) $\left(0, \frac{1}{e}\right)$ (4) none of these

7. Let $f(x) = x\sqrt{4ax - x^2}$, $(a > 0)$. Then $f(x)$ is
(1) increasing in $(0, 3a)$, decreasing in $(3a, 4a)$
(2) increasing in $(a, 4a)$, decreasing in $(5a, \infty)$
(3) increasing in $(0, 4a)$
(4) none of these

8. Function $f(x) = |x| - |x - 1|$ is monotonically increasing when
(1) $x < 0$ (2) $x > 1$ (3) $x < 1$ (4) $0 < x < 1$

9. If $f'(x) = |x| - \{x\}$, where $\{x\}$ denotes the fractional part of x, then $f(x)$ is decreasing in
(1) $\left(-\frac{1}{2}, 0\right)$ (2) $\left(-\frac{1}{2}, 2\right)$
(3) $\left(-\frac{1}{2}, 2\right)$ (4) $\left(\frac{1}{2}, \infty\right)$

10. The length of the largest continuous interval in which the function $f(x) = 4x - \tan 2x$ is monotonic is
(1) $\pi/2$ (2) $\pi/4$ (3) $\pi/8$ (4) $\pi/16$

11. $f(x) = (x - 2)|x - 3|$ is monotonically increasing in
(1) $(-\infty, 5/2) \cup (3, \infty)$ (2) $(5/2, \infty)$
(3) $(2, \infty)$ (4) $(-\infty, 3)$

12. $f(x) = (x-8)^4(x-9)^5, 0 \le x \le 10$, monotonically decreases in
(1) $\left(\frac{76}{9}, 10\right]$ (2) $\left(8, \frac{76}{9}\right)$
(3) $[0, 8)$ (4) $\left(\frac{76}{9}, 10\right]$

13. For all $x \in (0, 1)$,
(1) $e^x < 1 + x$ (2) $\log_e(1 + x) < x$
(3) $\sin x > x$ (4) $\log_e x > x$

14. If $f(x) = xe^{x(x-1)}$, then $f(x)$ is
(1) increasing on $[-1/2, 1]$ (2) decreasing on R
(3) increasing on R (4) decreasing on $[-1/2, \infty]$

15. If $f(x) = kx^3 - 9x^2 + 9x + 3$ is monotonically increasing in R, then
(1) $k < 3$ (2) $k \le 2$
(3) $k \ge 3$ (4) none of these

16. If the function $f(x) = \dfrac{K\sin x + 2\cos x}{\sin x + \cos x}$ is strictly increasing for all values of x, then
(1) $K < 1$ (2) $K > 1$ (3) $K < 2$ (4) $K > 2$

17. Let $f: R \to R$ be a function such that $f(x) = ax + 3\sin x + 4\cos x$. Then $f(x)$ is invertible if
(1) $a \in (-5, 5)$ (2) $a \in (-\infty, 5)$
(3) $a \in (-5, +\infty)$ (4) none of these

18. Which of the following statement is always true for differentiable functions $f(x)$ and $g(x)$?
(1) If $f(x)$ is increasing then $f^{-1}(x)$ is decreasing.
(2) If $f(x)$ is increasing then $1/f(x)$ is also increasing.
(3) If f and g are positive functions and f is increasing and g is decreasing then f/g is decreasing function
(4) If f and g are positive functions and f is decreasing and g is increasing then f/g is decreasing function

19. Let $f: R \to R$ be a differentiable function for all values of x and has the property that $f(x)$ and $f'(x)$ have opposite signs for all values of x. Then,

(1) $f(x)$ is an increasing function
(2) $f(x)$ is a decreasing function
(3) $f^2(x)$ is a decreasing function
(4) $|f(x)|$ is an increasing function

20. Let $f: R \to R$ be a differentiable function $\forall x \in R$. If the tangent drawn to the curve at any point $x \in (a, b)$ always lies below the curve, then

(1) $f'(x) > 0, f''(x) < 0 \ \forall x \in (a, b)$
(2) $f'(x) < 0, f''(x) < 0 \ \forall x \in (a, b)$
(3) $f'(x) > 0, f''(x) > 0 \ \forall x \in (a, b)$
(4) none of these

21. Let $f(x)$ be a function such that
$f'(x) = \log_{1/3} [\log_3(\sin x + a)]$.
If $f(x)$ is decreasing for all real values of x, then

(1) $a \in (1, 4)$ (2) $a \in [4, \infty)$
(3) $a \in (2, 3)$ (4) $a \in [2, \infty)$

22. If $f(x) = x^3 + 4x^2 + \lambda x + 1$ is a monotonically decreasing function of x in the largest possible interval $(-2, -2/3)$. Then

(1) $\lambda = 4$ (2) $\lambda = 2$
(3) $\lambda = -1$ (4) λ has no real value

23. $f(x) = |x \log_e x|$ monotonically decreases in

(1) $(0, 1/e)$ (2) $(1/e, 1)$
(3) $(1, \infty)$ (4) $(1/e, \infty)$

24. The set of value(s) of a for which the function $f(x) = \dfrac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ possesses a negative point of inflection is

(1) $(-\infty, -2) \cup (0, \infty)$ (2) $\{-4/5\}$
(3) $(-2, 0)$ (4) empty set

25. The maximum value of the function $f(x) = \dfrac{(1+x)^{0.6}}{1+x^{0.6}}$ in the interval $[0, 1]$ is

(1) $2^{0.4}$ (2) $2^{-0.4}$ (3) 1 (4) $2^{0.6}$

26. Suppose that f is a polynomial of degree 3 and that $f''(x) \neq 0$ at any of the stationary point. Then

(1) f has exactly one stationary point
(2) f must have no stationary point
(3) f must have exactly two stationary points
(4) f has either zero or two stationary points

27. A function $g(x)$ is defined as $g(x) = \dfrac{1}{4} f(2x^2 - 1) + \dfrac{1}{2} f(1 - x^2)$ and $f'(x)$ is an increasing function. Then $g(x)$ is increasing in the interval

(1) $(-1, 1)$ (2) $\left(-\sqrt{\dfrac{2}{3}}, 0\right) \cup \left(\sqrt{\dfrac{2}{3}}, \infty\right)$
(3) $\left(-\sqrt{\dfrac{2}{3}}, \sqrt{\dfrac{2}{3}}\right)$ (4) none of these

28. If $\phi(x)$ is a polynomial function and $\phi'(x) > \phi(x) \ \forall x \geq 1$ and $\phi(1) = 0$, then

(1) $\phi(x) \geq 0 \ \forall x \geq 1$ (2) $\phi(x) < 0 \ \forall x \geq 1$
(3) $\phi(x) = 0 \ \forall x \geq 1$ (4) none of these

29. If $f''(x) > 0 \ \forall x \in R, f'(3) = 0$, and $g(x) = f(\tan^2 x - 2\tan x + 4)$, $0 < x < \dfrac{\pi}{2}$, then $g(x)$ is increasing in

(1) $\left(0, \dfrac{\pi}{4}\right)$ (2) $\left(\dfrac{\pi}{6}, \dfrac{\pi}{3}\right)$
(3) $\left(0, \dfrac{\pi}{3}\right)$ (4) $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

30. If $f(x) = x + \sin x$, $g(x) = e^{-x}$, $u = \sqrt{c+1} - \sqrt{c}$, $v = \sqrt{c} - \sqrt{c-1}$, $(c > 1)$, then

(1) $fog(u) < fog(v)$ (2) $gof(u) < gof(v)$
(3) $gof(u) > gof(v)$ (4) $fog(u) < fog(v)$

31. The number of solutions of the equation $x^3 + 2x^2 + 5x + 2\cos x = 0$ in $[0, 2\pi]$ is

(1) one (2) two (3) three (4) zero

32. Let $f(x) = \cos \pi x + 10x + 3x^2 + x^3, -2 \leq x \leq 3$. The absolute minimum value of $f(x)$ is

(1) 0 (2) -15
(3) $3 - 2\pi$ (4) none of these

33. The global maximum value of $f(x) = \log_{10}(4x^3 - 12x^2 + 11x - 3)$, $x \in [2, 3]$, is

(1) $-\dfrac{3}{2}\log_{10} 3$ (2) $1 + \log_{10} 3$
(3) $\log_{10} 3$ (4) $\dfrac{3}{2}\log_{10} 3$

34. $f: R \to R, f(x)$ is differentiable such that $f(f(x)) = k(x^5 + x)$, $(k \neq 0)$. Then $f(x)$ is

(1) increasing
(2) decreasing
(3) either increasing or decreasing
(4) non-monotonic

35. The value of a for which the function $f(x) = a \sin x + (1/3) \sin 3x$ has an extremum at $x = \pi/3$ is

(1) 1 (2) -1 (3) 0 (4) 2

36. If $f(x) = a \log |x| + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then

(1) $a = 2, b = -1$ (2) $a = 2, b = -1/2$
(3) $a = -2, b = 1/2$ (4) none of these

37. If a function $f(x)$ has $f'(a) = 0$ and $f''(a) = 0$, then

(1) $x = a$ is a maximum for $f(x)$
(2) $x = a$ is a minimum for $f(x)$
(3) it is difficult to say (a) and (b)
(4) $f(x)$ is necessarily a constant function

38. Which of the following statements is true for the function

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 1 \\ x^3, & 0 \leq x \leq 1 \\ \dfrac{x^3}{3} - 4x, & x < 0 \end{cases}$$

(1) It is monotonic increasing $\forall x \in R$.
(2) $f'(x)$ fails to exist for three distinct real values of x.
(3) $f'(x)$ changes its sign twice as x varies from $-\infty$ to ∞.
(4) The function attains its extreme values at x_1 and x_2, such that $x_1 x_2 > 0$.

39. The function $f(x) = \sin^4 x + \cos^4 x$ increases if
(1) $0 < x < \pi/8$ (2) $\pi/4 < x < 3\pi/8$
(3) $3\pi/8 < x < 5\pi/8$ (4) $5\pi/8 < x < 3\pi/4$

40. If $f(x) = x^5 - 5x^4 + 5x^3 - 10$ has local maximum and minimum at $x = p$ and $x = q$, respectively, then $(p, q) \equiv$
(1) $(0, 1)$ (2) $(1, 3)$
(3) $(1, 0)$ (4) none of these

41. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n}$ be a polynomial in a real variable x with $0 < a_0 < a_1 < a_2 < \cdots < a_n$. The function $P(x)$ has
(1) neither a maximum nor a minimum
(2) only one maximum
(3) only one minimum
(4) only one maximum and only one minimum

42. Let $f(x) = \begin{cases} |x|, & \text{for } 0 < |x| \le 2 \\ 1, & \text{for } x = 0 \end{cases}$. Then at $x = 0$, f has
(1) a local maximum (2) no local maximum
(3) a local minimum (4) no extremum

43. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$,
(1) $f(x)$ is a strictly increasing function
(2) $f(x)$ has local maxima
(3) $f(x)$ is a strictly decreasing function
(4) $f(x)$ is bounded

44. If $f(x) = \begin{cases} \sin(x^2 - 3x), & x \le 0 \\ 6x + 5x^2, & x > 0 \end{cases}$, then at $x = 0$
(1) $f(x)$ has a local minima
(2) $f(x)$ has a local maxima
(3) $f(x)$ has point of inflection
(4) none of these

45. The greatest value of $f(x) = \cos(xe^{[x]} + 7x^2 - 3x)$, $x \in [-1, \infty)$, is (where $[\cdot]$ represents the greatest integer function)
(1) -1 (2) 1
(3) 0 (4) none of these

46. The function $f(x) = (4\sin^2 x - 1)^n (x^2 - x + 1)$, $n \in N$, has a local minimum at $x = \dfrac{\pi}{6}$. Then
(1) n is any even integer
(2) n is an odd integer
(3) n is odd prime number
(4) n is any natural number

47. All possible values of x for which the function, $f(x) = x\log_e x - x + 1$ is positive is
(1) $(1, \infty)$ (2) $(1/e, \infty)$
(3) $[e, \infty)$ (4) $(0, 1) \cup (1, \infty)$

48. The greatest value of $f(x) = (x + 1)^{1/3} - (x - 1)^{1/3}$ on $[0, 1]$ is
(1) 1 (2) 2 (3) 3 (4) $\dfrac{1}{3}$

49. The function $f(x) = \dfrac{\ln(\pi + x)}{\ln(e + x)}$ is
(1) increasing in $(0, \infty)$
(2) decreasing in $(0, \infty)$
(3) increasing in $(0, \pi/e)$, decreasing in $(\pi/e, \infty)$
(4) decreasing in $0, \pi/e)$, increasing in $(\pi/e, \infty)$

50. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its local maximum and minimum at p and q, respectively, such that $p^2 = q$, then a equals to
(1) 1 (2) 2 (3) $\dfrac{1}{2}$ (4) 3

51. Let $f(x) = \begin{cases} x + 2, & -1 \le x < 0 \\ 1, & x = 0 \\ \dfrac{x}{2}, & 0 < x \le 1 \end{cases}$

Then on $[-1, 1]$, this function has
(1) a point of minima
(2) a point of maxima
(3) both points of minima and maxima
(4) neither a point of minima nor that of maxima

52. If $f(x) = \begin{cases} \sin^{-1}(\sin x), & x > 0 \\ \dfrac{\pi}{2}, & x = 0 \\ \cos^{-1}(\cos x), & x < 0 \end{cases}$, then
(1) $x = 0$ is a point of maxima
(2) $x = 0$ is a point of minima
(3) $x = 0$ is a point of intersection
(4) none of these

53. A function f is defined by $f(x) = |x|^m |x - 1|^n \ \forall x \in R$. The local maximum value of the function is, $(m, n \in N)$,
(1) 1 (2) $m^n n^m$
(3) $\dfrac{m^m n^n}{(m+n)^{m+n}}$ (4) $\dfrac{(mn)^{mn}}{(m+n)^{m+n}}$

54. Let the function $f(x)$ be defined as follows:

$f(x) = \begin{cases} x^3 + x^2 - 10x, & -1 \le x < 0 \\ \cos x, & 0 \le x < \pi/2 \\ 1 + \sin x, & \pi/2 \le x \le \pi \end{cases}$ Then $f(x)$ has

(1) a local minimum at $x = \pi/2$
(2) a global maximum at $x = \pi/2$
(3) an absolute minimum at $x = -1$
(4) an absolute maximum at $x = \pi$

55. Consider the function f: $(-\infty, \infty) \to (-\infty, \infty)$ defined by

$f(x) = \frac{x^2 - a}{x^2 + a}$, $a > 0$. Which of the following is not true?

(1) Maximum value of f is not attained even though f is bounded.
(2) $f(x)$ is increasing on $(0, \infty)$ and has minimum at $x = 0$.
(3) $f(x)$ is decreasing on $(-\infty, 0)$ and has minimum at $x = 0$.
(4) $f(x)$ is increasing on $(-\infty, \infty)$ and has neither a local maximum nor a local minimum at $x = 0$.

56. $f(x) = 4 \tan x - \tan^2 x + \tan^3 x, x \neq n\pi + \frac{\pi}{2}$,

(1) is monotonically increasing
(2) is monotonically decreasing
(3) has a point of maxima
(4) has a point of minima

57. Let $h(x) = x^{m/n}$ for $x \in R$, where m and n are odd numbers and $0 < m < n$. Then $y = h(x)$ has

(1) no local extremums (2) one local maximum
(3) one local minimum (4) none of these

58. The greatest value of the function $f(x) = \frac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)}$ on the interval $\left(0, \frac{\pi}{2}\right)$ is

(1) $\frac{1}{\sqrt{2}}$ (2) $\sqrt{2}$ (3) 1 (4) $-\sqrt{2}$

59. The minimum value of $e^{(2x^2-2x+1)\sin^2 x}$ is

(1) e (2) $1/e$ (3) 1 (4) 0

60. The maximum value of $x^4 e^{-x^2}$ is

(1) e^2 (2) e^{-2} (3) $12e^{-2}$ (4) $4e^{-2}$

61. If $a^2x^4 + b^2y^4 = c^6 (a, b, x, y, c > 0)$, then the maximum value of xy is

(1) $\frac{c^2}{\sqrt{ab}}$ (2) $\frac{c^3}{ab}$ (3) $\frac{c^3}{\sqrt{2ab}}$ (4) $\frac{c^3}{2ab}$

62. The least natural number a for which $x + ax^{-2} > 2 \; \forall x \in (0, \infty)$ is

(1) 1 (2) 2
(3) 5 (4) none of these

63. $f(x) = \begin{cases} 4x - x^3 + \ln(a^2 - 3a + 3), & 0 \le x < 3 \\ x - 18, & x \ge 3 \end{cases}$

Complete set of values of a such that $f(x)$ as a local minima at $x = 3$ is

(1) $[-1, 2]$ (2) $(-\infty, 1) \cup (2, \infty)$
(3) $[1, 2]$ (4) $(-\infty, -1) \cup (2, \infty)$

64. $f(x) = \begin{cases} 2 - |x^2 + 5x + 6|, & x \neq -2 \\ a^2 + 1, & x = -2 \end{cases}$. Then the range of a, so that $f(x)$ has maxima at $x = -2$, is

(1) $|a| \ge 1$ (2) $|a| < 1$
(3) $a > 1$ (4) $a < 1$

65. A differentiable function $f(x)$ has a relative minimum at $x = 0$. Then the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$ for

(1) all a and all b (2) all b if $a = 0$
(3) all $b > 0$ (4) all $a > 0$

66. If $f(x) = 4x^3 - x^2 - 2x + 1$ and

$g(x) = \begin{cases} \min f(t), 0 \le t \le x; & 0 \le x \le 1 \\ 3 - x; & 1 < x \le 2 \end{cases}$ then the value of

$g\left(\frac{1}{4}\right) + g\left(\frac{3}{4}\right) + g\left(\frac{5}{4}\right)$ has the value equal to

(1) 7/4 (2) 9/4 (3) 13/4 (4) 5/2

67. If $f: R \to R$ and $g: R \to R$ are two functions such that $f(x) + f''(x) = -x\, g(x) f'(x)$ and $g(x) > 0 \; \forall x \in R$. Then the function $f^2(x) + (f'(x))^2$ has

(1) a maxima at $x = 0$
(2) a minima at $x = 0$
(3) a point of inflexion at $x = 0$
(4) none of these

68. If $A > 0, B > 0$, and $A + B = \frac{\pi}{3}$, then the maximum value of $\tan A \tan B$ is

(1) $\frac{1}{\sqrt{3}}$ (2) $\frac{1}{3}$ (3) 3 (4) $\sqrt{3}$

69. If $f(x) = \frac{t + 3x - x^2}{x - 4}$, where t is a parameter that has minimum and maximum, then the range of values of t is

(1) $(0, 4)$ (2) $(0, \infty)$
(3) $(-\infty, 4)$ (4) $(4, \infty)$

70. The least value of a for which the equation $\frac{4}{\sin x} + \frac{1}{1 - \sin x} = a$ has at least one solution in the interval $(0, \pi/2)$ is

(1) 9 (2) 4 (3) 8 (4) 1

71. If $f(x) = -x^3 - 3x^2 - 2x + a$, $a \in R$, then the real values of x satisfying $f(x^2 + 1) > f(2x^2 + 2x + 3)$ will be

(1) $(-\infty, \infty)$ (2) $(0, \infty)$
(3) $(-\infty, 0)$ (4) ϕ

72. Which of the following is the greatest?

(1) $\log_2 3$ (2) $\log_3 5$
(3) $\log_4 7$ (4) $\log_5 9$

73. If the equation $4x^3 + 5x + k = 0$ $(k \in R)$ has a negative real root then

(1) $k = 0$ (2) $-\infty < k < 0$
(3) $0 < k < \infty$ (4) $-\infty < k < \infty$

74. Tangent is drawn to ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3} \cos\theta, \sin\theta)$ [where $\theta \in (0, \pi/2)$]. Then the value of θ such that sum of intercepts on axes made by this tangent is minimum, is

(1) $\pi/3$ (2) $\pi/6$ (3) $\pi/8$ (4) $\pi/4$

75. The largest term in the sequence $a_n = \dfrac{n^2}{n^3+200}$ is given by

(1) $\dfrac{529}{49}$ (2) $\dfrac{8}{89}$

(3) $\dfrac{49}{543}$ (4) none of these

76. A factory D is to be connected by a road with a straight railway line on which a town A is situated. The distance DB of the factory to the railway line is $5\sqrt{3}$ km. Length AB of the railway line is 20 km. Freight charges on the road are twice the charges on the railway. The point P ($AP < AB$) on the railway line should the road DP be connected so as to ensure minimum freight charges from the factory to the town is

(1) $BP = 5$ km (2) $AP = 5$ km

(3) $BP = 7.5$ km (4) none of these

77. The volume of the greatest cylinder which can be inscribed in a cone of height 30 cm and semi-vertical angle 30° is

(1) $4000\,\pi/3$ cm^3 (2) $400\,\pi/3$ cm^3

(3) $4000\,\pi/\sqrt{3}$ cm^3 (4) none of these

78. A rectangle of the greatest area is inscribed in a trapezium $ABCD$, one of whose non-parallel sides AB is perpendicular to the base, so that one of the rectangle's side lies on the larger base of the trapezium. The base of trapezium are 6 cm and 10 cm and AB is 8 cm long. Then the maximum area of the rectangle is

(1) 24 cm^2 (2) 48 cm^2

(3) 36 cm^2 (4) none of these

79. A bell tent consists of a conical portion above a cylindrical portion near the ground. For a given volume and a circular base of a given radius, the amount of the canvas used is a minimum when the semi-vertical angle of the cone is

(1) $\cos^{-1}2/3$ (2) $\sin^{-1}2/3$

(3) $\cos^{-1}1/3$ (4) none of these

80. A rectangle is inscribed in an equilateral triangle of side length $2a$ units. The maximum area of this rectangle can be

(1) $\sqrt{3}a^2$ (2) $\dfrac{\sqrt{3}a^2}{4}$ (3) a^2 (4) $\dfrac{\sqrt{3}a^2}{2}$

81. Tangents are drawn to $x^2 + y^2 = 16$ from the point $P(0, h)$. These tangents meet the x-axis at A and B. If the area of triangle PAB is minimum, then

(1) $h = 12\sqrt{2}$ (2) $h = 6\sqrt{2}$

(3) $h = 8\sqrt{2}$ (4) $h = 4\sqrt{2}$

82. The largest area of a trapezium inscribed in a semi-circle of radius R, if the lower base is the diameter, is

(1) $\dfrac{3\sqrt{3}}{4}R^2$ (2) $\dfrac{\sqrt{3}}{2}R^2$

(3) $\dfrac{3\sqrt{3}}{8}R^2$ (4) R^2

83. In a ΔABC, $\angle B = 90°$ and $b + a = 4$. The area of the triangle is maximum when $\angle C$ is

(1) $\pi/4$ (2) $\pi/6$

(3) $\pi/3$ (4) none of these

84. Two runners A and B start at the origin and run along positive x-axis, with B running three times as fast as A. An observer, standing one unit above the origin, keeps A and B in view. Then the maximum angle θ of sight between the observer's view of A and B is

(1) $\pi/8$ (2) $\pi/6$

(3) $\pi/3$ (4) $\pi/4$

85. The fuel charges for running a train are proportional to the square of the speed generated in km/h, and the cost is ₹ 48 at 16 km/h. If the fixed charges amount to ₹ 300/h, the most economical speed is

(1) 60 km/h (2) 40 km/h

(3) 48 km/h (4) 36 km/h

86. A cylindrical gas container is closed at the top and open at the bottom. If the iron plate of the top is 5/4 times as thick as the plate forming the cylindrical sides, the ratio of the radius to the height of the cylinder using minimum material for the same capacity is

(1) 3 : 4 (2) 5 : 6

(3) 4 : 5 (4) none of these

87. The least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is

(1) $4\sqrt{3}r$ (2) $2\sqrt{3}r$ (3) $6\sqrt{3}r$ (4) $8\sqrt{3}r$

88. A given right cone has volume p, and the largest right circular cylinder that can be inscribed in the cone has volume q. Then $p : q$ is

(1) 9 : 4 (2) 8 : 3

(3) 7 : 2 (4) none of these

89. The cosine of the angle at the vertex of an isosceles triangle having the greatest area for the given constant length l of the median drawn to its lateral side is

(1) 0.4 (2) 0.5 (3) 0.6 (4) 0.8

90. A box, constructed from a rectangular metal sheet, is 21 cm by 16 cm by cutting equal squares of sides x from the corners of the sheet and then turning up the projected portions. The value of x so that volume of the box is maximum is

(1) 1 (2) 2 (3) 3 (4) 4

91. The vertices of a triangle are $(0, 0)$, $(x, \cos x)$, and $(\sin^3 x, 0)$, where $0 < x < \dfrac{\pi}{2}$. The maximum area for such a triangle in sq. units is

(1) $\dfrac{3\sqrt{3}}{32}$ (2) $\dfrac{\sqrt{3}}{32}$ (3) $\dfrac{4}{32}$ (4) $\dfrac{6\sqrt{3}}{32}$

92. The maximum area of the rectangle whose sides pass through the vertices of a given rectangle of sides a and b is

(1) $2(ab)$ (2) $\dfrac{1}{2}(a+b)^2$

(3) $\dfrac{1}{2}(a^2+b^2)$ (4) none of these

93. The base of prism is equilateral triangle. The distance from the center of one base to one of the vertices of the other base is ℓ. Then altitude of the prism for which the volume is greatest is

(1) $\dfrac{\ell}{2}$ (2) $\dfrac{\ell}{\sqrt{3}}$ (3) $\dfrac{\ell}{3}$ (4) $\dfrac{\ell}{4}$

Multiple Correct Answers Type

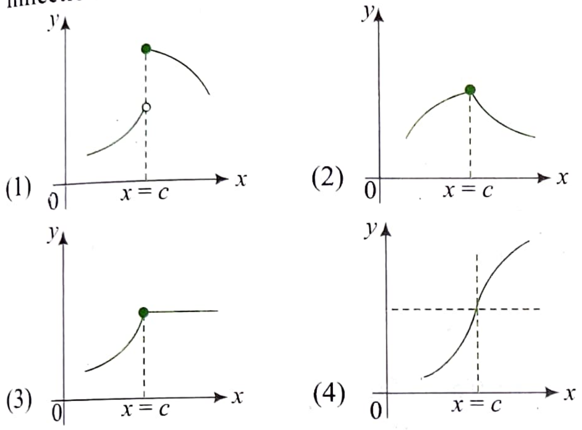
1. Let $f(x) = \begin{cases} x^2 + 3x, & -1 \le x < 0 \\ -\sin x, 0 & \le x < \pi/2 \\ -1 - \cos x, & \frac{\pi}{2} \le x \le \pi \end{cases}$. Then
   (1) $f(x)$ has global minimum value -2
   (2) global maximum value occurs at $x = 0$
   (3) global maximum value occurs at $x = \pi$
   (4) $x = \pi/2$ is point of local minima
2. Let $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$. Then,
   (1) f increases on $[1, \infty)$
   (2) f decreases on $[1, \infty)$
   (3) f has a minimum at $x = 1$
   (4) f has neither maximum nor minimum
3. Let $f(x) = 2x - \sin x$ and $g(x) = \sqrt[3]{x}$. Then
   (1) range of gof is R
   (2) gof is one-one
   (3) both f and g are one-one
   (4) both f and g are onto
4. If $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$, then $f(x)$
   (1) increases in $[0, \infty)$
   (2) decreases in $[0, \infty)$
   (3) neither increases nor decreases in $[0, \infty)$
   (4) increases in $(-\infty, \infty)$
5. Let $f(x) = |x^2 - 3x - 4|, -1 \le x \le 4$. Then
   (1) $f(x)$ is monotonically increasing in $[-1, 3/2]$
   (2) $f(x)$ is monotonically decreasing in $(3/2, 4]$
   (3) the maximum value of $f(x)$ is $\frac{25}{4}$
   (4) the minimum value of $f(x)$ is 0
6. If $f(x) = \int_0^x \frac{\sin t}{t} dt$, $x > 0$, then
   (1) $f(x)$ has a local maxima at $x = n\pi$ ($n = 2k$, $k \in I^+$)
   (2) $f(x)$ has a local minima at $x = n\pi$ ($n = 2k$, $k \in I^+$)
   (3) $f(x)$ has neither maxima nor minima at $x = n\pi$ ($n \in I^+$)
   (4) $f(x)$ has local maxima at $x = n\pi$ ($n = 2k - 1$, $k \in I^+$)
7. The possible values of parameter a for which the point of minimum of the function $f(x) = 1 + a^2x - x^3$ satisfies the inequality $\frac{x^2+x+2}{x^2+5x+6} < 0$ are
   (1) $(2\sqrt{3}, 3\sqrt{3})$ (2) $(-3\sqrt{3}, -2\sqrt{3})$
   (3) $(-2\sqrt{3}, 3\sqrt{3})$ (4) $(-3\sqrt{2}, 2\sqrt{3})$
8. Let $f(x) = ax^2 - b|x|$, where a and b are constants. Then at $x = 0$, $f(x)$ has
   (1) a maxima whenever $a > 0, b > 0$
   (2) a maxima whenever $a > 0, b < 0$
   (3) minima whenever $a > 0, b < 0$
   (4) neither a maxima nor a minima whenever $a > 0, b < 0$
9. The function $y = \frac{2x-1}{x-2}$, $(x \ne 2)$,
   (1) is its own inverse
   (2) decreases at all values of x in the domain
   (3) has a graph entirely above the x-axis
   (4) is unbounded
10. Let $g'(x) > 0$ and $f'(x) < 0 \ \forall x \in R$. Then
    (1) $g(f(x+1)) > g(f(x-1))$
    (2) $f(g(x-1)) > f(g(x+1))$
    (3) $g(f(x+1)) < g(f(x-1))$
    (4) $g(g(x+1)) < g(g(x-1))$
11. If $f(x) = x^3 - x^2 + 100x + 2002$, then
    (1) $f(1000) > f(1001)$ (2) $f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$
    (3) $f(x-1) > f(x-2)$ (4) $f(2x-3) > f(2x)$
12. If $f'(x) = g(x)(x-a)^2$, where $g(a) \ne 0$, and g is continuous at $x = a$, then
    (1) f is increasing in the neighborhood of a if $g(a) > 0$
    (2) f is increasing in the neighborhood of a if $g(a) < 0$
    (3) f is decreasing in the neighborhood of a if $g(a) > 0$
    (4) f is decreasing in the neighborhood of a if $g(a) < 0$
13. The value of a for which the function $f(x) = (4a - 3)(x + \log 5) + 2(a - 7)\cot\frac{x}{2}\sin^2\frac{x}{2}$ does not possess critical points is
    (1) $(-\infty, -4/3)$ (2) $(-\infty, -1)$
    (3) $[1, \infty)$ (4) $(2, \infty)$
14. Let $f(x) = (x-1)^4 (x-2)^n$, $n \in N$. Then $f(x)$ has
    (1) a maxima at $x = 1$ if n is odd
    (2) a maxima at $x = 1$ if n is even
    (3) a minima $x = 1$ if n is even
    (4) a minima at $x = 2$ if n is even
15. Let $f(x) = \sin x + ax + b$. Then which of the following is are true?
    (1) $f(x) = 0$ has only one real root which is positive if $a > 1, b < 0$.
    (2) $f(x) = 0$ has only one real root which is negative if $a > 1, b > 0$.
    (3) $f(x) = 0$ has only one real root which is negative if $a < -1, b < 0$.
    (4) None of these
16. The function $\frac{\sin(x+a)}{\sin(x+b)}$ has no maxima or minima if
    (1) $b - a = n\pi, n \in I$ (2) $b - a = (2n+1)\pi, n \in I$
    (3) $b - a = 2n\pi, n \in I$ (4) none of these
17. Consider $f(x) = ax^4 + cx^2 + dx + e$ has no point of inflection. Then which of the following is/are possible?
    (1) $a > 0, c < 0$ (2) $a < 0, c > 0$
    (3) $a, c < 0$ (4) $a, c > 0$

18. Let $f(x) = \begin{cases} \frac{(x-1)(6x-1)}{2x-1}, & \text{if } x \neq \frac{1}{2} \\ 0, & \text{if } x = \frac{1}{2} \end{cases}$

Then at $x = \frac{1}{2}$, which of the following is/are not true?

(1) f has a local maxima.
(2) f has a local minima.
(3) f has an inflection point.
(4) f has a removable discontinuity.

19. In which of the following graphs is $x = c$ the point of inflection?



20. Let $f(x)$ be an increasing function defined on $(0, \infty)$. If $f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$, then the possible integers in the range of a is/are

(1) 1 (2) 2 (3) 3 (4) 4

21. If $f(x) = (\sin^2 x - 1)^n$, $n \in N$, then $x = \frac{\pi}{2}$ is a point of

(1) local maximum, if n is odd
(2) local minimum, if n is odd
(3) local maximum, if n is even
(4) local minimum, if n is even

22. For the cubic function $f(x) = 2x^3 + 9x^2 + 12x + 1$, which one of the following statement/statements hold good?

(1) $f(x)$ is non-monotonic.
(2) $f(x)$ increases in $(-\infty, -2) \cup (-1, \infty)$ and decreases in $(-2, -1)$.
(3) $f: R \to R$ is bijective.
(4) Inflection point occurs at $x = -3/2$.

23. Let $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x$, where a_i's are real and $f(x) = 0$ has a positive root α_0. Then

(1) $f'(x) = 0$ has a root α_1 such that $0 < \alpha_1 < \alpha_0$
(2) $f'(x) = 0$ has at least two real roots
(3) $f''(x) = 0$ has at least one real root
(4) none of these

24. If $f(x)$ and $g(x)$ are two positive differentiable and increasing functions, then which of the following is not always true?

(1) $[f(x)]^{g(x)}$ is always increasing.
(2) $[f(x)]^{g(x)}$ is decreasing, when $f(x) < 1$.
(3) If $[f(x)]^{g(x)}$ is increasing, then $f(x) > 1$.
(4) If $f(x) > 1$, then $[f(x)]^{g(x)}$ is increasing.

25. An extremum of the function

$$f(x) = \frac{2-x}{\pi} \cos \pi(x+3) + \frac{1}{\pi^2} \sin \pi(x+3), \quad 0 < x < 4,$$

occurs at

(1) $x = 1$ (2) $x = 2$
(3) $x = 3$ (4) $x = \pi$

26. For the function $f(x) = x^4(12 \log_e x - 7)$,

(1) the point $(1, -7)$ is the point of inflection
(2) $x = e^{1/3}$ is the point of minima
(3) the graph is concave downwards in $(0, 1)$
(4) the graph is concave upwards in $(1, \infty)$

27. Let $f(x) = \log(2x - x^2) + \sin \frac{\pi x}{2}$. Then which of the following is/are true?

(1) Graph of f is symmetrical about the line $x = 1$.
(2) Maximum value of f is 1.
(3) Absolute minimum value of f does not exist.
(4) None of these.

28. Which of the following hold(s) good for the function $f(x) = 2x - 3x^{2/3}$?

(1) $f(x)$ has two points of extremum.
(2) $f(x)$ is concave upward $\forall x \in R$.
(3) $f(x)$ is non-differentiable function.
(4) $f(x)$ is continuous function.

29. For the function $f(x) = \frac{e^x}{1+e^x}$, which of the following hold good?

(1) f is monotonic in its entire domain.
(2) Maximum of f is not attained even though f is bounded.
(3) f has a point of inflection.
(4) f has one asymptote.

30. Which of the following is true about point of extremum $x = a$ of function $y = f(x)$?

(1) At $x = a$, function $y = f(x)$ may be discontinuous.
(2) At $x = a$, function $y = f(x)$ may be continuous but non-differentiable.
(3) At $x = a$, function $y = f(x)$ may have point of inflection.
(4) None of these.

31. Which of the following function has point of extremum at $x = 0$?

(1) $f(x) = e^{-|x|}$
(2) $f(x) = \sin |x|$
(3) $f(x) = \begin{cases} x^2 + 4x + 3, & x < 0 \\ -x, & x \geq 0 \end{cases}$
(4) $f(x) = \begin{cases} |x|, & x < 0 \\ \{x\}, & 0 \leq x < 1 \end{cases}$

(where $\{x\}$ represents fractional part function).

32. Which of the following function/functions has/have point of inflection?

(1) $f(x) = x^{6/7}$ (2) $f(x) = x^6$
(3) $f(x) = \cos x + 2x$ (4) $f(x) = x\,|x|$

33. The function $f(x) = x^2 + \dfrac{\lambda}{x}$ has a

(1) minimum at $x = 2$ if $\lambda = 16$
(2) maximum at $x = 2$ if $\lambda = 16$
(3) maximum for no real value of λ
(4) point of inflection at $x = 1$ if $\lambda = -1$

34. The function $f(x) = x^{1/3}(x - 1)$

(1) has two inflection points
(2) has one point of extremum
(3) is non-differentiable
(4) has range $[-3 \times 2^{-8/3}, \infty)$

35. Let f be the function $f(x) = \cos x - \left(1 - \dfrac{x^2}{2}\right)$. Then

(1) $f(x)$ is an increasing function in $(0, \infty)$
(2) $f(x)$ is a decreasing function in $(-\infty, \infty)$
(3) $f(x)$ is an increasing function in $(-\infty, \infty)$
(4) $f(x)$ is a decreasing function in $(-\infty, 0)$

36. Consider the function $f(x) = x \cos x - \sin x$, then identify the statement which is/are correct.

(1) f is odd
(2) f is monotonic decreasing at $x = 0$
(3) f has point of inflection at $x = 0$
(4) f has a maxima at $x = \pi$

37. If $f(x) = \dfrac{x^2}{2 - 2\cos x}$ and $g(x) = \dfrac{x^2}{6x - 6\sin x}$, where $0 < x < 1$, then

(1) f is increasing function
(2) g is increasing function
(3) f is decreasing function
(4) g is decreasing function

38. Greatest value of $f(x) = \dfrac{1}{2ax - x^2 - 5a^2}$ in $[-3, 5]$ is

(1) $f(5)$ if $a = 1$ (2) $f(-3)$ if $a = 1$
(3) $f(5)$ if $a < 1$ (4) $f(-3)$ if $a > 1$

39. For any acute angled ΔABC, $\dfrac{\sin A}{A} + \dfrac{\sin B}{B} + \dfrac{\sin C}{C}$ can take the value

(1) 1 (2) 2 (3) 3 (4) 4

40. Let $f(x)$ be a non–negative continuous and bounded function for all $x \geq 0$. If $(\cos x) f'(x) \leq (\sin x - \cos x) f(x)$ $\forall\, x \geq 0$, then which of the following is/are correct ?

(1) $f(6) + f(5) > 0$
(2) $x^2 - 3x + 2 + f(7) = 0$ has 2 distinct solutions
(3) $f(5) f(7) - f(6) f(5) = 0$
(4) $\lim\limits_{x \to 6} \dfrac{f(x) - \sin(\pi x)}{(x - 6)} = 1$

Linked Comprehension Type

For Problems 1 and 2

$f(x) = \sin^{-1} x + x^2 - 3x + \dfrac{x^3}{3}$, $x \in [0, 1]$

1. Which of the following is true about $f(x)$?

(1) $f(x)$ has a point of maxima.
(2) $f(x)$ has a point of minima.
(3) $f(x)$ is increasing.
(4) $f(x)$ is decreasing.

2. Which of the following is true for $x \in [0, 1]$?

(1) $\sin^{-1} x + x^2 - \dfrac{x(9 - x^2)}{3} \leq 0$
(2) $\sin^{-1} x + x^2 - \dfrac{x(9 - x^2)}{3} \geq 0$
(3) $\sin^{-1} x + x^2 - \dfrac{x(9 - x^2)}{3} \leq 1$
(4) $\sin^{-1} x + x^2 - \dfrac{x(9 - x^2)}{3} \geq 1$

For Problems 3 and 4

Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$ $\forall\, x \in \left(0, \dfrac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$.

3. Which of the following is true?

(1) g' is increasing.
(2) g' is decreasing.
(3) g' has a point of minima.
(4) g' has a point of maxima.

4. Which of the following is true?

(1) $g(x)$ is decreasing in $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$.
(2) $g(x)$ increasing in $\left(0, \dfrac{\pi}{4}\right)$.
(3) $g(x)$ is monotonically increasing in $\left(0, \dfrac{\pi}{2}\right)$
(4) None of these.

For Problems 5–8

Consider function $f(x) = \begin{cases} -x^2 + 4x + a, & x \leq 3 \\ ax + b, & 3 < x < 4 \\ -\dfrac{b}{4}x + 6, & x \geq 4 \end{cases}$

[For questions 6 to 8 consider $f(x)$ as a continuous function].

5. Which of the following is true?

(1) $f(x)$ is discontinuous function for all ordered pairs (a, b)
(2) $f(x)$ is continuous for finite number of ordered pairs (a, b)
(3) $f(x)$ can be differentiable
(4) $f(x)$ is continuous for infinite ordered pairs (a, b)

$x = 4$ is neither a point of maxima nor a point of minima, then which of the following can be true?

(1) $a > 0, b < 0$ (2) $a < 0, b < 0$

(3) $a > 0, b \in R$ (4) None of these

7. If $x = 4$ is the only point of maxima in its neighborhood but $x = 3$ is neither a point of maxima nor a point of minima, then which of the following can be true?

(1) $a < 0, b > 0$ (2) $a > 0, b < 0$

(3) $a > 0, b > 0$ (4) Not possible

8. If $x = 3$ is a point of minima and $x = 4$ is a point of maxima, then which of the following is true?

(1) $a < 0, b > 0$ (2) $a > 0, b < 0$

(3) $a > 0, b > 0$ (4) Not possible

or Problems 9 and 10

$\phi(x)$ is a differentiable real-valued function satisfying $\phi'(x)$ $2\phi(x) \le 1$, then it can be adjusted as $e^{2x}\phi'(x) + 2e^{2x}\phi(x) \le e^{2x}$

$$\frac{d}{dx}\left(e^{2x}\phi(x) - \frac{e^{2x}}{2}\right) \le 0 \text{ or } \frac{d}{dx}e^{2x}\left(\phi(x) - \frac{1}{2}\right) \le 0$$

ere, e^{2x} is called integrating factor which helps in creating single fferential coefficient as shown above. Answer the following uestions:

9. If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ for all $x \ge 1$, then

(1) $P(x) > 0\ \forall\ x > 1$ (2) $P(x)$ is a constant function

(3) $P(x) < 0\ \forall\ x > 1$ (4) none of these

10. If $H(x_0) = 0$ for some $x = x_0$ and $\frac{d}{dx}H(x) > 2cxH(x)$ for all $x \ge x_0$, where $c > 0$, then

(1) $H(x) = 0$ has root for $x > x_0$

(2) $H(x) = 0$ has no roots for $x > x_0$

(3) $H(x)$ is a constant function

(4) none of these

r Problems 11–13

t $h(x) = f(x) - a(f(x))^2 + a(f(x))^3$ for every real number x.

1. $h(x)$ increases as $f(x)$ increases for all real values of x if

(1) $a \in (0, 3)$ (2) $a \in (-2, 2)$

(3) $[3, \infty)$ (4) none of these

2. $h(x)$ increases as $f(x)$ decreases for all real values of x if

(1) $a \in (0, 3)$ (2) $a \in (-2, 2)$

(3) $(3, \infty)$ (4) none of these

3. If $f(x)$ is strictly increasing function, then $h(x)$ is non-monotonic function given

(1) $a \in (0, 3)$ (2) $a \in (-2, 2)$

(3) $(3, \infty)$ (4) $a \in (-\infty, 0) \cup (3, \infty)$

r Problems 14–16

$) = x^3 - 9x^2 + 24x + c = 0$ has three real and distinct roots α, and γ.

4. Possible values of c are

(1) $(-20, -16)$ (2) $(-20, -18)$

(3) $(-18, -16)$ (4) none of these

represents the greatest integer function, are

(1) $(-20, -16)$ (2) $(-20, -18)$

(3) $(-18, -16)$ (4) none of these

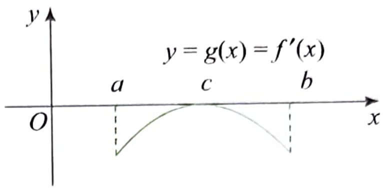
16. If $[\alpha] + [\beta] + [\gamma] = 7$, then the values of c, where $[\cdot]$ represents the greatest integer function, are

(1) $(-20, -16)$ (2) $(-20, -18)$

(3) $(-18, -16)$ (4) none of these

For Problems 17–21

Consider the graph of $y = g(x) = f'(x)$, given that $f(c) = 0$, where $y = f(x)$ is a polynomial function.



17. The graph of $y = f(x)$ will intersect the x-axis

(1) twice (2) once

(3) never (4) none of these

18. The equation $f(x) = 0$, $a \le x \le b$, has

(1) four real roots

(2) no real roots

(3) two distinct real roots

(4) at least three repeated roots

19. The graph of $y = f(x)$, $a \le x \le b$, has

(1) two points of inflection

(2) one point of inflection

(3) no point of inflection

(4) none of these

20. The function $y = f(x)$, $a < x < b$, has

(1) exactly one local maxima

(2) one local minima and one maxima

(3) exactly one local minima

(4) none of these

21. The equation $f''(x) = 0$

(1) has no real roots

(2) has at least one real root

(3) has at least two distinct real roots

(4) none of these

For Problems 22–24

Let $f(x) = 4x^2 - 4ax + a^2 - 2a + 2$ and the global minimum value of $f(x)$ for $x \in [0, 2]$ is equal to 3.

22. The number of values of a for which the global minimum value equal to 3 for $x \in [0, 2]$ occurs at the endpoint of interval $[0, 2]$ is

(1) 1 (2) 2 (3) 3 (4) 0

23. The number of values of a for which the global minimum value equal to 3 for $x \in [0, 2]$ occurs for the value of x lying in $(0, 2)$ is

(1) 1 (2) 2 (3) 3 (4) 0

24. The values of a for which $f(x)$ is monotonic for $x \in [0, 2]$ are

(1) $a \le 0$ or $a \ge 4$ (2) $0 \le a \le 4$

(3) $a > 0$ (4) none of these

For Problems 25–27

Let $f(x) = x^3 - 3(7-a)x^2 - 3(9-a^2)x + 2$.

25. The values of parameter a if $f(x)$ has a negative point of local minimum are

(1) ϕ (2) $(-3, 3)$

(3) $\left(-\infty, \frac{58}{14}\right)$ (4) none of these

26. The values of parameter a if $f(x)$ has a positive point of local maxima are

(1) ϕ (2) $(-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

(3) $\left(-\infty, \frac{58}{14}\right)$ (4) none of these

27. The values of parameter a if $f(x)$ has points of extrema which are opposite in sign are

(1) ϕ (2) $(-3, 3)$

(3) $\left(-\infty, \frac{58}{14}\right)$ (4) none of these

For Problems 28–30

Consider the function $f(x) = \left(1+\frac{1}{x}\right)^x$.

28. The domain of $f(x)$ is

(1) $(-1, 0) \cup (0, \infty)$ (2) $R - \{0\}$

(3) $(-\infty, -1) \cup (0, \infty)$ (4) $(0, \infty)$

29. The function $f(x)$

(1) has a maxima but no minima

(2) has a minima but no maxima

(3) has exactly one maxima and one minima

(4) is monotonic

30. The range of the function $f(x)$ is

(1) $(0, \infty)$ (2) $(-\infty, e)$

(3) $(1, \infty)$ (4) $(1, e) \cup (e, \infty)$

For Problems 31–33

Consider the function $f(x) = x + \cos x - a$.

31. Which of the following is not true about $y = f(x)$?

(1) It is an increasing function.

(2) It is a monotonic function.

(3) It has infinite points of inflections.

(4) None of these.

32. Values of a for which $f(x) = 0$ has exactly one positive root are

(1) $(0, 1)$ (2) $(-\infty, 1)$ (3) $(-1, 1)$ (4) $(1, \infty)$

33. Values of a for which $f(x) = 0$ has exactly one negative root are

(1) $(0, 1)$ (2) $(-\infty, 1)$ (3) $(-1, 1)$ (4) $(1, \infty)$

For Problems 34–36

Consider the function $f(x) = 3x^4 + 4x^3 - 12x^2$.

34. $y = f(x)$ increases in the interval

(1) $(-1, 0) \cup (2, \infty)$ (2) $(-\infty, 0) \cup (1, 2)$

(3) $(-2, 0) \cup (1, \infty)$ (4) none of these

35. The range of the function $y = f(x)$ is

(1) $(-\infty, \infty)$ (2) $[-32, \infty)$

(3) $[0, \infty)$ (4) none of these

36. The range of values of a for which $f(x) = a$ has no real roots is

(1) $(4, \infty)$ (2) $(10, \infty)$

(3) $(20, \infty)$ (4) none of these

For Problems 37–39

Consider the function $f: R \to R, f(x) = \frac{x^2 - 6x + 4}{x^2 + 2x + 4}$.

37. $f(x)$ is

(1) unbounded function (2) one-one function

(3) onto function (4) none of these

38. Which of the following is not true about $f(x)$?

(1) $f(x)$ has two points of extremum.

(2) $f(x)$ has only one asymptote.

(3) $f(x)$ is differentiable for all $x \in R$.

(4) None of these.

39. Range of $f(x)$ is

(1) $\left(-\infty, -\frac{2}{3}\right] \cup [2, \infty)$ (2) $\left[-\frac{1}{3}, 5\right]$

(3) $(-\infty, 2] \cup \left[\frac{7}{3}, \infty\right)$ (4) none of these

For Problems 40–42

Consider a polynomial $y = P(x)$ of the least degree passing through $A(-1, 1)$ and whose graph has two points of inflection $B(1, 2)$ and C with abscissa 0 at which the curve is inclined to the positive axis of abscissa at an angle of $\sec^{-1}\sqrt{2}$.

40. The value of $P(2)$ is

(1) -1 (2) $\frac{-3}{2}$ (3) $\frac{5}{2}$ (4) $\frac{7}{2}$

41. The value of $P(0)$ is

(1) 1 (2) 0 (3) $\frac{3}{4}$ (4) $\frac{1}{2}$

42. The equation $P'(x) = 0$ has

(1) three distinct real roots

(2) one real root

(3) three real roots such that one root is repeated

(4) none of these

For Problems 43–45

Let $f(x)$ be a real-valued continuous function on R defined as $f(x) = x^2 e^{-|x|}$

43. The values of k for which the equation $x^2 e^{-|x|} = k$ has four real roots are

(1) $0 < k < e$ (2) $0 < k < \frac{8}{e^2}$

(3) $0 < k < \frac{4}{e^2}$ (4) none of these

44. Which of the following is not true?
(1) $y = f(x)$ has two points of maxima.
(2) $y = f(x)$ has only one asymptote.
(3) $f'(x) = 0$ has three real roots.
(4) None of these.

45. Number of points of inflection for $y = f(x)$ is
(1) 1 (2) 2 (3) 3 (4) 4

For Problems 46 and 47

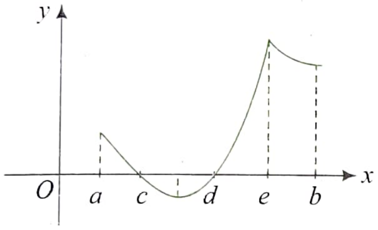
$p(x)$ be a polynomial of degree 3 satisfying $p(-1) = 10$, $p(1) = -6$ and $p(x)$ has $p(x)$ has maxima at $x = -1$ and $p'(x)$ has minima at $x = 1$, then

46. The value of $p(2)$ is
(1) −15 (2) −16 (3) −17 (4) −22

47. The value of $p'(1)$ is
(1) −12 (2) −10 (3) 15 (4) 21

For Problems 48–50

The graph of $y = g(x) = f'(x)$ is as shown in the following figure. Analyse this graph and answer the following questions



48. The graph of $y = f(x)$ for $a < x < b$ has
(1) no point of extrema (2) one point of extrema
(3) two points extrema (4) can't say anything

49. Number of points of inflection the graph of $y = f(x)$ for $a < x < b$ has
(1) 0 (2) 1
(3) 2 (4) can't say anything

50. Which of the following is not true about the graph of $y = f''(x)$, $a < x < b$
(1) always increasing
(2) discontinuous at one point
(3) first increases then decreases
(4) none of these

Matrix Match Type

1. Consider function $f(x) = x^4 - 14x^2 + 24x - 3$. Now, match the following lists:

List I: Equation $f(x) + p = 0$ has	List II
a. two negative real roots	**p.** for $p > 120$
b. two real roots of opposite sign	**q.** for $-8 \le p \le -5$
c. four real roots	**r.** for $3 < p \le < 120$
d. no real roots	**s.** for $p < -8$ or $-5 < p < 3$

2. Match the following lists:

List I	List II
a. $f(x) = x^2 \log x$	**p.** $f(x)$ has one point of minima
b. $f(x) = x \log_e x$	**q.** $f(x)$ has one point of maxima
c. $f(x) = \dfrac{\log x}{x}$	**r.** $f(x)$ increases in $(0, e)$
d. $f(x) = x^{-x}$	**s.** $f(x)$ decreases in $(0, 1/e)$

3. Let $f(x) = (x-1)^m (2-x)^n$; $m, n \in N$ and $m, n > 2$. Then match the following lists:

List I	List II
a. Both $x = 1$ and $x = 2$ are the points of minima if	**p.** m is even
b. $x = 1$ is a point of minima and $x = 2$ is a point of inflection if	**q.** m is odd
c. $x = 2$ is a point of minima and $x = 1$ is a point of inflection if	**r.** n is even
d. Both $x = 1$ and $x = 2$ are the points of inflection if	**s.** n is odd

4. The function $f(x) = \sqrt{(ax^3 + bx^2 + cx + d)}$ has its nonzero local minimum and maximum values at $x = -2$ and $x = 2$, respectively. If a is a root of $x^2 - x - 6 = 0$, then match the following lists:

List I	List II
a. The value/values of a	**p.** $= 0$
b. The value/values of b	**q.** $= 24$
c. The value/values of c	**r.** > 32
d. The value/values of d	**s.** -2

5. Match the following lists:

List I	List II
a. $f(x) = \sin x - x^2 + 1$	**p.** has point of minima
b. $f(x) = x \log_e x - x + e^{-x}$	**q.** has point of maxima
c. $f(x) = -x^3 + 2x^2 - 3x + 1$	**r.** is always increasing
d. $f(x) = \cos \pi x + 10x + 3x^2 + x^3$	**s.** is always decreasing

6. Match the following lists:

List I	List II
a. $f(x) = \lvert 2x-1\rvert + \lvert 2x-3\rvert$	**p.** has no points of extrema
b. $f(x) = 2\sin x - x$	**q.** has one point of maxima
c. $f(x) = \lvert x-1\rvert + \lvert 2x-3\rvert$	**r.** has one point of minima
d. $f(x) = \lvert x\rvert - \lvert 2x-3\rvert$	**s.** has infinite points of minima

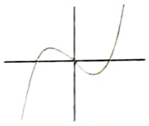
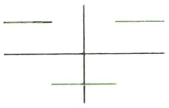
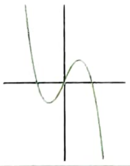
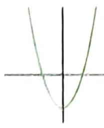
7. Match the following lists:

List I	List II
a. $f(x) = (x-1)^3(x-2)^5$	**p.** has points of maxima
b. $f(x) = 3\sin x + 4\cos x - 5x$	**q.** has point of minima
c. $f(x) = \begin{cases} \sin\dfrac{\pi x}{2}, 0<x\le 1 \\ x^2-4x+4, 1<x<2 \end{cases}$	**r.** has point of inflection
d. $f(x) = (x-1)^{3/5}$	**s.** has no point of extrema

8. Match the following lists:

List I	List II
a. At $x = 1$, $f(x) = \begin{cases} \log x, x<1 \\ 2x-x^2, x\ge 1 \end{cases}$	**p.** is increasing
b. At $x = 2$, $f(x) = \begin{cases} x-1, x<2 \\ 0, \quad x=2 \\ \sin x, x>2 \end{cases}$	**q.** is decreasing
c. At $x = 0$, $f(x) = \begin{cases} 2x+3, \ x<0 \\ 5, \quad x=0 \\ x^2+7, x>0 \end{cases}$	**r.** has point of maxima
d. At $x = 0$, $f(x) = \begin{cases} e^{-x} \ x<0 \\ 0, \quad x=0 \\ -\cos x, x>0 \end{cases}$	**s.** has point of minima

9. Match the following lists and then choose the correct code.

List I: Graph of $y = f(x)$	**List II:** Graph of $y = f'(x)$
a. 	**p.** 
b.	**q.** 
c.	**r.**
d.	**s.** 

Codes:

	a	b	c	d
(1)	p	s	q	r
(2)	s	r	q	p
(3)	r	p	s	q
(4)	q	p	r	s

10. $f(x)$ is polynomial function of degree 6, which satisfies $\lim_{x\to 0}\left(1+\dfrac{f(x)}{x^3}\right)^{1/x} = e^2$ and has local maximum at $x = 1$ and local minimum at $x = 0$ and $x = 2$. Now, match the following lists and then choose the correct code.

List I	List II
a. The coefficient of x^6	**p.** 0
b. The coefficient of x^5	**q.** 2
c. The coefficient of x^4	**r.** $-\dfrac{12}{5}$
d. The coefficient of x^3	**s.** $\dfrac{2}{3}$

Codes:

	a	b	c	d
(1)	p	s	q	r
(2)	s	r	q	p
(3)	r	p	s	q
(4)	s	r	q	p

Numerical Value Type

1. If α is an integer satisfying $|\alpha| \le 4 - |[x]|$, where x is a real number for which $2x\tan^{-1}x$ is greater than or equal to $\ln(1+x^2)$, then the number of maximum possible values of α (where $[\cdot]$ represents the greatest integer function) is_____.

2. From a given solid cone of height H, another inverted cone is carved whose height is h such that its volume is maximum. Then the ratio H/h is_____.

3. Let $f(x) = \begin{cases} \left|x^3+x^2+3x+\sin x\right|\left(3+\sin\dfrac{1}{x}\right), & x\ne 0, \\ 0 & , \ x=0 \end{cases}$

Then the number of points where $f(x)$ attains its minimum value is _____.

4. Let $f(x)$ be a cubic polynomial which has local maximum at $x = -1$ and $f'(x)$ has a local minimum at $x = 1$. If $f(-1) = 10$ and $f(3) = -22$, then the distance between its two horizontal tangents is _____.

5. Consider $P(x)$ to be a polynomial of degree 5 having extremum at $x = -1, 1$, and $\lim_{x\to 0}\left(\dfrac{P(x)}{x^3}-2\right) = 4$. Then the value of $P(1)$ is _____.

6. If m is the minimum value of $f(x, y) = x^2 - 4x + y^2 + 6y$ when x and y are subjected to the restrictions $0 \le x \le 1$ and $0 \le y \le 1$, then the value of $|m|$ is_____.

7. For a cubic function $y = f(x)$, $f''(x) = 4x$ at each point (x, y) on it and it crosses the x-axis at $(-2, 0)$ at an angle of 45° with positive direction of the x-axis. Then the value of $f(1)$ is _____.

8. Number of integral values of b for which the equation $\frac{x^3}{3} - x = b$ has three distinct solutions is ______.

9. Let $f(x) = \begin{cases} x+2, & x<-1 \\ x^2, & -1 \le x < 1 \\ (x-2)^2, & x \ge 1 \end{cases}$. Then number of times $f'(x)$ changes its sign in $(-\infty, \infty)$ is ______.

10. The number of nonzero integral values of a for which the function $f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$ is concave upward along the entire real line is ______.

11. Let $f(x) = \begin{cases} x^{3/5}, & \text{if } x \le 1 \\ -(x-2)^3, & \text{if } x > 1 \end{cases}$. Then the number of critical points on the graph of the function is ______.

12. A right triangle is drawn in a semicircle of radius $\frac{1}{2}$ with one of its legs along the diameter. If the maximum area of the triangle is M, then the value of $32\sqrt{3}M$ is ______.

13. A rectangle with one side lying along the x-axis is to be inscribed in the closed region of the xy plane bounded by the lines $y = 0$, $y = 3x$, and $y = 30 - 2x$. If M is the largest area of such a rectangle, then the value of M is______.

14. The least integral value of x where $f(x) = \log_{1/2}(x^2 - 2x - 3)$ is monotonically decreasing is______.

15. The least area of a circle circumscribing any right triangle of area $\frac{9}{\pi}$ is ______.

16. Let $f(x) = \begin{cases} |x^2 - 3x| + a, & 0 \le x < \frac{3}{2} \\ -2x + 3, & x \ge \frac{3}{2} \end{cases}$. If $f(x)$ has a local maxima at $x = \frac{3}{2}$, then greatest value of a is ______

17. Let $f(x) = 30 - 2x - x^3$, then the number of positive integral values of x which satisfies $f(f(f(x))) > f(f(-x))$ is ______.

18. Let $f(x) = \begin{cases} x(x-1)(x-2), & 0 \le x < n \\ \sin(\pi x), & n \le x \le 2n \end{cases}$. If $n \in N$, then the least value of n for which $f(x)$ has more points of minima than maxima in $[0, 2n]$ is ______.

19. Number of critical point of the function $f(x) = x + \sqrt{|x|}$ is ______.

20. Consider $f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-1|}$. Let x_1 and x_2 be points where $f(x)$ attains local minimum and global maximum respectively. If $k = f(x_1) + f(x_2)$, then $6k - 9 =$ ______.

Archives

JEE MAIN

Single Correct Answer Type

1. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$.
If $P(-1) < P(1)$, then in the interval $[-1, 1]$
(1) $P(-1)$ is the minimum and $P(1)$ is the maximum of P.
(2) $P(-1)$ is not minimum but $P(1)$ is the maximum of P.
(3) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P.
(4) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P.

(AIEEE 2009)

2. Let $f: B \to R$ be defined by
$f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$
If f has a local minimum at $x = -1$, then a possible value of k is
(1) -1 (2) 1 (3) 0 (4) $-\frac{1}{2}$

(AIEEE 2010)

3. Let $f: R \to R$ be a continuous function defined by
$f(x) = \frac{1}{e^x + 2e^{-x}}$.
Statement 1: $f(c) = 1/3$, for some $c \in R$.
Statement 2: $0 < f(x) \le 1/(2\sqrt{2})$, for all $x \in R$.
(1) Statement 1 is false, statement 2 is true.
(2) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.
(3) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 2.
(4) Statement 1 is true, statement 2 is false.

(AIEEE 2010)

4. Let $a, b \in R$ be such that the function f given by $f(x) = \ln|x| + bx^2 + ax, x \ne 0$ has extreme values at $x = -1$ and $x = 2$.
Statement 1: f has local maximum at $x = -1$ and at $x = 2$.
Statement 2: $a = \frac{1}{2}$ and $b = \frac{-1}{2}$.
(1) Statement 1 is false, statement 2 is true.
(2) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.
(3) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1.
(4) Statement 1 is true, statement 2 is false.

(AIEEE 2012)

5. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]
(1) lies between 1 and 2
(2) lies between 2 and 3
(3) lies between –1 and 0
(4) does not exist **(JEE Main 2013)**

6. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log |x| + \beta x^2 + x$, then
(1) $\alpha = -6, \beta = \frac{1}{3}$ (2) $\alpha = -6, \beta = -\frac{1}{2}$
(3) $\alpha = 2, \beta = -\frac{1}{2}$ (4) $\alpha = 2, \beta = \frac{1}{2}$
(JEE Main 2014)

7. Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim_{x\to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$, then $f(2)$ is equal to
(1) –8 (2) –4 (3) 0 (4) 4
(JEE Main 2015)

8. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then
(1) $(4 - \pi)x = \pi r$ (2) $x = 2r$
(3) $2x = r$ (4) $2x = (\pi + 4)r$
(JEE Main 2016)

9. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed is
(1) 30 (2) 12.5 (3) 10 (4) 25
(JEE Main 2017)

10. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is
(1) $2\sqrt{2}$ (2) 3 (3) –3 (4) $-2\sqrt{2}$
(JEE Main 2018)

JEE Advanced

Single Correct Answer Type

1. The function $f: [0, 3] \to [1, 29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is
(1) one-one and onto
(2) onto but not one-one
(3) one-one but not onto
(4) neither one-one nor onto **(IIT-JEE 2012)**

2. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is
(1) 6 (2) 4 (3) 2 (4) 0
(JEE Advanced 2013)

3. If $f: R \to R$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in R$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}, f(1) = 1$, then
(1) $0 < f'(1) \le \frac{1}{2}$ (2) $f'(1) \le 0$
(3) $f'(1) > 1$ (4) $\frac{1}{2} < f'(1) \le 1$
(JEE Advanced 2017)

Multiple Correct Answers Type

1. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are
(1) 24 (2) 32 (3) 45 (4) 60
(JEE Advanced 2013)

2. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at $x =$
(1) –2 (2) –2/3 (3) 2 (4) 2/3
(JEE Advanced 2013)

3. For every pair of continuous functions $f, g : [0, 1] \to R$ such that $\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0, 1]\}$, the correct statement(s) is(are)
(1) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(2) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(3) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0, 1]$
(4) $(f(c))^2 = (g(c))^2$ for some $c \in [0, 1]$
(JEE Advanced 2014)

4. Let $a \in R$ and let $f: R \to R$ be given by $f(x) = x^5 - 5x + a$, then
(1) $f(x)$ has three real roots if $a > 4$
(2) $f(x)$ has only one real root if $a > 4$
(3) $f(x)$ has three real roots if $a < -4$
(4) $f(x)$ has three real roots if $-4 < a < 4$
(JEE Advanced 2014)

5. Let $f: R \to (0, \infty)$ and $g: R \to R$ be twice differentiable functions such that f'' and g'' are continuous functions on R. Suppose $f'(2) = g(2) = 0$, $f''(2) \ne 0$ and $g''(2) \ne 0$. If $\lim_{x\to 2} \frac{f(x)\, g(x)}{f'(x)\, g'(x)} = 1$, then
(1) f has a local minimum at $x = 2$
(2) f has a local maximum at $x = 2$
(3) $f''(2) > f(2)$
(4) $f(x) - f''(x) = 0$ for at least one $x \in R$
(JEE Advanced 2016)

6. If $f: R$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in R$, and $f(0) = 1$, then
(1) $f(x) > e^{2x}$ in $(0, \infty)$
(2) $f(x)$ is decreasing in $(0, \infty)$
(3) $f(x)$ is increasing in $(0, \infty)$
(4) $f'(x) < e^{2x}$ in $(0, \infty)$ **(JEE Advanced 2017)**

7. If $f(x)=\begin{vmatrix}\cos(2x) & \cos(2x) & \sin(2x)\\ -\cos x & \cos x & -\sin x\\ \sin x & \sin x & \cos x\end{vmatrix}$, then

(1) $f'(x)=0$ at exactly three points in $(-\pi, \pi)$

(2) $f(x)$ attains its maximum at $x=0$

(3) $f(x)$ attains its minimum at $x=0$

(4) $f'(x)=0$ at more than three points in $(-\pi, \pi)$

(JEE Advanced 2017)

8. Let $f:(0,\pi)\to R$ be a twice differentiable function such that $\lim\limits_{t\to x}\dfrac{f(x)\sin t-f(t)\sin x}{t-x}=\sin^2 x$ for all $x\in(0,\pi)$.

If $f\left(\dfrac{\pi}{6}\right)=-\dfrac{\pi}{12}$, then which of the following statement(s) is (are) **TRUE**?

(1) $f\left(\dfrac{\pi}{4}\right)=\dfrac{\pi}{4\sqrt{2}}$

(2) $f(x)<\dfrac{x^4}{6}-x^2$ for all $x\in(0,\pi)$

(3) There exists $\alpha\in(0,\pi)$ such that $f'(\alpha)=0$

(4) $f''\left(\dfrac{\pi}{2}\right)+f\left(\dfrac{\pi}{2}\right)=0$

(JEE Advanced 2018)

Linked Comprehension Type

For Problem 1 and 2

Let $f[0,1]\to R$ (the set of all real numbers be a function. Suppose the function f is twice differentiable, $f(0)=f(1)=0$, and satisfies $f''(x)-2f'(x)+f(x)\geq e^x$, $x\in[0,1]$.

(JEE Advanced 2013)

1. Which of the following is true for $0<x<1$?

(1) $0<f(x)<\infty$

(2) $-\dfrac{1}{2}<f(x)<\dfrac{1}{2}$

(3) $-\dfrac{1}{4}<f(x)<1$

(4) $-\infty<f(x)<0$

2. If the function $e^{-x}f(x)$ assumes its minimum in the interval $[0,1]$ at $x=1/4$, which of the following is true?

(1) $f'(x)<f(x)$, $1/4<x<3/4$

(2) $f'(x)>f(x)$, $0<x<1/4$

(3) $f'(x)<f(x)$, $0<x<1/4$

(4) $f'(x)<f(x)$, $3/4<x<1$

Matrix Match Type

1. Match the statements/expressions in List I with the open intervals in List II.

List I	List II
a. Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2y'+y=0$	**p.** $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$
b. Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\,dx$	**q.** $\left(0,\dfrac{\pi}{2}\right)$
c. Interval in which at least one of the points local maximum of $\cos^2 x+\sin x$ lies	**r.** $\left(\dfrac{\pi}{8},\dfrac{5\pi}{4}\right)$
d. Interval in which $\tan^{-1}(\sin x+\cos x)$ is increasing	**s.** $\left(0,\dfrac{\pi}{8}\right)$
	t. $(-\pi,\pi)$

(IIT-JEE, 2009)

Direction for Questions 2-4:

Answer Q.2, Q.3 and Q.4 by appropriately matching the information given in the three columns of the following table.

Let $f(x)=x+\log_e x-x\log_e x$, $x\in(0,\infty)$.

- List I contains information about zero of $f(x)$, $f'(x)$ and $f''(x)$.
- List II contains information about the limiting behaviour of $f(x)$, $f'(x)$ and $f''(x)$ at infinity
- List III contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$

List I	List II	List III
(I) $f(x)=0$ for some $x\in(1,e^2)$	(i) $\lim\limits_{x\to\infty}f(x)=0$	(P) f is increasing in $(0,1)$
(II) $f'(x)=0$ for some $x\in(1,e)$	(ii) $\lim\limits_{x\to\infty}f(x)=-\infty$	(Q) f is decreasing in (e,e^2)
(III) $f'(x)=0$ for some $x\in(0,1)$	(iii) $\lim\limits_{x\to\infty}f'(x)=0$	(R) f' is increasing in $(0,1)$
(IV) $f''(x)=0$ for some $x\in(1,e)$	(iv) $\lim\limits_{x\to\infty}f''(x)=0$	(S) f' is decreasing in (e,e^2)

(JEE Advanced 2017)

2. Which of the following options is the only CORRECT combination?

(1) (IV) (i) (S) (2) (I) (ii) (R)

(3) (III) (iv) (P) (4) (II) (iii) (S)

3. Which of the following options is the only CORRECT combination?

(1) (III) (iii) (R) (2) (I) (i) (P)

(3) (IV) (iv) (S) (4) (II) (ii) (Q)

4. Which of the following options is the only INCORRECT combination?

(1) (II) (iii) (P) (2) (II) (iv) (Q)

(3) (I) (iii) (P) (4) (III) (i) (R)

Numerical Value Type

1. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010\,(x - 2009)\,(x - 2010)^2\,(x - 2011)^3\,(x - 2012)^4$, for all $x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x))$, for all $x \in R$, then the number of points in R at which g has a local maximum is_____.
(IIT-JEE, 2010)

2. Let $f: IR \to IR$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is_____.
(IIT-JEE, 2012)

3. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then $p'(0)$ is _____.
(IITJEE 2012)

4. A cylindrical container is to be made from certain solid material with the following constraints: It has fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.
If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\dfrac{V}{250\pi}$ is _____. **(JEE Advanced 2015)**

Answers Key

EXERCISES

Single Correct Answer Type

1. (4)	2. (4)	3. (1)	4. (4)	5. (1)
6. (3)	7. (1)	8. (4)	9. (1)	10. (2)
11. (1)	12. (2)	13. (2)	14. (1)	15. (3)
16. (4)	17. (4)	18. (4)	19. (3)	20. (3)
21. (2)	22. (1)	23. (2)	24. (1)	25. (3)
26. (4)	27. (2)	28. (1)	29. (4)	30. (3)
31. (4)	32. (2)	33. (2)	34. (3)	35. (4)
36. (2)	37. (3)	38. (3)	39. (2)	40. (2)
41. (3)	42. (1)	43. (1)	44. (1)	45. (2)
46. (1)	47. (4)	48. (2)	49. (2)	50. (2)
51. (4)	52. (1)	53. (3)	54. (3)	55. (4)
56. (1)	57. (1)	58. (3)	59. (3)	60. (4)
61. (3)	62. (2)	63. (2)	64. (1)	65. (2)
66. (4)	67. (1)	68. (2)	69. (3)	70. (1)
71. (1)	72. (1)	73. (3)	74. (2)	75. (3)
76. (1)	77. (1)	78. (2)	79. (1)	80. (4)
81. (4)	82. (1)	83. (3)	84. (2)	85. (2)
86. (3)	87. (3)	88. (1)	89. (4)	90. (3)
91. (1)	92. (2)	93. (2)		

Multiple Correct Answers Type

1. (1), (2), (3), (4)	2. (1), (3)
3. (1), (2), (3), (4)	4. (1), (4)
5. (1), (2), (3), (4)	6. (2), (4)
7. (1), (2)	8. (1), (3)
9. (1), (2), (4)	10. (2), (3)
11. (2), (3)	12. (1), (4)
13. (1), (4)	14. (1), (3), (4)
15. (1), (2), (3)	16. (1), (2), (3)
17. (3), (4)	18. (1), (2), (4)
19. (1), (2), (4)	20. (2), (3), (4)
21. (1), (4)	22. (1), (2), (4)
23. (1), (2), (3)	24. (1), (2), (3)
25. (1), (3)	26. (1), (2), (3), (4)
27. (1), (2), (3)	28. (1), (2), (3), (4)
29. (1), (2), (3)	30. (1), (2), (3)
31. (1), (2), (4)	32. (3), (4)
33. (1), (3), (4)	34. (1), (2), (3), (4)
35. (1), (4)	36. (1), (2), (3)
37. (1), (4)	38. (1), (2), (3), (4)
39. (1), (2)	40. (2), (3)

Linked Comprehension Type

1. (2)	2. (1)	3. (1)	4. (4)	5. (4)
6. (1)	7. (4)	8. (3)	9. (1)	10. (2)
11. (1)	12. (4)	13. (4)	14. (1)	15. (3)
16. (2)	17. (2)	18. (4)	19. (2)	20. (4)
21. (2)	22. (2)	23. (4)	24. (1)	25. (1)
26. (2)	27. (2)	28. (3)	29. (4)	30. (4)
31. (4)	32. (4)	33. (2)	34. (3)	35. (2)
36. (4)	37. (4)	38. (4)	39. (2)	40. (3)
41. (4)	42. (3)	43. (3)	44. (4)	45. (4)
46. (3)	47. (1)	48. (3)	49. (3)	50. (3)

Matrix Match Type

1. a → r; b → s; c → q; d → p
2. a → p, s; b → p, s; c → q, r; d → q
3. a → p, r; b → p, s; c → q, r; d → q, s
4. a → s; b → p; c → q; d → r
5. a → q, b → p, c → s, d → r
6. a → s; b → s; c → r; d → q
7. a → q, r; b → r, s; c → p, r; d → r, s
8. a → r; b → s; c → p; d → q
9. (2) 10. (4)

Numerical Value Type

1. (9)	2. (3)	3. (1)	4. (32)	5. (2.4)
6. (3)	7. (–15)	8. (1)	9. (4)	10. (4)
11. (3)	12. (9)	13. (67.5)	14. (4)	15. (9)
16. (–2.25)	17. (2)	18. (3)	19. (2)	20. (8)

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (2)	2. (1)	3. (2)	4. (2)	5. (4)
6. (3)	7. (3)	8. (2)	9. (4)	10. (1)

JEE ADVANCED

Single Correct Answer Type

1. (2)	2. (3)	3. (3)

Multiple Correct Answers Type

1. (1), (3)	2. (1), (2)
3. (1), (4)	4. (2), (4)
5. (1), (4)	6. (1), (3)
7. (2), (4)	8. (2), (3), (4)

Linked Comprehension Type

1. (4) 2. (3)

Matrix Match Type

1. d $\rightarrow$ s 2. (4) 3. (4) 4. (4)

Numerical Type

1. (1) 2. (5) 3. (9) 4. (4)

7 Indefinite Integration

INTEGRATION AS REVERSE PROCESS OF DIFFERENTIATION

The concept of integration originated in course of finding the area of a plane figure defined by some function. It is based on the limit of the sum of the series whose each term tends to zero and number of terms tends to infinity. In fact, it is called integration because of the process of summation. The integration is the inverse process of the differentiation and that is why it may also refer to the antiderivative of a function whose derivative is defined.

In integration, we find the function whose differential coefficient is given. For example, consider the function $5x^4$. We have to find the function whose differential coefficient w.r.t. x is $5x^4$. Few such functions are x^5, $x^5 + 1$ and $x^5 - 0.5$. So, we observe that it can be any function of the form $x^5 + c$, where c is an arbitrary constant. Therefore, the general form of the function whose differential coefficient is $5x^4$ is $x^5 + c$.

If differential coefficient of a function $F(x)$ is $f(x)$, i.e., if $\frac{d}{dx}F(x) = f(x)$, then we will say that one integral or primitive of $f(x)$ is $F(x)$. Symbolically, we write $\int f(x)dx = F(x)$. Here, $\int$ is the symbol of integration.

The process of finding the integral of a function is called the integration and the function which is integrated is called the integrand.

If $\frac{d}{dx}F(x) = f(x)$, then also $\frac{d}{dx}(F(x)+c) = f(x)$, where c is an arbitrary constant.

Thus, general value of $\int f(x)dx$ is $F(x) + c$ and c is called the constant of integration.

Clearly, integral will change if c changes. Thus, integral of a function is not unique. So, $\int f(x)dx$ will have infinite number of values and hence, it is called indefinite integral of $f(x)$.

INTEGRATION OF COMMONLY USED FUNCTIONS

Since $\frac{d}{dx}(F(x)) = f(x) \Leftrightarrow \int f(x)\,dx = F(x) + c$, based upon this definition and various standard differentiation formulas, we obtain the following integration formulae:

(i) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$

(ii) $\int \frac{1}{x}dx = \log|x| + c$, where $x \neq 0$

(iii) $\int e^x dx = e^x + c$

(iv) $\int a^x dx = \frac{a^x}{\log_e a} + c$

(v) $\int \sin x\,dx = -\cos x + c$

(vi) $\int \cos x\,dx = \sin x + c$

(vii) $\int \sec^2 x\,dx = \tan x + c$

(viii) $\int \text{cosec}^2 x\,dx = -\cot x + c$

(ix) $\int \sec x \tan x\,dx = \sec x + c$

(x) $\int \text{cosec}\,x \cot x\,dx = -\text{cosec}\,x + c$

(xi) $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + c$ or $-\cos^{-1}x + c$

(xii) $\int \frac{1}{x^2+1}dx = \tan^{-1}x + c$ or $-\cot^{-1}x + c$

(xiii) $\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x + c$ or $-\text{cosec}^{-1}x + c$

BASIC PROPERTIES OF INDEFINITE INTEGRATION

I. $\int kf(x)dx = k\int f(x)dx$, where k is constant.

II. $\int [f_1(x) \pm f_2(x) \pm \ldots \pm f_n(x)]dx$

$$= \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \ldots \pm \int f_n(x)\,dx$$

ILLUSTRATION 7.1

Evaluate $\int \frac{(1+x)^3}{\sqrt{x}}dx$.

Sol. $\int \frac{(1+x)^3}{\sqrt{x}}dx = \int \frac{1+3x+3x^2+x^3}{\sqrt{x}}dx$

$$= \int x^{-1/2}dx + 3\int x^{1/2}dx + 3\int x^{3/2}dx + \int x^{5/2}dx$$

$$= \frac{x^{1/2}}{1/2} + 3\frac{x^{3/2}}{3/2} + 3\cdot\frac{x^{5/2}}{5/2} + \frac{x^{7/2}}{7/2} + c$$

$$= 2\sqrt{x} + 2x^{3/2} + \frac{6}{5}x^{5/2} + \frac{2}{7}x^{7/2} + c$$

ILLUSTRATION 7.2

Evaluate $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx.$

Sol. $I = \int \frac{2^{x+1} - 5^{x-1}}{10^x} dx$

$$= \int \left[2\left(\frac{1}{5}\right)^x - \frac{1}{5}\left(\frac{1}{2}\right)^x\right] dx$$

$$= \frac{2\left(\frac{1}{5}\right)^x}{\log\left(\frac{1}{5}\right)} - \frac{1}{5}\frac{\left(\frac{1}{2}\right)^x}{\log\left(\frac{1}{2}\right)} + c$$

ILLUSTRATION 7.3

Evaluate $\int \sec^2 x \operatorname{cosec}^2 x dx.$

Sol. $I = \int \sec^2 x \operatorname{cosec}^2 x dx$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x} dx$$

$$= \int \left(\sec^2 x + \operatorname{cosec}^2 x\right) dx$$

$$= \tan x - \cot x + c$$

ILLUSTRATION 7.4

Evaluate $\int \frac{1 - \cos x}{1 + \cos x} dx.$

Sol. $\int \frac{1 - \cos x}{1 + \cos x} dx = \int \frac{(1 - \cos x)^2}{1 - \cos^2 x} dx$

$$= \int \frac{1 - 2\cos x + \cos^2 x}{\sin^2 x} dx$$

$$= \int (\operatorname{cosec}^2 x - 2\operatorname{cosec} x \cot x + \cot^2 x) dx$$

$$= \int (2\operatorname{cosec}^2 x - 2\operatorname{cosec} x \cot x - 1) dx$$

$$= -2\cot x + 2\operatorname{cosec} x - x + c$$

ILLUSTRATION 7.5

Evaluate $\int \frac{1}{1 + \sin x} dx.$

Sol. $\int \frac{1}{1 + \sin x} dx = \int \frac{1}{(1 + \sin x)} \cdot \frac{(1 - \sin x)}{(1 - \sin x)} dx$

$$= \int \frac{1 - \sin x}{1 - \sin^2 x} dx$$

$$= \int \frac{1 - \sin x}{\cos^2 x} dx$$

$$= \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx$$

$$= \int \sec^2 x dx - \int \tan x \sec x dx$$

$$= \tan x - \sec x + C$$

ILLUSTRATION 7.6

Evaluate $\int \tan^{-1}\left\{\sqrt{\left(\frac{1 - \cos 2x}{1 + \cos 2x}\right)}\right\} dx, \ 0 < x < \pi/2.$

Sol. $\int \tan^{-1}\left\{\sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}\right\} dx$

$$= \int \tan^{-1}\left\{\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right\} dx$$

$$= \int \tan^{-1}(\tan x) dx = \int x dx = \frac{x^2}{2} + C$$

ILLUSTRATION 7.7

Evaluate $\int \frac{\sec x}{\sec x + \tan x} dx.$

Sol. $\int \frac{\sec x}{\sec x + \tan x} dx = \int \frac{\sec x(\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)} dx$

$$= \int \frac{\sec^2 x - \sec x \tan x}{\sec^2 x - \tan^2 x} dx$$

$$= \int \sec^2 x dx - \int \sec x \tan x dx$$

$$= \tan x - \sec x + C$$

ILLUSTRATION 7.8

Evaluate $\int \frac{1 - x^2}{1 + x^2} dx.$

Sol. $\int \frac{1 - x^2}{1 + x^2} dx = \int \left(\frac{2}{1 + x^2} - 1\right) dx$

$$= 2\int \frac{1}{1 + x^2} dx - \int 1 dx$$

$$= 2\tan^{-1} x - x + c$$

CONCEPT APPLICATION EXERCISE 7.1

Evaluate the following:

1. $\int (\sec x + \tan x)^2 dx$
2. $\int \left(1 - \cos x\right) \operatorname{cosec}^2 x dx$
3. $\int a^{mx} b^{nx} dx$
4. $\int \frac{\tan x}{\sec x + \tan x} dx$
5. If $\int \frac{1}{x + x^5} dx = f(x) + c$, then $\int \frac{x^4}{x + x^5} dx.$
6. $\int \frac{(x^3 + 8)(x - 1)}{x^2 - 2x + 4} dx$
7. $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx$

8. $\int \tan^{-1}(\sec x + \tan x)dx, -\pi/2 < x < \pi/2$

9. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta}dx$

ANSWERS

1. $2(\sec x + \tan x) - x + C$
2. $\tan\frac{x}{2} + C$
3. $\frac{(a^m b^n)^x}{\log(a^m b^n)} + C$
4. $\sec x - \tan x + x + C$
5. $\log x - f(x) + C$
6. $\frac{x^3}{3} + \frac{x^2}{2} - 2x + C$
7. $\sec x - \operatorname{cosec} x + C$
8. $\frac{\pi}{4}x + \frac{x^2}{4} + C$
9. $\sin x + x\cos\theta + C$

INTEGRATION OF FUNCTION $f(ax + b)$

Let $\int f(x)dx = F(x) + c$. Then $\int f(ax+b)dx = \frac{F(ax+b)}{a} + c$.

Proof:

We have $I = \int f(ax+b)dx$.

Let $ax + b = t$. Then $a\,dx = dt$.

$\therefore$ $I = \int f(ax+b)dx$

$$= \int f(t)\frac{dt}{a}$$

$$= \frac{1}{a}\int f(t)dt$$

$$= \frac{1}{a}F(t) + c$$

$$= \frac{1}{a}F(ax+b) + c$$

i.e., $\int \sin 2x dx = \frac{1}{2}\cos 2x + c$

$\int \frac{1}{3x+4} = \frac{1}{3}\log_e|3x+4| + c$

$\int \frac{1}{1+(4x+5)^2} = \frac{1}{4}\tan^{-1}\frac{1}{1+(4x+5)^2} + c$

$\int e^{5x-3}dx = \frac{1}{5}e^{5x-3} + c$ etc.

ILLUSTRATION 7.9

Evaluate $\int \cos^3 x\, dx$.

Sol. $\int \cos^3 x\, dx = \frac{1}{4}\int(3\cos x + \cos 3x)dx$

$$= \frac{3}{4}\int \cos x dx + \frac{1}{4}\int \cos 3x dx$$

$$= \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + c$$

ILLUSTRATION 7.10

Evaluate $\int \frac{8x+13}{\sqrt{4x+7}}dx$.

Sol. $\int \frac{8x+13}{\sqrt{4x+7}}dx = \int \frac{8x+14-1}{\sqrt{4x+7}}dx$

$$= \int \frac{2(4x+7)-1}{\sqrt{4x+7}}dx$$

$$= 2\int\sqrt{4x+7}dx - \int\frac{1}{\sqrt{4x+7}}dx$$

$$= 2\left\{\frac{(4x+7)^{3/2}}{4\times\frac{3}{2}}\right\} - \left\{\frac{(4x+7)^{1/2}}{4\times\frac{1}{2}}\right\} + C$$

$$= \frac{1}{3}(4x+7)^{3/2} - \frac{1}{2}(4x+7)^{1/2} + C$$

ILLUSTRATION 7.11

Evaluate $\int \frac{1}{\sqrt{3x+4} - \sqrt{3x+1}}dx$.

Sol. $\int \frac{1}{\sqrt{3x+4} - \sqrt{3x+1}}dx$

$$= \int \frac{\sqrt{(3x+4)} + \sqrt{(3x+1)}}{(\sqrt{3x+4} + \sqrt{3x+1})(\sqrt{3x+4} - \sqrt{3x+1})}dx$$

$$= \int \frac{\sqrt{3x+4} + \sqrt{3x+1}}{(3x+4) - (3x+1)}dx$$

$$= \frac{1}{3}\int \sqrt{3x+4} + \sqrt{3x+1}\, dx$$

$$= \frac{1}{3}\int\sqrt{3x+4}dx + \frac{1}{3}\int\sqrt{3x+1}\,dx$$

$$= \frac{1}{3}\left\{\frac{(3x+4)^{3/2}}{3\times\frac{3}{2}}\right\} + \frac{1}{3}\left\{\frac{(3x+1)^{3/2}}{3\times\frac{3}{2}}\right\} + C$$

$$= \frac{2}{27}\{(3x+4)^{3/2} + (3x+1)^{3/2}\} + C$$

ILLUSTRATION 7.12

Evaluate $\int \sin^4 x dx$.

Sol. $\int \sin^4 x dx = \int\left(\frac{1-\cos 2x}{2}\right)^2 dx$

$$= \frac{1}{4}\int(1 - 2\cos 2x + \cos^2 2x)dx$$

$$= \frac{1}{4}\int\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{4}\int\left(\frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2}\right)dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right) + C$$

ILLUSTRATION 7.13

Evaluate $\int \sin 2x \sin 3x dx$.

Sol. $\int \sin 2x \sin 3x dx = \frac{1}{2}\int 2\sin 3x \sin 2x dx$

$$= \frac{1}{2}\int [\cos(3x-2x) - \cos(3x+2x)]dx$$

$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$

$$= \frac{1}{2}\int [\cos x - \cos 5x]dx$$

$$= \frac{1}{2}\int \cos x dx - \frac{1}{2}\int \cos 5x dx$$

$$= \frac{1}{2}\sin x - \frac{1}{2}\frac{\sin 5x}{5} + C$$

$$= \frac{\sin x}{2} - \frac{\sin 5x}{10} + C$$

ILLUSTRATION 7.14

Evaluate $\int \frac{dx}{(2x-7)\sqrt{(x-3)(x-4)}}$.

Sol. $I = \int \frac{dx}{(2x-7)\sqrt{(x-3)(x-4)}}$

$$= \int \frac{dx}{(2x-7)\sqrt{x^2-7x+12}}$$

$$= \int \frac{2dx}{(2x-7)\sqrt{(2x-7)^2-1}}$$

$$= \frac{1}{2}\sec^{-1}(2x-7) + c$$

ILLUSTRATION 7.15

Find the values of a and b such that $\int \frac{dx}{1+\sin x} = \tan\left(\frac{x}{2}+a\right) + b$.

Sol. $\int \frac{dx}{1+\sin x} = \int \frac{dx}{1+\cos(\pi/2 - x)}$

$$= \int \frac{dx}{2\cos^2(\pi/4 - x/2)}$$

$$= \frac{1}{2}\int \sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right)dx$$

$$= \frac{1}{2}\frac{\tan(\pi/4 - x/2)}{-\frac{1}{2}} + c$$

$$= \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + c$$

Given $\int \frac{dx}{1+\sin x} = \tan\left(\frac{x}{2}+a\right) + b$

or $\tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + c = \tan\left(\frac{x}{2}+a\right) + b$

$\therefore$ $a = -\frac{\pi}{4}$ and $b = c =$ an arbitrary constant

CONCEPT APPLICATION EXERCISE 7.2

Evaluate the following:

1. $\int \frac{dx}{\sqrt{x} + \sqrt{x-2}}$
2. $\int (1 + 2x + 3x^2 + 4x^3 + \cdots)\,dx$, $(0 < |x| < 1)$
3. $\int \frac{x^3}{x+1}dx$
4. $\int \frac{e^{3x} + e^{5x}}{e^x + e^{-x}}dx$
5. $\int \tan^2 x \sin^2 x dx$
6. $\int \frac{\cos x - \sin x}{\cos x + \sin x}(2 + 2\sin 2x)\,dx$
7. $\int \sin x \cos x \cos 2x \cos 4x \cos 8x\,dx$

ANSWERS

1. $\frac{1}{3}\left\{x^{3/2} - (x-2)^{3/2}\right\} + C$
2. $(1-x)^{-1} + C$
3. $\frac{x^3}{3} + x - \frac{x^2}{2} - \ln|x+1| + C$
4. $\frac{e^{4x}}{4} + C$
5. $\tan x - \frac{3}{2}x + \frac{\sin 2x}{4} + C$
6. $\sin 2x + C$
7. $-\frac{1}{256}\cos 16x + C$

INTEGRATION USING SUBSTITUTION

In previous section, we discussed one substitution technique. Now, we have two more standard substitution techniques for particular type of integration.

I. $\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)| + c$

Proof:

Let $f(x) = t$.

$\therefore$ $\frac{dt}{dx} = f'(x)$ or $dt = f'(x)dx$.

$\therefore$ $\int \frac{f'(x)}{f(x)}dx = \int \frac{dt}{t} = \log_e|t| + c$

$$= \log_e|f(x)| + c$$

11. $\int f'(x)(f(x))^n dx = \frac{(f(x))^{n+1}}{n+1} + c$

Proof:

Let $f(x) = t$.

$\therefore \frac{dt}{dx} = f'(x)$ or $dt = f'(x)dx$.

$$\therefore \int f'(x)(f(x))^n dx = \int t^n dt = \frac{t^{n+1}}{n+1} + c = \frac{(f(x))^{n+1}}{n+1} + c$$

ILLUSTRATION 7.16

Evaluate the following:

(i) $\int \frac{\sec^2 x}{3+\tan x} dx$ (ii) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

(iii) $\int \frac{1-\tan x}{1+\tan x} dx$ (iv) $\int \frac{1}{1+e^{-x}} dx$

Sol.

(i) $$\int \frac{\sec^2 x}{3+\tan x} dx = \int \frac{\frac{d}{dx}(3+\tan x)}{3+\tan x} dx$$

$= \log|3 + \tan x| + C$

(ii) $$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \frac{\frac{d}{dx}(e^x + e^{-x})}{e^x + e^{-x}} dx$$

$= \log_e(e^x + e^{-x}) + C$

(iii) $$\int \frac{1-\tan x}{1+\tan x} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

$$= \int \frac{\frac{d}{dx}(\cos x + \sin x)}{\cos x + \sin x} dx$$

$= \log_e|\cos x + \sin x| + C$

(iv) $$\int \frac{1}{1+e^{-x}} dx = \int \frac{e^x}{e^x + 1} dx$$

$$= \int \frac{\frac{d}{dx}(e^x + 1)}{e^x + 1} dx$$

$= \log_e(1 + e^x) + C$

ILLUSTRATION 7.17

Evaluate the following:

(i) $\int \left(x + \frac{1}{x}\right)^{3/2} \left(\frac{x^2 - 1}{x^2}\right) dx$ (ii) $\int \frac{\sqrt{2+\log x}}{x} dx$

(iii) $\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx$ (iv) $\int \frac{\cot x}{\sqrt{\sin x}} dx$

Sol.

(i) $$I = \int \left(x + \frac{1}{x}\right)^{3/2} \left(\frac{x^2 - 1}{x^2}\right) dx = \int \left(x + \frac{1}{x}\right)^{3/2} \left(1 - \frac{1}{x^2}\right) dx$$

let $t = x + \frac{1}{x} \Rightarrow dt = \left(1 - \frac{1}{x^2}\right) dx$

$$\Rightarrow I = \int t^{3/2} dt = \frac{2}{5} t^{5/2} + C = \frac{2}{5}\left(x + \frac{1}{x}\right)^{5/2} + C$$

(ii) $$\int \frac{\sqrt{2+\log x}}{x} dx = \int (2+\log x)^{\frac{1}{2}} \frac{d}{dx}(2+\log x) dx$$

$$= \frac{(2+\log x)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2(2+\log x)^{\frac{3}{2}}}{3} + c$$

(iii) $$\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx = \int (\sin^{-1} x)^3 \frac{d}{dx}(\sin^{-1} x) dx$$

$$= \frac{(\sin^{-1} x)^4}{4} + c$$

(iv) $$\int \frac{\cot x}{\sqrt{\sin x}} dx = \int \frac{\cos x}{\sin x \sqrt{\sin x}} dx$$

$$= \int (\sin x)^{-3/2} \cos x dx$$

$$= \frac{(\sin x)^{-1/2}}{-1/2} + c$$

$$= \frac{-2}{\sqrt{\sin x}} + c$$

ILLUSTRATION 7.18

Evaluate $\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x} dx$.

Sol. $\frac{d}{dx}\left(a^2 \sin^2 x + b^2 \cos^2 x\right) = (a^2 - b^2) \sin 2x$

Now, $$\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x} dx$$

$$= \frac{1}{(a^2 - b^2)} \int \frac{(a^2 - b^2)\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x} dx$$

$$= \frac{1}{(a^2 - b^2)} \int \frac{\frac{d}{dx}(a^2 \sin^2 x + b^2 \cos^2 x)}{a^2 \sin^2 x + b^2 \cos^2 x} dx$$

$$= \frac{1}{(a^2 - b^2)} \log|a^2 \sin^2 x + b^2 \cos^2 x| + C$$

ILLUSTRATION 7.19

Evaluate $\int \frac{1}{1-\tan x} dx$.

Sol. $$\int \frac{1}{1-\tan x} dx = \int \frac{\cos x}{\cos x - \sin x} dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\cos x - \sin x}dx$$

$$= \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}dx + \frac{1}{2}\int 1dx$$

$$= -\frac{1}{2}\int \frac{\frac{d}{dx}(\cos x - \sin x)}{\cos x - \sin x}dx + \frac{1}{2}x + c$$

$$= -\frac{1}{2}\log_e|\cos x - \sin x| + \frac{1}{2}x + c$$

ILLUSTRATION 7.20

Evaluate $\int \frac{\log\left(\tan\frac{x}{2}\right)}{\sin x}dx$.

Sol. $\frac{d}{dx}\left[\log\left(\tan\frac{x}{2}\right)\right] = \frac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}} = \frac{1}{\sin x}$

Now, $\int \frac{\log\left(\tan\frac{x}{2}\right)}{\sin x}dx = \int \log\left(\tan\frac{x}{2}\right)\frac{d}{dx}\left[\log\left(\tan\frac{x}{2}\right)\right]dx$

$$= \frac{\left[\log\left(\tan\frac{x}{2}\right)\right]^2}{2} + C$$

ILLUSTRATION 7.21

Evaluate $\int \tan^4 x dx$.

Sol. $\int \tan^4 x\, dx = \int \tan^2 x \tan^2 x dx = \int \tan^2 x(\sec^2 x - 1)dx$

$$= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx = \frac{\sec^3 x}{3} - \int (\sec^2 x - 1)dx$$

$$= \frac{\sec^3 x}{3} - \tan x + x + C$$

ILLUSTRATION 7.22

Evaluate $\int \sec^p x \tan x dx$.

Sol. $\int \sec^p x \tan x dx = \int \sec^{p-1} x(\sec x \tan x)dx$

$$= \int \sec^{p-1} x(\sec x)' dx$$

$$= \frac{\sec^p x}{p} + c$$

ILLUSTRATION 7.23

Evaluate $\int \frac{\log_e(x+\sqrt{x^2+1})}{\sqrt{x^2+1}}dx$.

Sol. Since $\frac{d}{dx}\log_e(x+\sqrt{x^2+1})$

$$= \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}(x+\sqrt{x^2+1})$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{\sqrt{x^2+1}}$$

$\therefore$ $\int \frac{\log_e(x+\sqrt{x^2+1})}{\sqrt{x^2+1}}dx$

$$= \int \log_e(x+\sqrt{x^2+1})\frac{d}{dx}(\log_e(x+\sqrt{x^2+1}))dx$$

$$= \frac{(\log_e(x+\sqrt{x^2+1}))^2}{2} + c$$

ILLUSTRATION 7.24

Evaluate $\int \frac{2x - \sqrt{\sin^{-1}x}}{\sqrt{1-x^2}}dx$.

Sol. $I = \int \frac{2x - \sqrt{\sin^{-1}x}}{\sqrt{1-x^2}}dx$

$$= -\int \frac{-2x}{\sqrt{1-x^2}}dx - \int \frac{\left(\sin^{-1}x\right)^{1/2}}{\sqrt{1-x^2}}dx$$

$$= -\int \left(1-x^2\right)^{-1/2}\left(1-x^2\right)' dx - \int \left(\sin^{-1}x\right)^{1/2}\left(\sin^{-1}x\right)' dx$$

$$= -\frac{\left(1-x^2\right)^{1/2}}{1/2} - \frac{\left(\sin^{-1}x\right)^{3/2}}{3/2} + c$$

ILLUSTRATION 7.25

Evaluate $\int (x^6+x^4+x^2)\sqrt{2x^4+3x^2+6}\,dx$.

Sol. $I = \int (x^6+x^4+x^2)\sqrt{2x^4+3x^2+6}$

$$= \int (x^5+x^3+x)\sqrt{2x^6+3x^4+6x^2}\,dx$$

Let $2x^6 + 3x^4 + 6x^2 = t^2$

$\therefore$ $12(x^5+x^3+x)\,dx = 2t\,dt$

$\therefore$ $I = \frac{1}{12}\int 2t^2 dt = \frac{1}{18}(2x^6+3x^4+6x^2)^{3/2} + c$

INTEGRATION OF tan, cot, sec AND cosec FUNCTIONS

We have following standard formulas for the integration of trigonometric functions:

(i) $\int \tan x dx = \log_e|\sec x| + c$

Proof:

$$\int \tan x dx = \int \frac{\sin x}{\cos x}dx$$
$$= -\int \frac{(\cos x)'}{\cos x}dx$$
$$= -\log_e|\cos x| + c$$
$$= \log_e|\sec x| + c$$

(ii) $\int \cot x dx = \log_e|\sin x| + c$

(iii) $\int \sec x dx = \log_e|\sec x + \tan x| + c = \log_e\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$

Proof:

$$\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}dx$$
$$= \int \frac{\sec^2 x + \sec x \tan x}{(\sec x + \tan x)}dx = \int \frac{(\sec x + \tan x)'}{\sec x + \tan x}dx$$
$$= \log_e|\sec x + \tan x| + c$$
$$= \log_e\left|\frac{1+\sin x}{\cos x}\right| + c$$
$$= \log_e\left|\frac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right| + c$$
$$= \log_e\left|\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right| + c$$
$$= \log_e\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$$

(iv) $\int \operatorname{cosec} x\, dx = \log_e|\operatorname{cosec} x - \cot x| + c$

$$= -\log_e|\operatorname{cosec} x + \cot x| + c$$
$$= \log_e\left|\tan\frac{x}{2}\right| + c$$

ILLUSTRATION 7.26

Evaluate $\int \sin 2x d(\tan x)$.

Sol. $I = \int \sin 2x\, d(\tan x)$

$$= \int \sin 2x \cdot \frac{d(\tan x)}{dx}dx$$
$$= \int \sin 2x \sec^2 x dx$$
$$= 2\int \tan x dx$$
$$= 2 \ln|\sec x| + c$$

ILLUSTRATION 7.27

Evaluate $\int \{1 + 2\tan x(\tan x + \sec x)\}^{1/2} dx$.

Sol. $\int \left(1 + 2\tan^2 x + 2\tan x \sec x\right)^{1/2} dx$

$$= \int \left(\sec^2 x + \tan^2 x + 2\tan x \sec x\right)^{1/2} dx$$
$$= \int (\sec x + \tan x)dx$$
$$= \log|\sec x + \tan x| + \log|\sec x| + c$$
$$= \log|\sec x(\sec x + \tan x)| + c$$

ILLUSTRATION 7.28

Evaluate $\int \tan x \tan 2x \tan 3x\, dx$.

Sol. $\tan 3x = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

$\therefore$ $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$

$\therefore$ $\int (\tan 3x - \tan 2x - \tan x)\, dx$

$$= -\frac{1}{3}\log|\cos 3x| + \frac{1}{2}\log|\cos 2x| + \log|\cos x| + c$$

ILLUSTRATION 7.29

Evaluate $\int \tan x \tan(x+1) dx$.

Sol. $\tan 1 = \dfrac{\tan(x+1) - \tan x}{1 + \tan x \tan(x+1)}$

$\therefore$ $\tan x \tan(x+1) = \tan(x+1) - \tan x - \tan 1$

$\therefore$ $\int \tan x \tan(x+1) dx$

$$= \int (\tan(x+1) - \tan x - \tan 1)dx$$
$$= \log_e|\sec(x+1)| - \log_e|\sec x| - (\tan 1)x + c$$

ILLUSTRATION 7.30

Evaluate $\int \frac{1}{\sqrt{3}\sin x + \cos x}dx$.

Sol. $\int \dfrac{1}{\sqrt{3}\sin x + \cos x}dx$

$$= \frac{1}{2}\int \frac{1}{\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x}dx$$
$$= \frac{1}{2}\int \frac{1}{\sin\left(x+\frac{\pi}{6}\right)}dx$$
$$= \frac{1}{2}\int \operatorname{cosec}\left(x+\frac{\pi}{6}\right)dx$$
$$= \frac{1}{2}\log_e\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right| + c$$

ILLUSTRATION 7.31

Evaluate $\int \frac{1}{\sin(x-a)\sin(x-b)}dx.$

Sol. $\int \frac{1}{\sin(x-a)\sin(x-b)}dx$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}$$

$$\times \int \frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int \{\cot(x-a)-\cot(x-b)\}dx$$

$$= \frac{1}{\sin(a-b)}\{\log|\sin(x-a)| - \log|\sin(x-b)|\} + c$$

$$= (\operatorname{cosec}(a-b))\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + c$$

SOME STANDARD TRIGONOMETRIC SUBSTITUTIONS

Expression	Substitution
a^2+x^2	$x = a\tan\theta$ or $a\cot\theta$
a^2-x^2	$x = a\sin\theta$ or $a\cos\theta$
x^2-a^2	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

ILLUSTRATION 7.32

Evaluate $\int \frac{dx}{\sqrt{1-x^2}(1+\sqrt{1-x^2})}$.

Sol. Put $x = \sin\theta$

$\therefore \quad dx = \cos\theta\, d\theta$

$$\therefore \quad I = \int \frac{dx}{\sqrt{1-x^2}(1+\sqrt{1-x^2})}$$

$$= \int \frac{\cos\theta\, d\theta}{\cos\theta(1+\cos\theta)}$$

$$= \frac{1}{2}\int \sec^2\frac{\theta}{2}\, d\theta$$

$$= \tan\frac{\theta}{2} + c$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} + c$$

$$= \frac{\sin\theta}{1+\cos\theta} + c$$

$$= \frac{x}{1+\sqrt{1-x^2}} + c$$

ILLUSTRATION 7.33

Evaluate $\int \frac{dx}{(a^2+x^2)^{3/2}}$.

Sol. Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

$$\therefore \quad I = \int \frac{dx}{(a^2+x^2)^{3/2}}$$

$$= \int \frac{a\sec^2\theta}{(a^2+a^2\tan^2\theta)^{3/2}}d\theta$$

$$= \int \frac{a\sec^2\theta}{a^3(\sec^2\theta)^{3/2}}d\theta$$

$$= \frac{1}{a^2}\int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{a^2}\int \cos\theta\, d\theta$$

$$= \frac{1}{a^2}\sin\theta + c$$

Now $\quad \tan\theta = \frac{x}{a}$

$$\therefore \quad \sin\theta = \frac{x}{\sqrt{x^2+a^2}}$$

$$\therefore \quad I = \frac{x}{a^2\sqrt{x^2+a^2}} + c$$

ILLUSTRATION 7.34

Evaluate $\int \frac{1}{(x^2+2x+2)^2}dx.$

Sol. $I = \int \frac{1}{(x^2+2x+2)^2}dx$

$$= \int \frac{1}{[(x+1)^2+1^2]^2}dx$$

Put $x + 1 = \tan\theta$

or $\quad dx = \sec^2\theta\, d\theta$

$$\therefore \quad I = \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$=\frac{1}{2}(\theta+\sin\theta\cos\theta)+C$$

$$=\frac{1}{2}\left\{\tan^{-1}(x+1)+\frac{x+1}{x^2+2x+2}\right\}+C$$

ILLUSTRATION 7.35

Evaluate $\int \frac{dx}{(x^3+3x^2+3x+1)\sqrt{x^2+2x-3}}$.

Sol. $I=\int \frac{dx}{(x+1)^3\sqrt{(x+1)^2-4}}$

Let $x+1=2\sec\theta$

$\therefore$ $dx=2\sec\theta\cdot\tan\theta\cdot d\theta$

$\therefore$ $I=\int \frac{2\sec\theta.\tan\theta d\theta}{\sec^3\theta 2\tan\theta}$

$$=\int \cos^2\theta d\theta$$

$$=\int \frac{1+\cos 2\theta}{2}d\theta$$

$$=\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+c$$

$$=\frac{1}{2}\sec^{-1}\frac{x+1}{2}+\frac{1}{2}\sin\theta.\cos\theta+c$$

$$=\frac{1}{2}\sec^{-1}\frac{x+1}{2}+\frac{1}{2}\sqrt{1-\frac{4}{(x+1)^2}}\cdot\frac{2}{(x+1)}+c$$

$$=\frac{1}{2}\sec^{-1}\frac{x+1}{2}+\frac{1}{(x+1)^2}\sqrt{x^2+2x-3}+c$$

ILLUSTRATION 7.36

Evaluate $\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$.

Sol. Put $x=\cos^2\theta$

$\Rightarrow$ $dx=-2\cos\theta\sin\theta d\theta$

$\therefore$ $I=\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

$$=\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\cdot\frac{-2\cos\theta\sin\theta}{\cos^2\theta}d\theta$$

$$=-\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cdot\frac{4\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta}d\theta$$

$$=-2\int\frac{2\sin^2\frac{\theta}{2}}{\cos\theta}d\theta$$

$$=-2\int\frac{1-\cos\theta}{\cos\theta}d\theta$$

$$=-2\int(\sec\theta-1)d\theta$$

$$=-2\left[\log_e|\sec\theta+\tan\theta|-\theta\right]+c$$

$$=-2\left[\log_e\left|\frac{1}{\sqrt{x}}+\frac{\sqrt{1-x}}{\sqrt{x}}\right|-\cos^{-1}\sqrt{x}\right]+c$$

$$=-2\left[\log_e\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right|-\cos^{-1}\sqrt{x}\right]+c$$

CONCEPT APPLICATION EXERCISE 7.3

Evaluate the following:

1. $\int \frac{dx}{x+x\log x}$
2. $\int \frac{x+1}{x(x+\log_e x)}dx$
3. $\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}dx$
4. $\int \frac{1+x^2\log_e x}{x+x^2\log_e x}dx$
5. $\int \frac{(1+\ln x)^5}{x}dx$
6. $\int \frac{\log_e(\log_e x)}{x\log_e x}dx$
7. $\int\left(\frac{x+1}{x}\right)(x+\log x)^2dx$
8. $\int \frac{\sqrt{\tan x}}{\sin x\cos x}dx$
9. $\int \operatorname{cosec}^4 x\, dx$
10. $\int \frac{\sin^6 x}{\cos^8 x}dx$
11. $\int \frac{dx}{(1+\sin x)^{1/2}}$
12. $\int \frac{\sin x}{\sin(x-a)}dx$
13. $\int \tan^3 x\, dx$
14. $\int \frac{dx}{\cot^2 x-1}$
15. $\int \frac{1}{x^2\sqrt{1+x^2}}dx$
16. $\int \frac{dx}{(2ax+x^2)^{3/2}}$

ANSWERS

1. $\log|1+\log x|+C$
2. $\log_e|x+\log_e x|+C$
3. $\log_e|4\sin^2 x-1|+C$
4. $\log_e x+x-\log_e|1+x\log_e x|+C$
5. $\frac{1}{6}(1+\ln x)^6+C$
6. $\frac{(\log_e(\log_e x))^2}{2}+C$
7. $\frac{(x+\log x)^3}{3}+C$
8. $2\sqrt{\tan x}+C$
9. $-\cot x-\frac{\cot^3 x}{3}+C$
10. $\frac{\tan^7 x}{7}+C$
11. $\sqrt{2}\log\left|\left(\tan\left(\frac{x}{4}+\frac{\pi}{x}\right)\right)\right|+C$
12. $(x-a)\cos a+\sin a\log|\sin(x-a)|+C$
13. $\frac{1}{2}\tan^2 x-\log|\sec x|+C$
14. $\frac{1}{4}\log_e|\sec 2x+\tan 2x|-\frac{x}{2}+C$
15. $\frac{-\sqrt{x^2+1}}{x}+C$
16. $-\frac{x+a}{a^2\sqrt{2ax+x^2}}+C$

INTEGRATION OF FUNCTION $f(g(x))g'(x)$

If $g(x)$ is a continuously differentiable function, then to evaluate integrals of the form,

$I=\int f(g(x))g'(x)dx$, we substitute $g(x)=t$,

and $(g'(x))\,dx=dt$.

The substitution reduces the integral to $\int f(t)\,dt$.

after evaluating this integral we substitute back the value of t.

ILLUSTRATION 7.37

Evaluate $\int \sin(e^x)d(e^x)$.

Sol. $I=\int \sin(e^x)d(e^x)$

Let $e^x=t$

$\therefore$ $I=\int \sin(e^x)d(e^x)$

$=\int \sin t\,dt=-\cos t+C=-\cos(e^x)+c$

ILLUSTRATION 7.38

Evaluate $\int \cos^3 x\sqrt{\sin x}\,dx$.

Sol. [Here, the power of $\cos x$ is 3, which is an odd positive integer. Therefore, put $z=\sin x$.]

Let $z=\sin x$. Then $dz=\cos x\,dx$.

Now, $\int \cos^3 x\sqrt{\sin x}\,dx=\int \cos^2 x\sqrt{\sin x}\cos x\,dx$

$=\int (1-\sin^2 x)\sqrt{\sin x}\cos x\,dx$

$=\int (1-z^2)\sqrt{z}\,dz$

$=\int (\sqrt{z}-z^{5/2})dz$

$=\frac{z^{3/2}}{3/2}-\frac{z^{7/2}}{7/2}+c=\frac{2}{3}z^{3/2}-\frac{2}{7}z^{7/2}+c$

$=\frac{2}{3}\sin^{3/2}x-\frac{2}{7}\sin^{7/2}x+c$

ILLUSTRATION 7.39

Evaluate $\int 2^{2^{2^x}}2^{2^x}2^x\,dx$.

Sol. $I=\int 2^{2^{2^x}}2^{2^x}2^x\,dx$

Let $2^{2^{2^x}}=t$

or $2^{2^{2^x}}2^{2^x}2^x(\log 2)^3\,dx=dt$

$\therefore$ $I=\int \frac{1}{(\log 2)^3}dt=\frac{1}{(\log 2)^3}t+c$

$=\frac{1}{(\log 2)^3}2^{2^{2^x}}+C$

ILLUSTRATION 7.40

Evaluate $\int \frac{e^{\sqrt{x}}\cos(e^{\sqrt{x}})}{\sqrt{x}}dx$.

Sol. Put $e^{\sqrt{x}}=t$ or $\frac{e^{\sqrt{x}}}{\sqrt{x}}dx=2dt$

$\therefore$ $\int \frac{e^{\sqrt{x}}\cos(e^{\sqrt{x}})}{\sqrt{x}}dx=2\int \cos t\,dt$

$=2\sin t+c$

$=2\sin e^{\sqrt{x}}+c$

ILLUSTRATION 7.41

Find $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$.

Sol. Let $z=xe^x$. Then $dz=(1e^x+xe^x)\,dx=e^x(1+x)\,dx$. Thus

$\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx=\int \frac{dz}{\cos^2 z}$

$=\int \sec^2 z\,dz$

$=\tan z+c=\tan(xe^x)+c$

ILLUSTRATION 7.42

$\int 5^{x+\tan^{-1}x}\cdot\left(\frac{x^2+2}{x^2+1}\right)dx$.

Sol. Let $x + \tan^{-1}x = t$

$\therefore \quad \left(1+\frac{1}{x^2+1}\right)dx = dt$ or $\left(\frac{x^2+2}{x^2+1}\right)dx = dt$

$\therefore \quad \int 5^{x+\tan^{-1}x}\cdot\left(\frac{x^2+2}{x^2+1}\right)dx = \int 5^t\,dt$

$$= \frac{5^t}{\log_e 5} + c$$

$$= \frac{5^{x+\tan^{-1}x}}{\log_e 5} + c$$

ILLUSTRATION 7.43

Evaluate $\int \frac{\sin^3 x\,dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}$.

Sol. $I = \int \frac{\sin^3 x\,dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}$

Put $\tan^{-1}(\sec x + \cos x) = t$

or $\frac{1}{1+(\sec x + \cos x)^2}(\sec x\tan x - \sin x)\,dx = dt$

or $\frac{\cos^2 x}{\cos^2 x + (1+\cos^2 x)^2}\sin x\left(\frac{1}{\cos^2 x} - 1\right)dx = dt$

or $\frac{\cos^2 x \sin x \frac{(1-\cos^2 x)}{\cos^2 x}}{\cos^2 x + 1 + \cos^4 x + 2\cos^2 x}dx = dt$

or $\frac{\sin^3 x}{\cos^4 x + 3\cos^2 x + 1}dx = dt$

$\therefore \quad I = \int \frac{dt}{t} = \log|t| + c$

$$= \log|\tan^{-1}(\sec x + \cos x)| + c$$

ILLUSTRATION 7.44

Evaluate $\int ((e/x)^x + (x/e)^x)\ln x\,dx$.

Sol. Let $\left(\frac{x}{e}\right)^x = t$

or $\left(\left(\frac{x}{e}\right)^x \ln\frac{x}{e} + x\cdot\left(\frac{x}{e}\right)^{x-1}\cdot\frac{1}{e}\right)dx = dt$

or $\left(\frac{x}{e}\right)^x\left(\ln\frac{x}{e} + 1\right)dx = dt$

or $\left(\frac{x}{e}\right)^x \ln x\,dx = dt$

So, $I = \int\left(\frac{1}{t^2} + 1\right)dt = -\frac{1}{t} + t + C$

$$= \left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + C$$

ILLUSTRATION 7.45

Evaluate $\int \frac{1}{\sqrt{e^{5x}}\sqrt[4]{(e^{2x}+e^{-2x})^3}}$.

Sol. $I = \int \frac{1}{\sqrt{e^{5x}}\sqrt[4]{(e^{2x}+e^{-2x})^3}}dx$

$$= \int \frac{1}{\sqrt{e^{5x}}\,e^{3x/2}(1+e^{-4x})^{3/4}}dx$$

$$= \int \frac{1}{e^{4x}(1+e^{-4x})^{3/4}}dx$$

Let $1 + e^{-4x} = t$

$\therefore \quad -4e^{-4x}dx = dt$

$\therefore \quad I = -\frac{1}{4}\int \frac{dt}{t^{3/4}}$

$$= -\frac{1}{4}\frac{t^{1/4}}{(1/4)} + c$$

$$= -(1+e^{-4x})^{1/4} + c$$

ILLUSTRATION 7.46

Find $\int \sin^5 x\,dx$.

Sol. [Here, power of $\sin x$ is 5 which is an odd positive integer. Therefore, put $z = \cos x$.]

Let $z = \cos x$. Then $dz = -\sin x\,dx$. Now,

$\int \sin^5 x\,dx \int \sin^4 x \sin x\,dx$

$= \int (\sin^2 x)^2 \sin x\,dx$

$= \int (1-\cos^2 x)^2 \sin x\,dx$

$= \int (1-z^2)^2(-dz)$ $\quad [\because z = \cos x]$

$= -\int (1-2z^2+z^4)dz$

$= -\left[z - 2\frac{z^3}{3} + \frac{z^5}{5}\right] + c$

$= -z + \frac{2}{3}z^3 - \frac{z^5}{5} + c$

$= -\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + c$

ILLUSTRATION 7.47

Find $\int \sin^3 x\cos^5 x\,dx$.

Sol. [Here, powers of both $\cos x$ and $\sin x$ are odd positive integers; therefore, put $z = \cos x$ or $z = \sin x$, but the power of $\cos x$ is greater. Therefore, it is convenient to put $z = \cos x$.]

Let $z = \cos x$. Then $dz = -\sin x\,dx$

$\int \sin^3 x\cos^5 x\,dx$

$$= \int \sin^2 x \cos^5 x \sin x dx$$

$$= \int (1-\cos^2 x)\cos^5 x \sin x dx$$

$$= \int (1-z^2)z^5(-dz)$$

$$= -\int (z^5 - z^7)dz$$

$$= -\left(\frac{z^6}{6} - \frac{z^8}{8}\right) + c = -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + c$$

ILLUSTRATION 7.48

Find $\int \frac{dx}{\sin x \cos^3 x}$.

Sol. [Here, the power of sin x is -1 and that of cos x is -3. Since the sum of powers of sin x and cos x is -4, which is even and negative, put $z = \tan x$.]

Let $z = \tan x$. Then $dz = \sec^2 x\, dx$. Now,

$$I = \int \frac{dx}{\sin x \cos^3 x}$$

$$= \int \frac{\sec^4 x\, dx}{\tan x}$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x\, dx}{\tan x}$$

Let $\tan x = z$. Then, $\sec^2 x dx = dz$. Therefore,

$$I = \int \frac{1+z^2}{z} dz = \int \left(\frac{1}{z} + z\right) dz$$

$$= \log|z| + \frac{z^2}{2} + c$$

$$= \log|\tan x| + \frac{\tan^2 x}{2} + c$$

ILLUSTRATION 7.49

Evaluate $\int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}}$.

Sol. $I = \int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}} = \int \frac{dx}{x^2.x^3\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}}$

$$= \int \frac{\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}} dx}{x^5}$$

Let $1+\frac{1}{x^4} = t$ or $\frac{-4}{x^5} dx = dt$

$\therefore$ $I = -\frac{1}{4}\int t^{-\frac{3}{4}} dt = -\frac{1}{4}\frac{t^{\frac{1}{4}}}{\frac{1}{4}} + C = -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$

ILLUSTRATION 7.50

Evaluate $\int \frac{dx}{x^2\sqrt{1+x^2}}$.

Sol. $I = \int \frac{dx}{x^2\sqrt{1+x^2}} = \int \frac{dx}{x^3\sqrt{1+\frac{1}{x^2}}}$

Let $t = \sqrt{1+\frac{1}{x^2}}$ or $\frac{dt}{dx} = \frac{1\left(-\frac{2}{x^3}\right)}{2\sqrt{1+\frac{1}{x^2}}}$

or $\frac{dx}{x^3} = -t dt$

or $I = -\int \frac{t dt}{t} = -t + C = -\sqrt{1+\frac{1}{x^2}} + C$

$$= -\frac{1}{x}\sqrt{1+x^2} + C$$

ILLUSTRATION 7.51

Evaluate $\int x^{-11}(1+x^4)^{-1/2} dx$.

Sol. $I = \int \frac{dx}{x^{11}(1+x^4)^{1/2}} = \int \frac{dx}{x^{11}\cdot x^2(1+1/x^4)^{1/2}}$

Let $1 + \frac{1}{x^4} = t^2$ or $\frac{-4}{x^5} dx = 2t\, dt$

$\therefore$ $I = \int \frac{dx}{x^{13}(1+1/x^4)^{1/2}}$

$$= -\frac{1}{4}\int \frac{2t\, dt}{x^8\, t}$$

$$= -\frac{1}{2}\int (t^2-1)^2 dt$$

$$= -\frac{1}{2}\int (t^4 - 2t^2 + 1) dt$$

$$= -\frac{1}{2}\left[\frac{t^5}{5} - \frac{2t^3}{3} + t\right] + C, \text{ where } t = \sqrt{1+\frac{1}{x^4}}$$

ILLUSTRATION 7.52

Evaluate $\int \frac{1+x^4}{(1-x^4)^{3/2}} dx$.

Sol. $I = \int \frac{1+x^4}{(1-x^4)^{3/2}} dx = \int \frac{x^3(x+1/x^3)\, dx}{x^3\left(x^2 - \frac{1}{x^2}\right)^{3/2}}$

$$= \int \frac{(x+1/x^3)\, dx}{\left(\frac{1}{x^2} - x^2\right)^{3/2}}$$

Let $\frac{1}{x^2} - x^2 = t$ or $\left(\frac{-2}{x^3} - 2x\right) dx = dt$

or $\left(x+\frac{1}{x^3}\right)dx=-\frac{1}{2}dt$

$\therefore \quad I=-\frac{1}{2}\int\frac{dt}{t^{3/2}}=\frac{1}{\sqrt{t}}+C=\frac{1}{\sqrt{\frac{1}{x^2}-x^2}}+C$

ILLUSTRATION 7.53

Evaluate $\int\frac{1}{[(x-1)^3(x+2)^5]^{1/4}}dx.$

Sol. $I=\int\frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}dx$

Let $\frac{x-1}{x+2}=t$ or $\frac{3dx}{(x+2)^2}=dt$

$\therefore \quad I=\frac{1}{3}\int\frac{1}{t^{3/4}}dt=\frac{1}{3}\left(\frac{t^{1/4}}{1/4}\right)+C$

$=\frac{4}{3}t^{1/4}+C=\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+C$

CONCEPT APPLICATION EXERCISE 7.4

Evaluate the following:

1. $\int\frac{\sin 2x}{(a+b\cos x)^2}dx$
2. $\int\frac{x^2\tan^{-1}x^3}{1+x^6}dx$
3. $\int\frac{\sqrt{x}\,dx}{1+x}$
4. $\int\frac{dx}{x+\sqrt{x}}$
5. $\int\frac{e^{2x}-2e^x}{e^{2x}+1}dx$
6. $\int\frac{dx}{e^x+e^{-x}}$
7. $\int\frac{\log\left(1+\frac{1}{x}\right)}{x(1+x)}dx$
8. $\int\frac{ax^3+bx}{x^4+c^2}dx$
9. $\int\frac{dx}{x^{2/3}(1+x^{2/3})}$
10. $\int e^{3\log x}\left(x^4+1\right)^{-1}dx$
11. $\int\frac{\sec x\,dx}{\sqrt{\cos 2x}}$
12. $\int\sin^3 x\cos^2 x\,dx$
13. $\int\frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$
14. $\int\frac{(x-x^3)^{1/3}}{x^4}dx$
15. $\int\frac{dx}{x^2(1+x^5)^{4/5}}$
16. $\int\frac{1}{x^2(x^4+1)^{3/4}}dx$
17. $\int\frac{(x^4-x)^{1/4}}{x^5}dx$
18. $\int x^x\ln(ex)\,dx$
19. $\int\frac{dx}{(x-p)\sqrt{(x-p)(x-q)}}$
20. $\int\frac{\left[\sqrt{1+x^2}+x\right]^n}{\sqrt{1+x^2}}dx$

ANSWERS

1. $-\frac{2}{b^2}\left[\log|a+b\cos x|+\frac{a}{a+b\cos x}\right]+C$
2. $\frac{(\tan^{-1}x^3)^2}{6}+C$
3. $2\sqrt{x}-2\tan^{-1}\sqrt{x}+C$
4. $2\log\left(\sqrt{x}+1\right)+C$
5. $\frac{1}{2}\log\left(e^{2x}+1\right)-2\tan^{-1}\left(e^x\right)+C$
6. $\tan^{-1}(e^x)+C$
7. $-\frac{\left(\log\left(1+\frac{1}{x}\right)\right)^2}{2}+C$
8. $\frac{a}{4}\log\left(x^4+c^2\right)+\frac{b}{2}\frac{1}{c}\tan^{-1}\frac{x^2}{c}+k$
9. $3\tan^{-1}\left(x^{1/3}\right)+C$
10. $\frac{1}{4}\log\left(x^4+1\right)+C$
11. $\sin^{-1}(\tan x)+C$
12. $-\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+C$
13. $1+\sqrt{1+x^2}+c$
14. $-\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{4/3}+C$
15. $-\frac{\left(1+x^5\right)^{1/5}}{x}+C$
16. $-\left(1+\frac{1}{x^4}\right)^{1/4}+C$
17. $\frac{4}{15}\left(1-\frac{1}{x^3}\right)^{5/4}+C$
18. x^x+C
19. $-\frac{2}{p-q}\sqrt{\frac{x-q}{x-p}}+C$
20. $\frac{1}{n}\left(\sqrt{1+x^2}+x\right)^n+C$

INTEGRATION OF FUNCTION WHICH IS RECIPROCAL OF QUADRATIC

We know that $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

Now, let us study more formulas for integration of function in which denominator is quadratic function.

(i) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

Proof:

Let $x = a\tan\theta$, then $dx = a\sec^2\theta\, d\theta$

$$\therefore \int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta}{a^2+a^2\tan^2\theta}d\theta$$

$$= \int \frac{a\sec^2\theta}{a^2(1+\tan^2\theta)}d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log_e\left|\frac{x-a}{a+x}\right|+c$

Proof:

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}(\log_e|x-a| - \log_e|x+a|)+c$$

$$= \frac{1}{2a}\log_e\left|\frac{x-a}{a+x}\right|+c$$

ILLUSTRATION 7.54

Evaluate $\int \frac{1}{x^2-x+1}dx.$

Sol. $\int \frac{1}{x^2-x+1}dx$

$$= \int \frac{1}{(x-1/2)^2+3/4}dx$$

$$= \int \frac{1}{(x-1/2)^2+(\sqrt{3}/2)^2}dx$$

$$= \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right)+C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

ILLUSTRATION 7.55

Evaluate $\int \frac{1}{2x^2+x-1}dx.$

Sol. $\int \frac{1}{2x^2+x-1}dx$

$$= \frac{1}{2}\int \frac{1}{x^2+\frac{x}{2}-\frac{1}{2}}dx$$

$$= \frac{1}{2}\int \frac{1}{(x+1/4)^2-(3/4)^2}dx$$

$$= \frac{1}{2}\cdot\frac{1}{2\,(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right|+C$$

$$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right|+C = \frac{1}{3}\log\left|\frac{2x-1}{2\,(x+1)}\right|+C$$

ILLUSTRATION 7.56

Evaluate $\int \frac{\cos x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}dx.$

Sol. $I = \int \frac{\cos x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}dx$

$$= \int \frac{\cos x}{\sin^2 x - \sin^2\frac{\pi}{6}}dx$$

Let $\sin x = t$

or $dt = \cos x\, dx$

$$\therefore \quad I = \int \frac{dt}{t^2-\frac{1}{4}}$$

$$= \frac{1}{2\frac{1}{2}}\log\left|\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right|+C$$

$$= \log\left|\frac{2t-1}{2t+1}\right|+C$$

$$= \log\left|\frac{2\sin x-1}{2\sin x+1}\right|+C$$

ILLUSTRATION 7.57

Evaluate $\int \frac{x}{x^4+2x^2+3}dx.$

Sol. $I = \int \frac{x}{x^4+2x^2+3}dx$

Let $x^2 = t.$

$\therefore \quad 2x dx = dt$

$$\therefore \quad I = \frac{1}{2}\int \frac{dt}{t^2+2t+3}$$

$$= \frac{1}{2}\int \frac{dt}{(t+1)^2+2}$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}\tan^{-1}\frac{t+1}{\sqrt{2}}+c$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x^2+1}{\sqrt{2}}+c$$

ILLUSTRATION 7.58

Evaluate $\int \frac{2x+1}{x^4+2x^3+x^2-1}dx$.

Sol. $I = \int \frac{2x+1}{x^4+2x^3+x^2-1}dx$

$$= \int \frac{2x+1}{(x^2+x)^2-1}dx$$

Let $x^2+x=t$

$\therefore$ $dt=(2x+1)dx$

$\therefore$ $I = \int \frac{dt}{t^2-1}$

$$= \frac{1}{2}\log_e\left|\frac{x^2+x-1}{x^2+x+1}\right|+C$$

ILLUSTRATION 7.59

Evaluate $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

Sol. $I = \int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $x=\sin\theta$

$\therefore$ $I = \int \frac{\cos\theta d\theta}{(1+\sin^2\theta)(\cos\theta)}$

$$= \int \frac{d\theta}{1+\sin^2\theta}$$

$$= \int \frac{\operatorname{cosec}^2\theta d\theta}{2+\cot^2\theta}$$

Put $\cot\theta = t$

$\therefore$ $-\operatorname{cosec}^2\theta d\theta = dt$

$\therefore$ $I = \int \frac{-dt}{2+t^2}$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+c$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\cot\theta}{\sqrt{2}}+c$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{1-x^2}}{\sqrt{2}x}+c$$

INTEGRATION OF FUNCTION IN WHICH LINEAR FUNCTION IS DIVIDED BY QUADRATIC

Form: $\int \frac{px+q}{ax^2+bx+c}dx$

Working Rule:

Let $px+q=\lambda(ax^2+bx+c)'+\mu$

$\Rightarrow$ $px+q=\lambda(2ax+b)+\mu$

Now find λ and μ,

ILLUSTRATION 7.60

Evaluate $\int \frac{4x+1}{x^2+3x+2}dx$.

Sol. $I = \int \frac{4x+1}{x^2+3x+2}dx$

$$= \int \frac{2(2x+3)-5}{x^2+3x+2}dx$$

$$= 2\int \frac{2x+3}{x^2+3x+2}dx - 5\int \frac{1}{x^2+3x+2}dx$$

$$= 2\log|x^2+3x+2| - 5\int \frac{1}{x^2+3x+(9/4)-(9/4)+2}dx$$

$$= 2\log|x^2+3x+2| - 5\int \frac{1}{(x+3/2)^2-(1/2)^2}dx$$

$$= 2\log|x^2+3x+2| - 5\frac{1}{2(1/2)}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right|+C$$

$$= 2\log|x^2+3x+2| - 5\log\left|\frac{x+1}{x+2}\right|+C$$

INTEGRATION OF EXPRESSIONS INVOLVING BIQUADRATIC FUNCTION

ILLUSTRATION 7.61

Evaluate $\int \frac{x^2+1}{x^4+1}dx$.

Sol. $I = \int \frac{x^2+1}{x^4+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx$

Let $x-\frac{1}{x}=t$ or $d\left(x-\frac{1}{x}\right)=dt$ or $\left(1+\frac{1}{x^2}\right)dx=dt$

$\therefore$ $I = \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+C$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right)+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right)+C$$

ILLUSTRATION 7.62

Evaluate $\int \frac{x^2-1}{x^4+x^2+1}dx$.

Sol. $I = \int \frac{x^2-1}{x^4+x^2+1}dx$

$$= \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx$$

$$= \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-1^2}dx$$

Let $x+\frac{1}{x}=u$. Then $d\left(x+\frac{1}{x}\right)=du$ or $\left(1-\frac{1}{x^2}\right)dx=du$

or $I = \int \frac{du}{u^2-1^2}$

$$= \frac{1}{2(1)}\log\left|\frac{u-1}{u+1}\right|+C$$

$$= \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right|+C = \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|+C$$

ILLUSTRATION 7.63

Evaluate $\int \frac{x^2+4}{x^4+16}dx$.

Sol. $I = \int \frac{x^2+4}{x^4+16}dx$

$$= \int \frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}dx$$

$$= \int \frac{1+\frac{4}{x^2}}{x^2+\left(\frac{4}{x}\right)^2-8+8}dx$$

$$= \int \frac{1+\frac{4}{x^2}}{\left(x-\frac{4}{x}\right)^2+8}dx$$

Let $x-\frac{4}{x}=t$. Then $d\left(x-\frac{4}{x}\right)=dt$ or $\left(1+\frac{4}{x^2}\right)dx=dt$.

Therefore,

$$I = \int \frac{dt}{t^2+\left(2\sqrt{2}\right)^2} = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{2\sqrt{2}}\right)+C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{4}{x}}{2\sqrt{2}}\right)+C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-4}{2x\sqrt{2}}\right)+C$$

ILLUSTRATION 7.64

Evaluate $\int \sqrt{\tan\theta}\,d\theta$.

Sol. Let $I = \int \sqrt{\tan\theta}\,d\theta$.

Let $\tan\theta = x^2$. Then, $d(\tan\theta) = d(x^2)$ or $\sec^2\theta\,d\theta = 2x\,dx$

or $d\theta = \frac{2x\cdot dx}{\sec^2\theta} = \frac{2x\,dx}{1+\tan^2\theta} = \frac{2x\,dx}{1+x^4}$

$$I = \int \sqrt{x^2}\times\frac{2x\,dx}{1+x^4}$$

$$= \int \frac{2x^2}{x^4+1}dx$$

$$= \int \frac{2}{x^2+1/x^2}dx$$

$$= \int \frac{1+1/x^2+1-1/x^2}{x^2+1/x^2}dx$$

$$= \int \frac{1+1/x^2}{x^2+1/x^2}dx + \int \frac{1-1/x^2}{x^2+1/x^2}dx$$

$$= \int \frac{1+1/x^2}{(x-1/x)^2+2}dx + \int \frac{1-1/x^2}{(x+1/x)^2-2}dx$$

Putting $x-\frac{1}{x}=u$ in first integral and $x+\frac{1}{x}=v$ in second integral, we get

$$I = \int \frac{du}{u^2+\left(\sqrt{2}\right)^2} + \int \frac{dv}{v^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right|+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right|+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\log\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right|+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan\theta-1}{\sqrt{2\tan\theta}}\right)+\frac{1}{2\sqrt{2}}\log\left|\frac{\tan\theta-\sqrt{2\tan\theta}+1}{\tan\theta+\sqrt{2\tan\theta}+1}\right|+C$$

CONCEPT APPLICATION EXERCISE 7.5

Evaluate the following:

1. $\int \frac{dx}{1-x-x^2}$
2. $\int \frac{x^2-\sqrt{3}x+1}{x^4-x^2+1}dx$
3. $\int \frac{x^2}{x^6-a^6}dx$
4. $\int \frac{\sin x}{\cos 2x}dx$
5. $\int \frac{e^x}{e^{2x}+6e^x+5}dx$
6. $\int \frac{x^3+x+1}{x^2-1}dx$
7. $\int \frac{1}{x^4+1}dx$
8. $\int \frac{1}{\sin^4 x+\cos^4 x}dx$
9. $\int \frac{x^4+1}{x^6+1}dx$

ANSWERS

1. $-\frac{1}{\sqrt{5}}\log_e\left|\frac{2x+1-\sqrt{5}}{2x+1+\sqrt{5}}\right|+C$
2. $2\tan^{-1}(2x+\sqrt{3})+C$
3. $\frac{1}{3}\frac{1}{2a^3}\log_e\left|\frac{x^3-a^3}{x^3+a^3}\right|+C$
4. $-\frac{1}{2\sqrt{2}}\log_e\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right|+C$
5. $\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+C$
6. $\frac{x^2}{2}+\log|x^2-1|+\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C$
7. $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)-\frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|+C$
8. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x-1}{\sqrt{2}\tan x}\right)+C$
9. $\tan^{-1}\left(x-\frac{1}{x}\right)-\frac{2}{3}\tan^{-1}(x^3)+C$

INTEGRATION OF FUNCTION WHICH IS RECIPROCAL OF SQUARE ROOT OF QUADRATIC

We know that $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x+c$

Now let us see some more formulas for integration of function in which denominator is square root of quadratic.

(i) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$

Proof:

Let $x = a\sin\theta$

$\therefore\ dx = a\cos\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}d\theta$$

$$= \int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}d\theta$$

$$= \int \frac{a\cos\theta}{a\cos\theta}d\theta$$

$$= \int d\theta = \theta + c = \sin^{-1}\frac{x}{a}+c$$

$$\left[\because \sin\theta = \frac{x}{a} \therefore \theta = \sin^{-1}\frac{x}{a}\right]$$

(ii) $\int \frac{dx}{\sqrt{a^2+x^2}} = \log|x+\sqrt{a^2+x^2}|+c$

Proof:

Let $x = a\tan\theta$.

$\therefore\ dx = a\sec^2\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\sec^2\theta}{\sqrt{a^2+a^2\tan^2\theta}}d\theta$$

$$= \int \frac{a\sec^2\theta}{\sqrt{a^2(1+\tan^2\theta)}}d\theta$$

$$= \int \frac{a\sec^2\theta}{a\sec\theta}d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \log|\sec\theta+\tan\theta|+c \quad \ldots(1)$$

Now $x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a}$

$\Rightarrow\ \sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{x^2}{a^2}} = \frac{\sqrt{a^2+x^2}}{a}$

$\therefore$ from (1), $I = \log\left|\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}\right|+c$

$$= \log\left|\frac{\sqrt{a^2+x^2}+x}{a}\right|+c$$

$$= \log|\sqrt{a^2+x^2}+x| - \log|a| + c$$

$$= \log|x+\sqrt{a^2+x^2}| + c$$

(iii) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|\sqrt{x^2-a^2}+x\right| + c$

ILLUSTRATION 7.65

Evaluate $\int \frac{1}{\sqrt{(x-1)(x-2)}}dx.$

Sol. $I = \int \frac{1}{\sqrt{x^2-3x+2}}dx$

$$= \int \frac{1}{\sqrt{x^2-3x+\frac{9}{4}-\frac{9}{4}+2}}dx$$

$$= \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right| + C$$

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right| + C$$

ILLUSTRATION 7.66

Evaluate $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx.$

Sol. Since derivative of $\tan x$ is $\sec^2 x$.

Let $\tan x = t$ or $\sec^2 x\, dx = dt$

$\therefore$ $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int \frac{dt}{\sqrt{t^2+2^2}}$

$$= \log\left|t+\sqrt{t^2+4}\right| + C$$

$$= \log\left|\tan x+\sqrt{\tan^2 x+4}\right| + C$$

ILLUSTRATION 7.67

Evaluate $\int \frac{e^x}{\sqrt{4-e^{2x}}}dx.$

Sol. $I = \int \frac{e^x}{\sqrt{4-e^{2x}}}dx = \int \frac{e^x}{\sqrt{2^2-(e^x)^2}}dx$

Let $e^x = t$ or $e^x\, dx = dt$

$\therefore$ $I = \int \frac{dt}{\sqrt{4-t^2}} = \int \frac{dt}{\sqrt{2^2-t^2}}$

$$= \sin^{-1}\left(\frac{t}{2}\right) + C = \sin^{-1}\left(\frac{e^x}{2}\right) + C$$

ILLUSTRATION 7.68

Evaluate $\int \frac{\tan x\, dx}{\sqrt{2+3\tan^2 x}}$.

Sol. $\int \frac{\tan x\, dx}{\sqrt{2+3\tan^2 x}} = \int \frac{\sin x\, dx}{\sqrt{2\cos^2 x+3\sin^2 x}}$

$$= \int \frac{\sin x\, dx}{\sqrt{2\cos^2 x+3(1-\cos^2 x)}}$$

$$= \int \frac{\sin x\, dx}{\sqrt{3-\cos^2 x}}$$

$$= -\int \frac{dt}{\sqrt{3-t^2}}$$

$$= -\sin^{-1}\frac{t}{\sqrt{3}} + c$$

$$= -\sin^{-1}\frac{\cos x}{\sqrt{3}} + c$$

ILLUSTRATION 7.69

Evaluate $\int \sqrt{\sec x - 1}\, dx$.

Sol. $I = \int \sqrt{\sec x-1}\, dx = \int \sqrt{\frac{1-\cos x}{\cos x}}dx$

$$= \int \sqrt{\frac{(1-\cos x)}{\cos x}\times\frac{(1+\cos x)}{(1+\cos x)}}dx$$

$$= \int \sqrt{\frac{1-\cos^2 x}{\cos x+\cos^2 x}}dx$$

$$= \int \frac{\sin x}{\sqrt{\cos^2 x+\cos x}}dx$$

Let $\cos x = t$. Then $d(\cos x) = dt$ or $-\sin x\, dx = dt$. Therefore.

$$I = \int \frac{-dt}{\sqrt{t^2+t}}$$

$$= -\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$= -\log\left|\left(t+\frac{1}{2}\right)+\sqrt{\left(t+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right| + C$$

$$= -\log\left|\left(t+\frac{1}{2}\right)+\sqrt{t^2+t}\right| + C$$

$$= -\log\left|\left(\cos x+\frac{1}{2}\right)+\sqrt{\cos^2 x+\cos x}\right| + C$$

Integration of Function in Which Linear Function is Divided by Square Root of Quadratic

Form: $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Working Rule:

Let $px+q = \lambda(ax^2+bx+c)' + \mu$

$\Rightarrow$ $px+q = \lambda(2ax+b)+\mu$

Now find λ and μ,

Illustration 7.70

Evaluate $\int \sqrt{\frac{1+x}{x}}dx$.

Sol. $\int \sqrt{\frac{1+x}{x}}dx = \int \sqrt{\frac{1+x}{x}}\sqrt{\frac{1+x}{1+x}}dx$

$$= \int \frac{1+x}{\sqrt{x(1+x)}}dx$$

$$= \int \frac{1+x}{\sqrt{x^2+x}}dx$$

$$= \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x}}dx + \frac{1}{2}\int \frac{1}{\sqrt{x^2+x}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{t}}dt + \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}dx,$$

where $t = x^2 + x$

$$= \sqrt{t} + \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right| + C$$

$$= \sqrt{x^2+x} + \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right| + C$$

Concept Application Exercise 7.6

Evaluate the following:

1. $\int \frac{x^2}{\sqrt{1-x^6}}dx$
2. $\int \sqrt{\frac{x}{a^3-x^3}}dx$
3. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$
4. $\int \frac{2x+3}{\sqrt{x^2+4x+1}}dx$
5. $\int \frac{x^{5/2}}{\sqrt{1+x^7}}dx$
6. $\int \sqrt{1+\operatorname{cosec} x}\, dx, (\pi/2 < x < \pi)$

Answers

1. $\frac{1}{3}\sin^{-1}(x^3)+C$
2. $\frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)+C$
3. $-\log\left|e^{-x}+\sqrt{e^{-2x}-1}\right|+C$
4. $2\sqrt{x^2+4x+1} - \log\left|x+2+\sqrt{x^2+4x+1}\right|+C$
5. $\frac{2}{7}\log\left(x^{7/2}+\sqrt{1+x^7}\right)+C$
6. $2\sin^{-1}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)+c$

Integration of Functions of Some Standard Forms

Form I: $\int \frac{dx}{a+b\cos^2 x + c\sin^2 x}$, $\int \frac{dx}{a+b\sin 2x}$

Working rule:

Divide numerator and denominator both by $\cos^2 x$, replace $\sec^2 x$, if any, in denominator by $(1+\tan^2 x)$ and put $\tan x = t$.

Illustration 7.71

Evaluate $\int \frac{\sin x}{\sin 3x}dx$.

Sol. $I = \int \frac{\sin x}{\sin 3x}dx = \int \frac{\sin x}{3\sin x - 4\sin^3 x}dx$

$$= \int \frac{1}{3-4\sin^2 x}dx$$

$$= \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x}dx$$

[Dividing N′ and D′ by $\cos^2 x$]

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$I = \int \frac{dt}{3(1+t^2)-4t^2} = \int \frac{dt}{3-t^2} = \int \frac{1}{(\sqrt{3})^2 - t^2}dt$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + C = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$$

Illustration 7.72

Evaluate $\int \frac{1}{3+\sin 2x}dx$.

Sol. $I = \int \frac{1}{3+\sin 2x}dx$

$$= \int \frac{1}{3(\sin^2 x + \cos^2 x) + 2\sin x\cos x}dx$$

$$= \int \frac{\sec^2 x}{3\tan^2 x + 2\tan x + 3} dx$$

[Dividing N′ and D′ by $\cos^2 x$]

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$I = \int \frac{dt}{3t^2 + 2t + 3} = \frac{1}{3}\int \frac{dt}{t^2 + \frac{2}{3}t + 1}$$

$$= \frac{1}{3}\int \frac{dt}{\left(t + \frac{1}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2}$$

$$\therefore \quad I = \frac{1}{3}\frac{1}{\left(\frac{2\sqrt{2}}{3}\right)}\tan^{-1}\left(\frac{t + \frac{1}{3}}{\frac{2\sqrt{2}}{3}}\right) + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{3\tan x + 1}{2\sqrt{2}}\right) + C$$

Form II: $\int \frac{dx}{a + b\cos x + c\sin x}$

Working rule:

Write $\sin x$ and $\cos x$ in terms of $\tan(x/2)$ and then substitute for $\tan(x/2) = t$.

ILLUSTRATION 7.73

Evaluate $\int \frac{dx}{2 + \sin x + \cos x}$.

Sol. $I = \int \frac{dx}{2 + \sin x + \cos x}$

$$= \int \frac{dx}{2 + \frac{2\tan x/2}{1 + \tan^2 x/2} + \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}}$$

$$= \int \frac{1 + \tan^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 3} dx$$

$$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 3} dx$$

Put $\tan\frac{x}{2} = t$. $\therefore$ $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$\therefore \quad I = \int \frac{2dt}{t^2 + 2t + 3}$$

$$= 2\int \frac{dt}{(t+1)^2 + 2}$$

$$= 2\frac{1}{\sqrt{2}}\tan\frac{t+1}{\sqrt{2}} + c$$

$$= \sqrt{2}\tan\frac{\tan\frac{x}{2} + 2}{\sqrt{2}} + c$$

Form III: $\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c} dx$

Working Rule:

Let $(p\cos x + q\sin x + r) = \lambda(a\cos x + b\sin x + c) + \mu(a\cos x + b\sin x + c)' + \gamma$

Find λ, μ and γ by comparing coefficients of $\sin x$, $\cos x$ and constant term and split the integral into sum of three integrals.

$$\lambda\int dx + \mu\int \frac{(a\cos x + b\sin x + c)'dx}{a\cos x + b\sin x + c} + \gamma\int \frac{dx}{a\cos x + b\sin x + c}$$

ILLUSTRATION 7.74

Evaluate $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} dx$.

Sol. We have $I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} dx$

Let $3\sin x + 2\cos x = \mu\frac{d}{dx}(3\cos x + 2\sin x) + \lambda(3\cos x + 2\sin x)$

$= \mu(-3\sin x + 2\cos x) + \lambda(3\cos x + 2\sin x)$

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$-3\mu + 2\lambda = 3$ and $2\mu + 3\lambda = 2$

or $\lambda = \frac{12}{13}$ and $\mu = -\frac{5}{13}$

$$\therefore \quad I = \int \frac{\mu(-3\sin x + 2\cos x) + \lambda(3\cos x + 2\sin x)}{3\cos x + 2\sin x} dx$$

$$= \lambda\int 1\,dx + \mu\int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x} dx$$

$= \lambda x + \mu\int \frac{dt}{t}$, where $t = 3\cos x + 2\sin x$

$$= \lambda x + \mu\log|t| + C$$

$$= \frac{12}{13}x + \frac{-5}{13}\log|3\cos x + 2\sin x| + C$$

Form IV: $\int \frac{1}{(\text{Linear})_1\sqrt{\text{Linear}}}dx$, $\int \frac{(\text{Linear})_1}{\sqrt{\text{Linear}}}dx$, $\int \frac{\sqrt{\text{Linear}}}{(\text{Linear})_1}dx$

Working Rule:

Substitute (Linear) $= t^2$

ILLUSTRATION 7.75

Evaluate $\int \frac{1}{(x-3)\sqrt{x+1}} dx$.

Sol. Let $I = \int \frac{1}{(x-3)\sqrt{x+1}} dx$

Let $x + 1 = t^2$ or $dx = 2t\,dt$

$$\therefore \quad I = \int \frac{1}{(t^2 - 1 - 3)}\frac{2t}{\sqrt{t^2}} dt$$

$$= 2\int \frac{dt}{t^2 - 2^2} = 2 \times \frac{1}{2(2)} \log\left|\frac{t-2}{t+2}\right| + C$$

$$= \frac{1}{2} \log\left|\frac{\sqrt{x+1} - 2}{\sqrt{x+1} + 2}\right| + C$$

Form V: $\int \frac{1}{\text{Linear}\sqrt{\text{Quadratic}}} dx$,

Working Rule:

Substitute for Linear $= \frac{1}{t}$

ILLUSTRATION 7.76

Evaluate $\int \frac{1}{(x+1)\sqrt{x^2-1}} dx$.

Sol. Let $I = \int \frac{1}{(x+1)\sqrt{x^2-1}} dx$

Putting $x + 1 = \frac{1}{t}$ and $dx = -\frac{1}{t^2} dt$, we get

$$I = \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t} - 1\right)^2 - 1}} \left(-\frac{1}{t^2}\right) dt$$

$$= -\int \frac{dt}{\sqrt{1-2t}} = -\int (1-2t)^{-1/2} dt$$

$$= -\frac{(1-2t)^{1/2}}{(-2)\left(\frac{1}{2}\right)} + C = \sqrt{1-2t} + C$$

$$= \sqrt{1 - \frac{2}{x+1}} + C = \sqrt{\frac{x-1}{x+1}} + C$$

ILLUSTRATION 7.77

Evaluate $\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}} dx$.

Sol. $I = \int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}} dx$

$$= \int \frac{x^2(1-1/x^2)}{x^2(x+1/x)\sqrt{x^2+1/x^2}} dx$$

$$= \int \frac{(1-1/x^2)dx}{(x+1/x)\sqrt{(x+1/x)^2 - 2}}$$

Putting $x + 1/x = t$, we have $I = \int \frac{dt}{t\sqrt{t^2-2}}$.

Again putting $t^2 - 2 = y^2$, $2t\, dt = 2y\, dy$,

$$I = \int \frac{y dy}{(y^2+2)y} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{y}{\sqrt{2}} = \frac{1}{2} \tan^{-1} \frac{\sqrt{x^2+1/x^2}}{\sqrt{2}} + c.$$

Form VI: $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$

Working Rule:

Substitute for $x = \frac{1}{t}$ then I reduces to $\int \frac{t dt}{(pt^2+q)\sqrt{rt^2+s}}$.

Now substitute for $rt^2 + s = u^2$

ILLUSTRATION 7.78

Evaluate $\int \frac{1}{(1-x^2)\sqrt{1+x^2}} dx$.

Sol. Putting $x = \frac{1}{t}$ and $dx = -\frac{1}{t^2} dt$, we get

$$I = \int \frac{\left(-\frac{1}{t^2}\right)dt}{\left(1 - \frac{1}{t^2}\right)\sqrt{1 + \frac{1}{t^2}}} = -\int \frac{t\, dt}{(t^2-1)\sqrt{t^2+1}}$$

Let $t^2 + 1 = u^2$, or $2tdt = 2udu$

$$\therefore \quad I = -\int \frac{du}{u^2 - (\sqrt{2})^2}$$

$$= -\frac{1}{2\sqrt{2}} \log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| + C$$

$$= -\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{t^2+1} - \sqrt{2}}{\sqrt{t^2+1} + \sqrt{2}}\right| + C$$

$$= -\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{\frac{1}{x^2}+1} - \sqrt{2}}{\sqrt{\frac{1}{x^2}+1} + \sqrt{2}}\right| + C$$

$$= -\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{1+x^2} - \sqrt{2}x}{\sqrt{1+x^2} + \sqrt{2}x}\right| + C$$

Form VII: Substitution $\sin x \pm \cos x = t$ when there is $\sin 2x$ in the denominator

ILLUSTRATION 7.79

Evaluate $\int \frac{(\sin x + \cos x)dx}{\sqrt{3 + \sin 2x}}$.

Sol. $\int \frac{(\sin x + \cos x)dx}{\sqrt{3+\sin 2x}} = \int \frac{(\sin x + \cos x)dx}{\sqrt{4 - (\sin x - \cos x)^2}}$

$$= \int \frac{dt}{\sqrt{4-t^2}} \quad (\text{Putting } \sin x - \cos x = t)$$

$$= \sin^{-1} \frac{t}{2} + c$$

$$= \sin^{-1} \frac{\sin x - \cos x}{2} + c$$

ILLUSTRATION 7.80

Evaluate $\int \frac{\sin x}{2+\sin 2x}dx.$

Sol. $\int \frac{\sin x}{2+\sin 2x}dx$

$$= \frac{1}{2}\int \frac{\sin x+\cos x-(\cos x-\sin x)}{2+\sin 2x}dx$$

$$= \frac{1}{2}\int \frac{\sin x+\cos x}{2+\sin 2x}dx - \frac{1}{2}\int \frac{\cos x-\sin x}{2+\sin 2x}dx$$

$$= \frac{1}{2}\int \frac{\sin x+\cos x}{3-(\sin x-\cos x)^2}dx - \frac{1}{2}\int \frac{\cos x-\sin x}{1+(\sin x+\cos x)^2}dx$$

$$= \frac{1}{2}\int \frac{dt}{3-t^2} - \frac{1}{2}\int \frac{du}{1+u^2} \quad \left(\begin{array}{l}\text{where } t=\sin x-\cos x\\ \text{and } u=\sin x+\cos x\end{array}\right)$$

$$= \frac{1}{2}\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}-t}{\sqrt{3}+t}\right| - \frac{1}{2}\tan^{-1}u+c$$

$$= \frac{1}{4\sqrt{3}}\log\left|\frac{\sqrt{3}-(\sin x-\cos x)}{\sqrt{3}+(\sin x-\cos x)}\right| - \frac{1}{2}\tan^{-1}(\sin x+\cos x)+c$$

CONCEPT APPLICATION EXERCISE 7.7

Evaluate the following:

1. $\int \frac{dx}{3+\sin 2x}$

2. $\int \frac{dx}{9+16\sin^2 x}$

3. $\int \frac{x+1}{(x-1)\sqrt{x+2}}dx$

4. $\int \frac{dx}{\sqrt{1+e^x+e^{2x}}}$

5. $\int \frac{1}{(x+1)\sqrt{x^2-1}}dx$

6. $\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}dx$

7. $\int \frac{x}{(x^2+4)\sqrt{x^2+1}}dx$

8. $\int \frac{1}{(x+1)\sqrt{x^2+x+1}}dx$

9. $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx$

10. $\int (\sqrt{\tan x}+\sqrt{\cot x})dx$

ANSWERS

1. $\frac{1}{\sqrt{8}}\tan^{-1}\frac{3\tan x+1}{\sqrt{8}}+c$

2. $\frac{1}{15}\tan^{-1}\frac{5\tan x}{3}+C$

3. $2\sqrt{x+2}+\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+C$

4. $-\log_e\left|\left(e^{-x}+\frac{1}{2}\right)+\sqrt{e^{-2x}+e^{-x}+1}\right|+c$

5. $\sqrt{\frac{x-1}{x+1}}+C$

6. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{x^2+\frac{1}{x^2}}}{\sqrt{2}}+C$

7. $\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\frac{x^2+1}{3}}+C$

8. $-\log\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{x+1}\right|+C$

9. $2\tan^{-1}\sqrt{x+\frac{1}{x}+1}+C$

10. $\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$

INTEGRATION BY PARTIAL FRACTIONS

Method of partial fraction is used to integrate rational functions. Rational functions are of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$.

If the degree of $P(x)$ is less than the degree of $Q(x)$, then the rational function is called proper, otherwise, it is called improper. The improper rational functions can be reduced to the proper rational functions by long division process. Thus if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = R(x)+\frac{P_1(x)}{Q(x)}$, where $R(x)$ is polynomial in x and $\frac{P_1(x)}{Q(x)}$ is proper rational function.

Now we want to evaluate $\int \frac{P(x)}{Q(x)}dx$, where $\frac{P(x)}{Q(x)}$ is proper fraction. Here we are assuming that $Q(x)$ can be factorized in to linear and quadratic expressions.

Expression $\dfrac{P(x)}{Q(x)}$ can be decomposed into simpler rational expressions that we can add or subtract to get the original rational expression. This process is called is called **partial fraction decomposition**. After this, the integration can be carried out easily using the known methods.

Some of the examples of simple partial fractions we have been using till now are

(i) $\dfrac{1}{x^2-x-2}=\dfrac{1}{(x-2)(x+1)}=\dfrac{1}{3}\left[\dfrac{1}{x-2}-\dfrac{1}{x+1}\right]$

(ii) $\dfrac{1}{x^4-1}=\dfrac{1}{(x^2-1)(x^2+1)}$

$$=\frac{1}{2}\left[\frac{1}{x^2-1}-\frac{1}{x^2+1}\right]$$

$$=\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)-\frac{1}{x^2+1}\right]$$

Here fractions on R.H.S. are called partial fractions.

The partial fractions of proper rational expression $\dfrac{P(x)}{Q(x)}$ depend on the nature of the factors of $Q(x)$. We have to deal with the following different type, where the factors of $Q(x)$ are:

(i) Linear and non-repeated.
(ii) Linear and repeated.
(iii) Quadratic and non-repeated.

Case I : When denominator is expressible as the product of non-repeated linear factors:

Let $Q(x) = (x - a_1)(x - a_2)(x - a_3) \dots (x - a_n)$.

Then we have

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a_1)}+\frac{A_2}{(x-a_2)}+\frac{A_3}{(x-a_3)}+\dots+\frac{A_n}{(x-a_n)}$$

where $A_1, A_2 \dots, A_n$ are constants and can be determined by equating the numerator on RHS to numerator on LHS and then substituting $x = a_1, a_2, \dots, a_n$

Short cut method: Consider $x - a_1 = 0$, then $x = a_1$, put this value of x in all the expressions other than $x - a_1$ and so on

e.g.,

$$\frac{x^2+1}{x(x-1)(x+1)}=\frac{0+1}{x(0-1)(0+1)}+\frac{1+1}{1(x-1)(1+1)}+\frac{1+1}{-1(-1-1)(x+1)}$$

Case II: When the denominator $g(x)$ is expressible as the product of the linear factor such that some of them are repeating. **(Linear and Repeated)**

Let, $Q(x) = (x - a)^k (x - a_1)(x - a_2) \dots (x - a_r)$.

Then we assume that

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\dots\frac{A_k}{(x-a)^k}+\frac{B_1}{(x-a_1)}$$

$$+\frac{B_2}{(x-a_2)}+\dots+\frac{B_r}{(x-a_r)}$$

Case III: When some of the factors in denominator are quadratic but nonrepeating.

Corresponding to each quadratic factor $ax^2 + bx + c$, we assume the partial fraction of the type $\dfrac{Ax+B}{ax^2+bx+C}$, where A and B are constants to be determined by comparing coefficients of similar power of x in numerator of both sides.

ILLUSTRATION 7.81

Evaluate $\displaystyle\int\frac{2x-1}{(x-1)(x+2)(x-3)}dx.$

Sol. Since all the factors in the denominator are linear, we have

$$\int\frac{2x-1}{(x-1)(x+2)(x-3)}dx$$

$$=\int\left[\frac{1}{(x-1)(3)(-2)}+\frac{-5}{(-3)(x+2)(-5)}+\frac{5}{(2)(5)(x-3)}\right]dx$$

$$=-\frac{1}{6}\log|x-1|-\frac{1}{3}\log|x+2|+\frac{1}{2}\log|x-3|+C$$

ILLUSTRATION 7.82

Evaluate $\displaystyle\int\frac{2x}{(x^2+1)(x^2+2)}dx.$

Sol. Let $I=\displaystyle\int\frac{2x}{(x^2+1)(x^2+2)}dx$

Putting $x^2 = t$ and $2x\,dx = dt$, we get

$$I=\int\frac{dt}{(t+1)(t+2)}$$

$$=\int\left(\frac{1}{t+1}-\frac{1}{t+2}\right)dt$$

$$=\log|t+1|-\log|t+2|+C$$

$$=\log|x^2+1|-\log|x^2+2|+C$$

ILLUSTRATION 7.83

Evaluate $\displaystyle\int\frac{1}{\sin x-\sin 2x}dx.$

Sol. $I=\displaystyle\int\frac{1}{\sin x-\sin 2x}dx$

$$=\int\frac{1}{(\sin x-2\sin x\cos x)}dx$$

$$=\int\frac{1}{\sin x(1-2\cos x)}dx$$

$$=\int\frac{\sin x}{\sin^2 x(1-2\cos x)}dx$$

$$=\int\frac{\sin x}{(1-\cos^2 x)(1-2\cos x)}dx$$

Putting $\cos x = t$ and $-\sin x\,dx = dt$

or $\sin x\,dx = -dt$, we get

$$I=\int\frac{-dt}{\left(1-t^2\right)(1-2t)}$$

$$=\int\frac{1}{(t-1)(1+t)(1-2t)}dt$$

$$=\int\left(\frac{1}{(t-1)(2)(-1)}+\frac{1}{(-2)(1+t)(3)}+\frac{1}{(-1/2)(3/2)(1-2t)}\right)dt$$

$$=-\frac{1}{2}\log|1-t|-\frac{1}{6}\log|1+t|+\frac{2}{3}\log|1-2t|+C$$

$$=-\frac{1}{2}\log|1-\cos x|-\frac{1}{6}\log|1+\cos x|+\frac{2}{3}\log|1-2\cos x|+C$$

ILLUSTRATION 7.84

Evaluate $\int\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}dx$.

Sol. Improper fraction

$$\int\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}dx$$ Proper fraction

$$I=\int\left[1+\frac{(x-1)(x-2)(x-3)-(x-4)(x-5)(x-6)}{(x-4)(x-5)(x-6)}\right]dx$$

(Adding and subtracting 1)

$$=\int\left[1+\frac{3\times2\times1}{(x-4)(-1)(-2)}+\frac{4\times3\times2}{1(x-5)(-1)}+\frac{5\times4\times3}{(2)(1)(x-6)}\right]dx$$

$$=1+3\log|x-4|-24\log|x-5|+30\log|x-6|+C$$

ILLUSTRATION 7.85

Evaluate $\int\frac{x^2}{(x^2+1)(x^2+4)}dx$.

Sol. $\int\frac{x^2}{(x^2+1)(x^2+4)}dx=\frac{1}{3}\int\left[\frac{4}{x^2+4}-\frac{1}{x^2+1}\right]dx$

$$=-\frac{1}{3}\int\frac{1}{x^2+1}dx+\frac{4}{3}\int\frac{1}{x^2+4}dx$$

$$=-\frac{1}{3}\tan^{-1}x+\frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$=-\frac{1}{3}\tan^{-1}x+\frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right)+C$$

ILLUSTRATION 7.86

Evaluate $\int\frac{\sin x}{\sin 4x}dx$.

Sol. $I=\int\frac{\sin x}{\sin 4x}dx=\int\frac{\sin x}{2\sin 2x\cos 2x}dx$

$$=\int\frac{\sin x}{4\sin x\cos x\cos 2x}dx$$

$$=\frac{1}{4}\int\frac{1}{\cos x\cos 2x}dx=\frac{1}{4}\int\frac{\cos x}{\cos^2 x\cos 2x}dx$$

$$=\frac{1}{4}\int\frac{\cos x}{\left(1-\sin^2 x\right)\left(1-2\sin^2 x\right)}dx$$

Putting $\sin x=t$ and $\cos x\,dx=dt$, we get

$$I=\frac{1}{4}\int\frac{dt}{\left(1-t^2\right)\left(1-2t^2\right)}$$

$$=\frac{1}{4}\int\left[\frac{2}{1-2t^2}-\frac{1}{1-t^2}\right]dt$$

$$=-\frac{1}{4}\int\frac{1}{1-t^2}dt+\frac{2}{4}\int\frac{1}{1-\left(\sqrt{2}t\right)^2}dt$$

$$=-\frac{1}{4}\times\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|+\frac{1}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\,t}{1-\sqrt{2}\,t}\right|+C$$

$$=-\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right|+\frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|+C$$

ILLUSTRATION 7.87

Evaluate $\int\frac{x^2+1}{(x-1)^2(x+3)}dx$.

Sol. $I=\int\frac{x^2+1}{(x-1)^2(x+3)}dx$

Let $\frac{x^2+1}{(x-1)^2(x+3)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+3}$ (1)

or $x^2+1=A(x-1)(x+3)+B(x+3)+C(x-1)^2$ (2)

Putting $x-1=0$, i.e., $x=1$ in equation (2), we get

$2=4B$ or $B=\frac{1}{2}$. Putting $x+3=0$,

i.e., $x=-3$ in equation (2), we get $10=16C$ or $C=\frac{5}{8}$.

Equating the coefficients of x^2 on both the sides of the identity of equation (2), we get

$$1=A+C \text{ or } A=1-C=1-\frac{5}{8}=\frac{3}{8}$$

Substituting the values of A, B in equation(1), we get

$$\frac{x^2+1}{(x-1)^2(x+3)}=\frac{3}{8}\frac{1}{x-1}+\frac{1}{2}\frac{1}{(x-1)^2}+\frac{5}{8}(x+3)$$

or $I=\frac{3}{8}\int\frac{1}{x-1}dx+\frac{1}{2}\int\frac{1}{(x-1)^2}dx+\frac{5}{8}\int\frac{1}{x+3}dx$

$$=\frac{3}{8}\log|x-1|-\frac{1}{2(x-1)}+\frac{5}{8}\log|x+3|+C$$

ILLUSTRATION 7.88

Evaluate $\int \frac{x}{(x-1)(x^2+4)}dx$.

Sol. $\int \frac{x}{(x-1)(x^2+4)}dx$

Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$ (1)

or $x = A(x^2+4) + (Bx+C)(x-1)$ (2)

Putting $x = 1$ in equation (2), we get $1 = 5A$.

Putting $x = 0$ in equation (2), we get $0 = 4A - C$.

Putting $x = -1$ in equation (2), we get $-1 = 5A + 2B - 2C$.

Solving these equations, we obtain $A = \frac{1}{5}, B = -\frac{1}{5}$, and $C = \frac{4}{5}$.

Substituting the values of A, B, and C in equation (1), we obtain

$$\frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} + \frac{-\frac{1}{5}x + \frac{4}{5}}{x^2+4}$$

$$= \frac{1}{5(x-1)} - \frac{1}{5}\frac{(x-4)}{(x^2+4)}$$

or $I = \frac{1}{5}\int \frac{1}{x-1}dx - \frac{1}{5}\int \frac{x-4}{x^2+4}dx$

$$= \frac{1}{5}\int \frac{1}{x-1}dx - \frac{1}{10}\int \frac{2x}{x^2+4}dx + \frac{4}{5}\int \frac{1}{x^2+4}dx$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\log(x^2+4) + \frac{4}{5}\times\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\log(x^2+4) + \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C$$

ILLUSTRATION 7.89

Evaluate $\int \frac{\log_{ex} e \cdot \log_{e^2x} e \cdot \log_{e^3x} e}{x}dx$.

Sol. $I = \int \frac{\log_{ex} e \cdot \log_{e^2x} e \cdot \log_{e^3x} e}{x}dx$

$$= \int \frac{1}{x\log_e ex \cdot \log_e e^2x \cdot \log_e e^3x}dx$$

$$= \int \frac{1}{x(\log_e e + \log_e x)(\log_e e^2 + \log_e x)(\log_e e^3 + \log_e x)}dx$$

$$= \int \frac{1}{(1+t)(2+t)(3+t)}dt, \text{ where } t = \log_e x$$

$$= \int\left(\frac{1}{2}\cdot\frac{1}{1+t} - \frac{1}{2+t} + \frac{1}{3+t}\right)dt$$

[Using partial fractions]

$$= \frac{1}{2}\log|1+t| - \log|2+t| + \log|3+t| + C$$

$$= \frac{1}{2}\log|1+\log_e x| - \log|2+\log_e x| + \log|3+\log_e x| + C$$

CONCEPT APPLICATION EXERCISE 7.8

Evaluate the following:

1. $\int \frac{1}{(x^2-4)\sqrt{x+1}}dx$
2. $\int \frac{x^2+1}{x(x^2-1)}dx$
3. $\int \frac{1}{x^4-1}dx$
4. $\int \frac{x^3}{(x-1)(x-2)}dx$
5. $\int \frac{dx}{\sin x(3+\cos^2 x)}$
6. $\int \frac{\cos 2x \sin 4x dx}{\cos^4 x(1+\cos^2 2x)}$
7. $\int \frac{1-\cos x}{\cos x(1+\cos x)}dx$

ANSWERS

1. $\frac{1}{4\sqrt{3}}\log\left|\frac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}\right| - \frac{1}{2}\tan^{-1}\sqrt{x+1} + C$
2. $\log\left|\frac{x^2-1}{x}\right| + C$
3. $\frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + C$
4. $\frac{x^2}{3} + 3x - \log|x-1| + 8\log|x-2| + C$
5. $\frac{1}{4}\log\left|\frac{y-1}{y+1}\right| - \frac{1}{4\sqrt{3}}\tan^{-1}\frac{y}{\sqrt{3}} + C$
6. $-\frac{1}{2}\log|\cos 2x| - \frac{1}{2(1+\cos 2x)} + \frac{1}{4}\log(1+\cos^2 2x) + C$
7. $\log|\sec x + \tan x| - 2\tan x/2 + C$

INTEGRATION BY PARTS

This is method of integration is useful in integrating function which is product of two functions. The rule is as follows.

If u and v are two functions of x, then

$$\int uv dx = u \times \int v dx - \int\left\{\frac{du}{dx}\int v\, dx\right\}dx$$

i.e., the integral of product of two functions

= (first function) × (integral of second function)

− integral of (differential of first function × integral of second).

Proof:

For any two functions $f(x)$ and $g(x)$, we have

$$\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot\frac{d}{dx}(g(x)) + g(x)\cdot\frac{d}{dx}(f(x))$$

$$\therefore \quad \int\left(f(x).\frac{d}{dx}\{g(x)\} + g(x).\frac{d}{dx}\{f(x)\}\right)dx = f(x)\cdot g(x)$$

$$\Rightarrow \quad \int\left(f(x).\frac{d}{dx}\{g(x)\}\right)dx$$

$$+\int\left(g(x).\frac{d}{dx}\{f(x)\}\right)dx = f(x).\,g(x)$$

$$\Rightarrow \quad \int\left(f(x).\frac{d}{dx}\{g(x)\}\right)dx$$

$$= f(x).\,g(x) - \int\left(g(x).\frac{d}{dx}\{f(x)\}\right)dx$$

Let, $f(x) = u$ and $\frac{d}{dx}\{g(x)\} = v$, so that $g(x) = \int vdx$

$$\therefore \quad \int uvdx = u \times \int vdx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

Important points for choosing u (first function):

While applying the above rule, we should be careful in selecting the first function (u) and the second function (v). Normally, we use the following methods:

(i) If in the product of the two functions, one of the functions is not directly integrable (e.g., $\log|x|$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ …, etc.), then we take it as the first function and the other function as the second function. e.g, in $\int x\tan^{-1}xdx$, $\tan^{-1}x$ is taken as the first function and x as the second function.

(ii) If these is no other function then unity is taken as the second function, e.g., in the integration of $\int\sin^{-1}xdx$, $\int\log_e xdx$, 1 is taken as the second function.

(iii) If both of the functions are directly integrable then the first function is chosen in such a way that the product of derivatives of the function and the second function obtained under integral sign is easily integrable.

Usually, the preference order for the first function is:

Inverse Trigonometric, Logarithmic, Algebraic, Trigonometric, Exponential

In above stated order, the function on the left is always chosen as the first function. This rule can be remembered by the mnemonic **ILATE.**

ILLUSTRATION 7.90

Evaluate $\int x\sin 3xdx$.

Sol. Here, both the functions, viz., x and $\sin 3x$ are easily integrable and the derivative of x is one, a less complicated function. Therefore, we take x as the first function and $\sin 3x$ as the second function. Thus,

$$\int \underset{\text{I}}{x}\,\underset{\text{II}}{\sin 3x}\,dx$$

$$= x\left\{\int \sin 3x\,dx\right\} - \int\left\{\frac{d}{dx}(x)\int \sin 3x\,dx\right\}dx$$

$$= -x\frac{\cos 3x}{3} - \int 1\left\{-\frac{\cos 3x}{3}\right\}dx$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

ILLUSTRATION 7.91

Evaluate $\int x\log xdx$.

Sol. $\int \underset{\text{II}}{x}\,\underset{\text{I}}{\log x}\,dx$

$$= \log x\left\{\int x\,dx\right\} - \int\left\{\frac{d}{dx}(\log x)\int x\,dx\right\}dx$$

$$= (\log x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C = \frac{x^2}{2}\log x - \frac{1}{4}x^2 + C$$

ILLUSTRATION 7.92

Evaluate $\int \sin^{-1}xdx$.

Sol. $\int \sin^{-1}x\,dx$

$$= \sin^{-1}x\int 1dx - \int\left((\sin^{-1}x)'\int 1dx\right)dx$$

$$= x\sin^{-1}x - \int\frac{x}{\sqrt{1-x^2}}dx$$

$$= x\sin^{-1}x + \frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}(-2x)dx$$

$$= x\sin^{-1}x + \frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

ILLUSTRATION 7.93

Evaluate $\int 2x^3e^{x^2}dx$.

Sol. $I = \int x^2e^{x^2}(2x)dx$

Let $\quad x^2 = t$

$$\therefore \quad I = \int te^t dt$$

$$= t\cdot e^t - \int 1\cdot e^t dt$$

$$= t\cdot e^t - e^t + c$$

$$= x^2\cdot e^{x^2} - e^{x^2} + c$$

ILLUSTRATION 7.94

Evaluate $\int \frac{x^2 \tan^{-1} x}{1+x^2} dx$.

Sol. Let $x = \tan\theta$.

$dx = \sec^2\theta \, d\theta$

$$\therefore \quad I = \int \frac{x^2 \tan^{-1} x}{1+x^2} dx$$

$$= \int \frac{\theta \tan^2\theta}{1+\tan^2\theta} \cdot \sec^2\theta d\theta$$

$$= \int \theta \tan^2\theta d\theta$$

$$= \int \theta(\sec^2\theta - 1) d\theta$$

$$= \int \theta \sec^2\theta d\theta - \int \theta d\theta$$

$$= \theta \tan\theta - \int 1 \cdot \tan\theta d\theta - \frac{\theta^2}{2} + c$$

$$= \theta \tan\theta - \log_e |\cos\theta| - \frac{\theta^2}{2} + c, \text{ where } \theta = \tan^{-1} x.$$

ILLUSTRATION 7.95

Evaluate $\int \frac{x - \sin x}{1 - \cos x} dx$.

Sol. $\int \frac{x - \sin x}{1 - \cos x} dx$

$$= \int \frac{x}{1-\cos x} dx - \int \frac{\sin x}{1 - \cos x} dx$$

$$= \frac{1}{2}\int x \operatorname{cosec}^2 \frac{x}{2} dx - \int \frac{2 \sin x/2 \cos x/2}{2 \sin^2 x/2} dx$$

$$= \frac{1}{2} \int \underset{I}{x} \underset{II}{\operatorname{cosec}^2} \frac{x}{2} dx - \int \cot \frac{x}{2} dx$$

$$= \frac{1}{2}\left\{x\left(-2 \cot \frac{x}{2}\right) - \int 1 \cdot \left(-2 \cot \frac{x}{2}\right) dx\right\} - \int \cot \frac{x}{2} dx + c$$

$$= -x \cot \frac{x}{2} + \int \cot \frac{x}{2} dx - \int \cot \frac{x}{2} dx + c$$

$$= -x \cot \frac{x}{2} + c.$$

ILLUSTRATION 7.96

If $f(x)$ is a polynomial function of the nth degree, prove that $\int e^x f(x) dx = e^x[f(x) - f'(x) + f''(x) - f'''(x) + \cdots + (-1)^n f^{(n)}(x)]$ where $f^{(n)}(x)$ denotes $\frac{d^n f}{dx^n}$.

Sol. $I = \int e^x f(x) dx$

$$= f(x) e^x - \int e^x f'(x) dx$$

$$= f(x) e^x - f'(x) e^x + \int e^x f''(x) dx$$

$$= f(x) e^x - f'(x) e^x + f''(x) e^x - \int e^x f'''(x) dx$$

$$= f(x) e^x - f'(x) e^x + f''(x) e^x - f'''(x) e^x + \int e^x f^{iv}(x) dx$$

Continuing this way, we get

$$I = e^x[f(x) - f'(x) + f''(x) - f'''(x) \cdots + (-1)^n f^{(n)}(x)]$$

ILLUSTRATION 7.97

Evaluate $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$.

Sol. $I = \int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$

Let $x = a \tan^2\theta$ or $dx = 2a \tan\theta \sec^2\theta \, d\theta$

$$\therefore \quad \int \sin^{-1} \sqrt{\sin^2\theta} \, 2a \sec^2\theta \tan\theta d\theta$$

$$= 2a \int \underset{I}{\theta} \underset{II}{\sec^2\theta \tan\theta} d\theta$$

$$= 2a\left[\theta \frac{\tan^2\theta}{2} - \int \frac{\tan^2\theta}{2} d\theta\right]$$

$$= a\left[\theta \tan^2\theta - \int (\sec^2\theta - 1) d\theta\right]$$

$$= a[\theta \tan^2\theta - \tan\theta + \theta] + c, \text{ where } \theta = \tan^{-1}\sqrt{\frac{x}{a}}$$

ILLUSTRATION 7.98

Evaluate $\int \frac{\log_e (1 + \sin^2 x)}{\cos^2 x} dx$

Sol. $I = \int \log_e (1 + \sin^2 x) . \sec^2 x dx$

$$= \log_e (1 + \sin^2 x) \cdot \tan x - \int \frac{2 \sin x \cos x}{1 + \sin^2 x} \tan x dx$$

$$= \tan x \times \log_e (1 + \sin^2 x) - 2\int \frac{\sin^2 x}{1 + \sin^2 x} dx$$

$$= \tan x \times \log_e (1 + \sin^2 x) - 2\int \left(1 - \frac{1}{1 + \sin^2 x}\right) dx$$

$$= \tan x \times \log_e (1 + \sin^2 x) - 2x + 2 \int \frac{\sec^2 x dx}{\sec^2 x + \tan^2 x}$$

$$= \tan x \times \log_e (1 + \sin^2 x) - 2x + 2 \int \frac{dt}{1 + 2t^2}$$

(where $t = \tan x$)

$$= \tan x \times \log_e (1 + \sin^2 x) - 2x + 2 . \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{2} t + c$$

$$= \tan x \times \log_e (1 + \sin^2 x) - 2x + \sqrt{2} \tan^{-1}(\sqrt{2} \tan x) + c$$

INTEGRATION OF $e^{ax}\sin bx$ AND $e^{ax}\cos bx$

(i) $I=\int e^{ax}\sin bx\,dx$

$$= e^{ax}\int \sin bx\,dx - \int ae^{ax}\left(\frac{-\cos bx}{b}\right)dx + c$$

$$= -e^{ax}\frac{\cos bx}{b}+\frac{a}{b}\int e^{ax}\cos bx\,dx + c$$

$$= -e^{ax}\frac{\cos bx}{b}+\frac{a}{b}\left[e^{ax}\int \cos bx\,dx - \int ae^{ax}\frac{\sin bx}{b}dx\right] + c$$

$$= -e^{ax}\frac{\cos bx}{b}+\frac{a}{b}\left[e^{ax}\frac{\sin bx}{b}-\frac{a}{b}I\right] + c$$

$$\Rightarrow \left(1+\frac{a^2}{b^2}\right)I = -e^{ax}\frac{\cos bx}{b}+\frac{a}{b^2}e^{ax}\sin bx + c$$

$$\Rightarrow I=\frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

(ii) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c$

ILLUSTRATION 7.99

Evaluate $\int e^{2x}\sin 3x\,dx$.

Sol. We can use direct formula or can get it by the method in which we derived the formula.

Let us use the direct formula,

$$I=\int e^{2x}\sin 3x\,dx$$

$$=\frac{e^{2x}}{2^2+3^2}(2\sin 3x - 3\cos 3x) + C$$

$$=\frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

ILLUSTRATION 7.100

Evaluate $\int \sin(\log x)dx$.

Sol. Let $\log x = t$. Then,

$$x=e^t \Rightarrow dx = e^t dt$$

$$\therefore \quad I=\int \sin(\log x)dx$$

$$=\int \sin t\, e^t dt$$

$$=\frac{e^t}{2}(\sin t - \cos t) + C$$

Hence, $\int \sin(\log x)dx = \frac{x}{2}$ [sin (log x) – cos (log x)] + C

ILLUSTRATION 7.101

Evaluate $\int e^{\sin^{-1}x}dx$.

Sol. Let $\sin^{-1}x = t \Rightarrow x=\sin t \Rightarrow dx = \cos t\,dt$

$$\therefore \quad I=\int e^{\sin^{-1}x}dx$$

$$=\int e^t\cos t\,dt$$

$$=\frac{e^t}{2}(\sin t+\cos t)+c$$

$$=\frac{e^{\sin^{-1}x}}{2}\left(x+\sqrt{1-x^2}\right)+c$$

INTEGRATION OF SQUARE ROOT OF QUADRATIC

We have following standard formulas for this form:

(i) $\int\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log_e\left|x+\sqrt{a^2+x^2}\right|+c$

Proof:

$$I=\int\sqrt{a^2+x^2}\,dx$$

$$=\sqrt{a^2+x^2}\int 1dx-\int\left[\frac{d}{dx}\left(\sqrt{a^2+x^2}\right)\int 1dx\right]dx + c$$

$$=x\sqrt{a^2+x^2}-\int\frac{x}{\sqrt{a^2+x^2}}x\,dx + c$$

$$=x\sqrt{a^2+x^2}-\int\frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}}dx + c$$

$$=x\sqrt{a^2+x^2}-\int\sqrt{a^2+x^2}\,dx+\int\frac{a^2}{\sqrt{a^2+x^2}}dx + c$$

$$\Rightarrow 2I = x\sqrt{a^2+x^2}+a^2\log_e\left|x+\sqrt{a^2+x^2}\right| + c$$

$$\Rightarrow I=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log_e\left|x+\sqrt{a^2+x^2}\right| + c$$

(ii) $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log_e\left|x+\sqrt{x^2-a^2}\right|+c$

(iii) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

ILLUSTRATION 7.102

Evaluate $\int\sqrt{1+3x-x^2}\,dx$.

Sol. $\int\sqrt{1+3x-x^2}\,dx$

$$=\int\sqrt{1+\frac{9}{4}-\left(x^2-2x\left(\frac{3}{2}\right)+\frac{9}{4}\right)}dx$$

$$=\int\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}dx$$

$$= \frac{x-\frac{3}{2}}{2}\sqrt{1+3x-x^2}+\frac{13}{4\times 2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right)+C$$

$$= \frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C$$

ILLUSTRATION 7.103

Evaluate $\int \sec^3 x\, dx$.

Sol. Let $I = \int \sec^3 x\, dx \; d = \int \sec x \sec^2 x\, dx$

$$= \int \sqrt{1+\tan^2 x}\sec^2 x\, dx$$

Put $\tan x = z$ and $\sec^2 x\, dx = dz$

$$\therefore \quad I = \int \sqrt{1+z^2}\, dz$$

$$= \frac{z\sqrt{z^2+1}}{2}+\frac{1}{2}\log|z+\sqrt{z^2+1}|+C$$

$$= \frac{\tan x \sec x}{2}+\frac{1}{2}\log|\tan x+\sec x|+C$$

$$= \frac{1}{2}[\sec x\tan x+\log|\sec x+\tan x|]+C$$

ILLUSTRATION 7.104

Evaluate $\int (x-5)\sqrt{x^2+x}dx$.

Sol. Let $(x-5) = \lambda \frac{d}{dx}(x^2+x)+\mu$.

Then, $x-5 = \lambda(2x+1)+\mu$

Comparing coefficients of like power of x, we get

$1 = 2\lambda$ and $\lambda+\mu = -5$

or $\lambda = \frac{1}{2}$ and $\mu = -\frac{11}{2}$. Therefore,

$$\int (x-5)\sqrt{x^2+x}\,dx = \int\left[\frac{1}{2}(2x+1)-\frac{11}{2}\right]\sqrt{x^2+x}\,dx$$

$$= \frac{1}{2}\int (2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int\sqrt{x^2+x}\,dx$$

$$= \frac{1}{2}\int\sqrt{t}\,dt - \frac{11}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx,$$

where $t = x^2+x$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2}-\frac{11}{2}\left[\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right.$$

$$\left.-\frac{1}{2}\left(\frac{1}{2}\right)^2\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|\right]+C$$

$$= \frac{(x^2+x)^{3/2}}{3}-\frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x}-\frac{1}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right]+C$$

INTEGRATION BY CANCELLATION

One of the interesting properties of the integration by parts method is that when we integrate the addition of two functions, one the of the integrals which can not be evaluated gets cancelled. This is the reason we call this method "Integration by cancellation". One standard formula in this category is

$$\int e^x(f(x)+f'(x))dx = e^x f(x)+c$$

as $\int e^x(f(x)+f'(x))dx$

$$= \int e^x f(x)dx + \int e^x f'(x)dx$$

$$= f(x)\int e^x dx - \int f'(x)e^x dx + \int e^x f'(x)dx + c$$

$$= e^x f(x)+c$$

ILLUSTRATION 7.105

Evaluate $\int\left(\frac{\cos x}{x}-\log x^{\sin x}\right)dx$.

Sol. $\int\left(\frac{\cos x}{x}-\log x^{\sin x}\right)dx$

$$= \int\frac{\cos x}{x}dx - \int \sin x\log x dx$$

$$= \cos x\log x - \int -\sin x\log x dx - \int \sin x \log x dx$$

$$= \cos x \log x + c$$

ILLUSTRATION 7.106

Evaluate $\int\left(3x^2\tan\frac{1}{x}-x\sec^2\frac{1}{x}\right)dx$.

Sol. $\int\left(3x^2\tan\frac{1}{x}-x\sec^2\frac{1}{x}\right)dx$

$$= \int 3x^2\tan\frac{1}{x}dx - \int x\sec^2\frac{1}{x}dx$$

$$= \tan\frac{1}{x}x^3 - \int\left(\sec^2\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)x^3dx - \int x\sec^2\frac{1}{x}dx$$

$$= x^3\tan\frac{1}{x}+c$$

ILLUSTRATION 7.107

Evaluate $\int\left(\log(\log x)+\frac{1}{(\log x)^2}\right)dx$.

Sol. $\int\left(\log(\log x)+\frac{1}{(\log x)^2}\right)dx$

$$= \int 1\cdot\log(\log x)dx + \int\frac{dx}{(\log x)^2}$$

$$= x\log(\log x) - \int\frac{1}{x\log x}x\,dx + \int\frac{dx}{(\log x)^2}+c$$

$= x\log(\log x) - \int(\log x)^{-1}dx + \int\frac{dx}{(\log x)^2} + c$

$= x\log(\log x) - [x(\log x)^{-1} + \int(\log x)^{-2}dx]$

$+ \int(\log x)^{-2}dx + c$

$= x\log(\log x) - x(\log x)^{-1} + c$

ILLUSTRATION 7.108

Evaluate $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$.

Sol. $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \int e^x\left(\frac{1}{x} + \left(\frac{1}{x}\right)'\right)dx$

$= \frac{1}{x}e^x + C$

ILLUSTRATION 7.109

Evaluate $\int e^x\left(\log x + \frac{1}{x^2}\right)dx$.

Sol. $I = \int e^x\left(\log x + \frac{1}{x^2}\right)dx = \int e^x\left(\left(\log x - \frac{1}{x}\right) + \left(\frac{1}{x} + \frac{1}{x^2}\right)\right)dx$

$= e^x\left(\log x - \frac{1}{x}\right) + c \quad \left[\because \frac{d}{dx}\left(\log x - \frac{1}{x}\right) = \frac{1}{x} + \frac{1}{x^2}\right]$

ILLUSTRATION 7.110

Evaluate $\int e^x \frac{x}{(x+1)^2}dx$.

Sol. $\int e^x \frac{x}{(x+1)^2}dx = \int e^x \frac{x+1-1}{(x+1)^2}dx$

$= \int e^x\left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)dx$

$= \int e^x\left\{\frac{1}{x+1} + \left(\frac{1}{x+1}\right)'\right\}dx$

$= \frac{1}{x+1}e^x + c$

ILLUSTRATION 7.111

Evaluate $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$.

Sol. $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$= \int e^x\left(\frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$

$= \int e^x\left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$

$= -\int e^x\left(\cot\frac{x}{2} + \left(\cot\frac{x}{2}\right)'\right)dx$

$= -e^x\cot\frac{x}{2} + c$

ILLUSTRATION 7.112

Evaluate $\int\frac{\log x}{(1+\log x)^2}dx$.

Sol. $I = \int\frac{\log x}{(1+\log x)^2}dx$

Let $\log x = t$. Then $x = e^t$

or $\quad dx = e^t dt$. Thus,

$I = \int\frac{te^t}{(t+1)^2}dt$

$= \int e^t\left(\frac{1}{(t+1)} - \frac{1}{(t+1)^2}\right)dt$

$= \frac{e^t}{t+1} + c$

$= \frac{x}{(\log x + 1)} + c$

CONCEPT APPLICATION EXERCISE 7.9

Evaluate the following:

1. $\int x\sin^2 x\,dx$
2. $\int\cos\sqrt{x}\,dx$
3. $\int\tan^{-1}\sqrt{x}\,dx$
4. If $\int f(x)\,dx = g(x)$, then $\int f^{-1}(x)\,dx$.
5. $\int[f(x)g''(x) - f''(x)g(x)]\,dx$
6. If $\int g(x)dx = g(x)$, then $\int g(x)\{f(x) + f'(x)\}dx$.
7. $\int(\log_e x)^2\,dx$
8. $\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$
9. $\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$
10. $\int\cos x\log\left(\tan\frac{x}{2}\right)dx$
11. $\int\sin^2(\log x)\,dx$

12. $\int e^x(1+\tan x+\tan^2 x)\,dx$

13. $\int \frac{e^x(2-x^2)dx}{(1-x)\sqrt{1-x^2}}$

14. $\int\left(\frac{\log x-1}{1+(\log x)^2}\right)^2 dx$

15. $\int\sqrt{x^2+2x+5}\,dx$

Answers

1. $\frac{1}{4}x^2-\frac{x}{4}\sin 2x-\frac{1}{8}\cos 2x+C$
2. $2\left[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}\right]+C$
3. $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$
4. $xf^{-1}(x)-g\{f^{-1}(x)\}+C$
5. $f(x)g'(x)-f'(x)g(x)+C$
6. $f(x)\,g(x)+C$
7. $x\cdot(\log_e x)^2-2(x\log_e x-x)+c$
8. $x-(\sin^{-1}x)\sqrt{1-x^2}+C$
9. $\theta\cos 2\theta-\frac{\sin 2\theta}{2}+C$
10. $\sin x\log\left(\tan\frac{x}{2}\right)-x+C$
11. $\frac{1}{10}x(5-2\sin(2\log x)-\cos(2\log x))+C$
12. $e^x\tan x+C$
13. $e^x\sqrt{\frac{1+x}{1-x}}+C$
14. $\frac{x}{1+(\log x)^2}+C$
15. $\frac{1}{2}(x+1)\sqrt{x^2+2x+5}+2\log\left|(x+1)+\sqrt{x^2+2x+5}\right|+C$

Solved Examples

Example 7.1

Evaluate $\int\frac{1}{x^{1/2}+x^{1/3}}dx.$

Sol. $\frac{1}{x^{1/2}+x^{1/3}}=\frac{1}{x^{1/3}\left(1+x^{1/6}\right)}$

Let $x=t^6 \Rightarrow dx=6t^5dt$

$$\therefore \int\frac{1}{x^{1/2}+x^{1/3}}dx=\int\frac{1}{x^{1/3}\left(1+x^{1/6}\right)}dx$$

$$=\int\frac{6t^5}{t^2(1+t)}dt$$

$$=6\int\frac{t^3}{(1+t)}dt$$

On dividing, we obtain

$$\int\frac{1}{x^{1/2}+x^{1/3}}dx=6\int\left\{(t^2-t+1)-\frac{1}{1+t}\right\}dt$$

$$=6\left[\left(\frac{t^3}{3}\right)-\left(\frac{t^2}{2}\right)+t-\log|1+t|\right]+C$$

$$=2x^{1/2}-3x^{1/3}+6x^{1/6}-6\log\left(1+x^{1/6}\right)+C$$

Example 7.2

Evaluate $\int\frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}dx.$

Sol. $I=\int\frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}dx$

$$=\int\frac{1-\frac{1}{x^2}}{\left(x^2+\frac{1}{x^2}+3\right)\tan^{-1}\left(x+\frac{1}{x}\right)}dx$$

Put $x+\frac{1}{x}=t$

$$\therefore \left(1-\frac{1}{x^2}\right)dx=dt \text{ and } x^2+\frac{1}{x^2}+3=t^2+1$$

$$\therefore I=\int\frac{dt}{(t^2+1)\tan^{-1}t}=\ln|\tan^{-1}t|+C$$

$$=\ln\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right|+C$$

Example 7.3

Evaluate $\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$.

Sol. $\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

$$=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}\cdot\frac{1-\sqrt{x}}{1-\sqrt{x}}}dx$$

$$= \int \frac{1-\sqrt{x}}{\sqrt{1-x}}dx$$

$$= \int \left(\frac{1}{\sqrt{1-x}} - \frac{x}{\sqrt{x-x^2}} \right) dx$$

$$= \int \left((1-x)^{-1/2} + \frac{1}{2}\frac{(1-2x)-1}{\sqrt{x-x^2}} \right) dx$$

$$= -\frac{(1-x)^{1/2}}{1/2} + \frac{1}{2}\int \frac{(1-2x)}{\sqrt{x-x^2}}dx - \frac{1}{2}\int \frac{1}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}}dx$$

$$= -2\sqrt{1-x} + \sqrt{x-x^2} - \frac{1}{2}\sin^{-1}(2x-1) + c$$

EXAMPLE 7.4

Evaluate $\int \frac{\sqrt{x^2+1}\,\{\log_e (x^2+1) - 2\log x\}dx}{x^4}$.

Sol. $I = \int \frac{\sqrt{x^2+1}\cdot \log\left(\frac{x^2+1}{x^2}\right)dx}{x^4}$

$$= \int \frac{\sqrt{1+\frac{1}{x^2}}\cdot \log\left(1+\frac{1}{x^2}\right)dx}{x^3}$$

Let $1 + \frac{1}{x^2} = t$

$\therefore$ $-\frac{2}{x^3}\,dx = dt$

$\therefore$ $I = -\frac{1}{2}\int \sqrt{t}\log t\,dt$

$$= -\frac{1}{2}\left[\log t.\frac{t^{3/2}}{3/2} - \int \frac{1}{t}.\frac{t^{3/2}}{3/2}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{3/2}\log_e t - \frac{2}{3}\frac{t^{3/2}}{3/2}\right] + c$$

$$= -\frac{1}{3}t^{3/2}\log_e t + \frac{2}{9}t^{3/2} + c, \text{ where } t = 1+\frac{1}{x^2}$$

EXAMPLE 7.5

Evaluate $\int \frac{(x-1)^2}{x^4+x^2+1}dx$.

Sol. $I = \int \frac{x^2-2x+1}{x^4+x^2+1}dx$

$$= \int \frac{x^2+1}{x^4+x^2+1}dx - \int \frac{2x\,dx}{x^4+x^2+1}$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+1+\frac{1}{x^2}} - \int \frac{dt}{t^2+t+1}, \text{ where } t = x^2$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3} - \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{3}} - \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x^2-1}{\sqrt{3}x} - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x^2+1}{\sqrt{3}} + c$$

EXAMPLE 7.6

Evaluate $\int \sqrt{\left(\frac{1+x^2}{x^2-x^4}\right)}dx$.

Sol. $\int \sqrt{\left(\frac{1+x^2}{x^2-x^4}\right)}dx = \int \frac{1+x^2}{x\sqrt{(1-x^4)}}dx$

$$= \int \frac{dx}{x\sqrt{(1-x^4)}} + \int \frac{x\,dx}{\sqrt{(1-x^4)}}$$

$$= \int \frac{x^3dx}{x^4\sqrt{(1-x^4)}} + \int \frac{x\,dx}{\sqrt{(1-x^4)}}$$

$$= -\frac{1}{2}\int \frac{u\,du}{(1-u^2)u} + \frac{1}{2}\int \frac{dv}{\sqrt{(1-v^2)}}$$

(Putting $1-x^4 = u^2$, $-4x^3\,dx = 2u\,du$ in the first integral and $x^2 = v$, $2x\,dx = dv$ in the second integral)

$$= \frac{1}{2}\int \frac{du}{u^2-1} + \frac{1}{2}\sin^{-1}v$$

$$= \frac{1}{2}\cdot\frac{1}{2\times 1}\log\left|\frac{u-1}{u+1}\right| + \frac{1}{2}\sin^{-1}v + c$$

$$= \frac{1}{4}\log\left|\frac{\sqrt{(1-x^4)}-1}{\sqrt{(1-x^4)}+1}\right| + \frac{1}{2}\sin^{-1}(x^2) + c$$

EXAMPLE 7.7

If $I_n = \int \cos^n x\,dx$, prove that

$$I_n = \frac{1}{n}\left(\cos^{n-1}x\sin x\right) + \left(\frac{n-1}{n}\right)I_{n-2}.$$

Sol. $I_n = \int \cos^n x\,dx$

$$= \cos^{n-1} x \int \cos x dx + (n-1) \int (\sin^2 x) \cos^{n-2} x dx$$

$$= (\cos^{n-1} x \sin x) + (n-1) \int \cos^{n-2} x (1-\cos^2 x) dx$$

$$= (\cos^{n-1} x \sin x) + (n-1) \int \left[\cos^{n-2} x - \cos^n x\right] dx$$

or $I_n + (n-1) I_n = (\cos^{n-1} x \sin x) + (n-1)(I_{n-2})$

or $I_n = \frac{1}{n} (\cos^{n-1} x \sin x) + \left(\frac{n-1}{n}\right) I_{n-2}$

EXAMPLE 7.8

Evaluate $\int \frac{(1-x\sin x)\,dx}{x\left(1-x^3 e^{3\cos x}\right)}$.

Sol. Here, $I = \int \frac{(1-x\sin x)\,dx}{x\left(1-\left(xe^{\cos x}\right)^3\right)}$

Put $xe^{\cos x} = t$ so that $(xe^{\cos x}(-\sin x) + e^{\cos x})\,dx = dt$

$\therefore \quad I = \int \frac{dt}{t(1-t^3)} = \int \frac{dt}{t^4\left(\frac{1}{t^3}-1\right)}$

Let $\frac{1}{t^3} - 1 = y$ or $dy = \frac{-3}{t^4} dt$

$\therefore \quad I = -\frac{1}{3}\int \frac{dy}{y} = -\frac{1}{3}\log|y| + C = -\frac{1}{3}\log\left|\frac{1}{t^3}-1\right| + C$

$$= -\frac{1}{3}\log_e\left|\frac{1}{x^3 e^{3\cos x}} - 1\right| + C$$

EXAMPLE 7.9

Evaluate

$$\int \frac{e^{\tan^{-1}x}}{(1+x^2)}\left[\left(\sec^{-1}\sqrt{1+x^2}\right)^2 + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right] dx \ (x > 0).$$

Sol. For $x > 0$, $\sec^{-1}\sqrt{1+x^2} = \tan^{-1}x$

and $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$

$\therefore \quad I = \int \frac{e^{\tan^{-1}x}}{1+x^2}\left[(\tan^{-1}x)^2 + 2\tan^{-1}x\right] dx \quad$ [Put $\tan^{-1}x = t$]

$$= \int e^t (t^2 + 2t)\,dt$$

$$= e^t t^2 + C = e^{\tan^{-1}x}(\tan^{-1}x)^2 + C$$

EXAMPLE 7.10

Evaluate $\int \frac{2x}{(1-x^2)\sqrt{x^4-1}} dx.$

Sol. $\int \frac{2x}{(1-x^2)\sqrt{x^4-1}} dx = \int \frac{-2x}{(x^2-1)^{3/2}\sqrt{x^2+1}} dx$

Put $\sqrt{\frac{x^2+1}{x^2-1}} = z$

$\therefore \quad \frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{1/2} \frac{(x^2-1)2x - (x^2+1)2x}{(x^2-1)^2} dx = dz$

or $\frac{\sqrt{x^2-1}}{\sqrt{x^2+1}} \frac{-2x}{(x^2-1)^2} dx = dz$

or $\frac{-2x}{(x^2-1)^{3/2}\sqrt{x^2+1}} dx = dz$

$\therefore \quad$ Given integral $= \int dz = z + c = \sqrt{\frac{x^2+1}{x^2-1}} + c$

EXAMPLE 7.11

Evaluate $\int \frac{dx}{x^3\sqrt{x^2-1}}$.

Sol. Write $I = \int \frac{x\ dx}{x^4\sqrt{x^2-1}}$ and put $x^2 - 1 = t^2$, so that

$2x\,dx = 2t\,dt$

$\therefore \quad I = \int \frac{t}{(t^2+1)^2 t} dt = \int \frac{dt}{(t^2+1)^2}$

But $\tan^{-1} t = \int \frac{dt}{t^2+1} = \int 1 \cdot \frac{1}{t^2+1} dt$

$$= \frac{t}{t^2+1} + \int t \frac{2t}{(t^2+1)^2} dt$$

$$= \frac{t}{t^2+1} + 2\int \frac{t^2+1-1}{(t^2+1)^2} dt$$

$$= \frac{t}{t^2+1} + 2\tan^{-1} t - 2I$$

$\therefore \quad I = \frac{1}{2}\frac{t}{t^2+1} + \frac{1}{2}\tan^{-1} t + C$

$$= \frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x^2} + \tan^{-1}\sqrt{x^2-1}\right) + C$$

EXAMPLE 7.12

Evaluate $\int \sqrt{\frac{3-x}{3+x}} \cdot \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right) dx$.

Sol. Here, $I = \int \sqrt{\frac{3-x}{3+x}} \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right) dx$

Put $x = 3\cos 2\theta$ and $dx = -6\sin 2\theta\, d\theta$

$$I = \int \sqrt{\frac{3-3\cos 2\theta}{3+3\cos 2\theta}} \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-3\cos 2\theta}\right) \times (-6\sin 2\theta)\,d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta} \sin^{-1}(\sin\theta)\ (-6\sin 2\theta)\,d\theta$$

$$= -6\int\theta\,(2\sin^2\theta)\,d\theta = -6\int\theta\,(1-\cos 2\theta)\,d\theta$$

$$= -6\left\{\frac{\theta^2}{2} - \int\theta\cos 2\theta\,d\theta\right\}$$

$$= -6\left\{\frac{\theta^2}{2} - \left(\theta\frac{\sin 2\theta}{2} - \int 1\left(\frac{\sin 2\theta}{2}\right)d\theta\right)\right\}$$

$$= -3\theta^2 + 6\left\{\frac{\theta\sin 2\theta}{2} + \frac{\cos 2\theta}{4}\right\} + c$$

$$= \frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}\cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + c$$

Example 7.13

Evaluate $\int\cos 2\theta\log_e\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$.

Sol. $I = \int\cos 2\theta\log_e\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$

$$= \log_e\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\int\cos 2\theta\,d\theta - \int\left(\left[\log_e\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right]'\int\cos 2\theta\,d\theta\right)d\theta$$

$$= \frac{\sin 2\theta}{2}\log_e\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \int\frac{\sin 2\theta(\cos\theta-\sin\theta)}{2(\sin\theta+\cos\theta)}\cdot\frac{2}{(\cos\theta-\sin\theta)^2}d\theta$$

$$= \frac{\sin 2\theta}{2}\log_e\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \int\tan 2\theta\,d\theta$$

$$\therefore\quad I = \sin 2\theta\log_e\sqrt{\frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}} - \frac{1}{2}\log_e|\sec 2\theta| + c$$

Example 7.14

Evaluate $\int x^{-1/2}(2+3x^{1/3})^{-2}\,dx$.

Sol. $I = \int x^{-1/2}(2+3x^{1/3})^{-2}\,dx$

Let $x = t^6$ or $dx = 6t^5dt$

$\therefore\quad I = \int t^{-3}(2+3t^2)^{-2}\cdot 6t^5dt$

$$= 6\int\frac{t^2}{(2+3t^2)^2}dt$$

$$= \frac{6}{9}\int\frac{t^2dt}{\left(\frac{2}{3}+t^2\right)^2}$$

Now, let $t = \sqrt{\left(\frac{2}{3}\right)}\tan\theta$

$\therefore\quad dt = \sqrt{\left(\frac{2}{3}\right)}\sec^2\theta\,d\theta$

$$\therefore\quad I = \frac{6}{9}\int\frac{\frac{2}{3}\tan^2\theta\cdot\sqrt{\left(\frac{2}{3}\right)}\sec^2\theta\,d\theta}{\frac{4}{9}\sec^4\theta}$$

$$= \sqrt{\frac{2}{3}}\int\sin^2\theta\,d\theta$$

$$= \frac{1}{\sqrt{6}}\int(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{\sqrt{6}}\left\{\theta - \frac{\sin 2\theta}{2}\right\} + c$$

$$= \frac{1}{\sqrt{6}}\left\{\theta - \frac{\tan\theta}{1+\tan^2\theta}\right\} + c$$

$$= \frac{1}{\sqrt{6}}\left\{\tan^{-1}\left\{\sqrt{\frac{3}{2}}t\right\} - \frac{\sqrt{\frac{3}{2}}\cdot t}{1+\frac{3}{2}t^2}\right\} + c$$

$$= \frac{1}{\sqrt{6}}\left\{\tan^{-1}\left\{\sqrt{\frac{3}{2}}x^{1/6}\right\} - \frac{\sqrt{6}x^{1/6}}{2+3x^{1/3}}\right\} + c$$

Example 7.15

Evaluate $\int\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$.

Sol. $I = \int\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

$$= \int\sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right)dx$$

[Put $2x+2 = 3\tan\theta$ or $2\,dx = 3\sec^2\theta$

$$= \int\sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{3}{2}\int\theta\sec^2\theta\,d\theta$$

$$= \frac{3}{2}\left\{\theta\tan\theta - \int\tan\theta\,d\theta\right\} + C$$

$$= \frac{3}{2}\{\theta\tan\theta - \log|\sec\theta|\} + C$$

$$= \frac{3}{2}\left\{\frac{2x+2}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) - \log\left(\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right)\right.$$

$$= \frac{3}{2}\left\{\frac{2}{3}(x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \log\sqrt{4x^2+8x+13}\right.$$

EXAMPLE 7.16

Evaluate for, $m \in N$,

$\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}\,dx, x>0.$

Sol. $I=\int \left(x^{3m}+x^{2m}+x^m\right)\left(2x^{2m}+3x^m+6\right)^{1/m} dx$

$$=\int \left(\frac{x^{3m}+x^{2m}+x^m}{x}\right)(x)\left(2x^{2m}+3x^m+6\right)^{1/m} dx$$

$$=\int \left(x^{3m-1}+x^{2m-1}+x^{m-1}\right)\left(2x^{3m}+3x^{2m}+6x^m\right)^{1/m} dx$$

Let $2x^{3m}+3x^{2m}+6x^m=t$. Then $dt=6m(x^{3m-1}+x^{2m-1}+x^{m-1})dx$.

$$\therefore \quad I=\int t^{1/m}\frac{dt}{6m}=\frac{1}{6m}\frac{t^{\frac{1}{m}+1}}{\frac{1}{m}+1}+C$$

$$=\frac{1}{6(m+1)}\left(2x^{3m}+3x^{2m}+6x^m\right)^{(m+1)/m}+C$$

EXAMPLE 7.17

Evaluate $\int \frac{dx}{\cos x\sqrt{\cos 2x}}$.

Sol. $I=\int \frac{dx}{\cos x\sqrt{\cos 2x}}$

$$=\int \frac{\cos x dx}{\cos^2 x\sqrt{1-2\sin^2 x}}$$

$$=\int \frac{\cos x dx}{(1-\sin^2 x)\sqrt{1-2\sin^2 x}}$$

$$=\int \frac{dt}{(1-t^2)\sqrt{1-2t^2}} \qquad \text{(Putting } \sin x = t\text{)}$$

Put $t=\frac{1}{y}$

$$\therefore \quad dt=-\frac{1}{y^2}dy$$

$$\therefore \quad I=-\int \frac{dy}{y^2\left(1-\frac{1}{y^2}\right)\sqrt{1-\frac{2}{y^2}}}$$

$$=-\int \frac{ydy}{(y^2-1)\sqrt{y^2-2}}$$

Now put $y^2-2=z^2$

$$\therefore \quad ydy=zdz$$

$$\therefore \quad I=-\int \frac{dz}{z^2+1}$$

$$=-\tan^{-1} z+c$$

$$=-\tan^{-1}\sqrt{y^2-2}+c$$

$$=-\tan^{-1}\sqrt{\frac{1}{t^2}-2}+c$$

$$=-\tan^{-1}\sqrt{\frac{1}{\sin^2 x}-2}+c$$

EXAMPLE 7.18

Evaluate $\int \sin(101x).\sin^{99} x\,dx$.

Sol. $I=\int \sin(100x+x)\sin^{99} x\,dx$

$$=\int \sin 100x\cos x\sin^{99} x\,dx+\int \sin x\cos 100x\sin^{99} x\,dx$$

$$=\int \sin 100x\cos x\sin^{99} x\,dx+\int \cos 100x.\sin^{100} x\,dx$$

$$=\int \sin 100x.\cos x\sin^{99} x\,dx+\sin^{100} x\cdot\frac{\sin 100x}{100}-\int \frac{100\sin^{99} x\cos x.\sin 100x}{100}dx$$

$$=\frac{\sin^{100} x\sin 100x}{100}+c$$

Exercises

Single Correct Answer Type

1. $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$ is equal to

(1) $\frac{1}{2}\sin 2x + C$ (2) $-\frac{1}{2}\sin 2x + C$

(3) $-\frac{1}{2}\sin x + C$ (4) $-\sin^2 x + C$

2. If $\int \frac{\cos 4x + 1}{\cot x - \tan x} dx = A\cos 4x + B$, then

(1) $A = -1/2$ (2) $A = -1/8$

(3) $A = -1/4$ (4) none of these

3. $\int \left(\sqrt{\frac{a+x}{a-x}} + \sqrt{\frac{a-x}{a+x}}\right) dx$ is equal to

(1) $2\sin^{-1}(x/a) + c$ (2) $2a\sin^{-1}(x/a) + c$

(3) $2\cos^{-1}(x/a) + c$ (4) $2a\cos^{-1}(x/a) + c$

4. $\int \sqrt{1 + \sin x}\, dx$ is equal to

(1) $-2\sqrt{1 - \sin x} + C$

(2) $\sin(x/2) + \cos(x/2) + C$

(3) $\cos(x/2) - \sin(x/2) + C$

(4) $2\sqrt{1 - \sin x} + C$

5. $\int 4\sin x \cos\frac{x}{2} \cos\frac{3x}{2} dx$ is equal to

(1) $\cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x + C$

(2) $\cos x - \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x + C$

(3) $\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + C$

(4) $\cos x - \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + C$

6. If $\int \sqrt{1 + \sin x}\, f(x)\, dx = \frac{2}{3}(1 + \sin x)^{3/2} + c$, then $f(x)$ equals

(1) $\cos x$ (2) $\sin x$

(3) $\tan x$ (4) 1

7. $\int \frac{\sqrt{x-1}}{x\sqrt{x+1}} dx$ is equal to

(1) $\ln|x - \sqrt{x^2-1}| - \tan^{-1} x + c$

(2) $\ln|x + \sqrt{x^2-1}| - \tan^{-1} x + c$

(3) $\ln|x - \sqrt{x^2-1}| - \sec^{-1} x + c$

(4) $\ln|x + \sqrt{x^2-1}| - \sec^{-1} x + c$

8. If $I = \int \frac{dx}{\sec x + \mathrm{cosec}\, x}$, then I equals

(1) $\frac{1}{2}\left(\cos x + \sin x - \frac{1}{\sqrt{2}}\log(\mathrm{cosec}\, x - \cos x)\right) + C$

(2) $\frac{1}{2}\left(\sin x - \cos x - \frac{1}{\sqrt{2}}\log|\mathrm{cosec}\, x + \cot x|\right) + C$

(3) $\frac{1}{\sqrt{2}}\left(\sin x + \cos x + \frac{1}{2}\log|\mathrm{cosec}\, x - \cos x|\right) + C$

(4) $\frac{1}{2}[\sin x - \cos x] - \frac{1}{\sqrt{2}}\log|\mathrm{cosec}(x + \pi/4) - \cot(x + \pi/4)| + C$

9. $\int \frac{\sin x}{\sin\left(x - \frac{\pi}{4}\right)} dx$ is equal to

(1) $\frac{1}{\sqrt{2}}\left(x + \log_e\left|\cos\left(x - \frac{\pi}{4}\right)\right|\right) + c$

(2) $\frac{1}{\sqrt{2}}\left(x - \log_e\left|\sin\left(x - \frac{\pi}{4}\right)\right|\right) + c$

(3) $\sqrt{2}\left(x + \log_e\left|\sin\left(x - \frac{\pi}{4}\right)\right|\right) + c$

(4) $\frac{1}{\sqrt{2}}\left(x + \log_e\left|\sin\left(x - \frac{\pi}{4}\right)\right|\right) + c$

10. $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx$ is equal to

(1) $\frac{\sin 2x}{2} + \cos x + c$ (2) $\frac{\sin 2x}{2} - \cos x + c$

(3) $-\frac{\sin 2x}{2} - \sin x + c$ (4) $\frac{\sin 2x}{2} - \cos x + c$

11. If $I = \int \sqrt{\frac{5-x}{2+x}} dx$, then I equals

(1) $\sqrt{x+2}\sqrt{5-x} + 3\sin^{-1}\sqrt{\frac{x+2}{3}} + C$

(2) $\sqrt{x+2}\sqrt{5-x} + 7\sin^{-1}\sqrt{\frac{x+2}{7}} + C$

(3) $\sqrt{x+2}\sqrt{5-x} + 5\sin^{-1}\sqrt{\frac{x+2}{5}} + C$

(4) none of these

12. $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$ is equal to

(1) $\log\sin 3x - \log\sin 5x + c$

(2) $\frac{1}{3}\log\sin 3x + \frac{1}{5}\log\sin 5x + c$

(3) $\frac{1}{3}\log\sin 3x - \frac{1}{5}\log\sin 5x + c$

(4) $3\log\sin 3x - 5\log\sin 5x + c$

13. $\int \frac{dx}{x(x^n+1)}$ is equal to

(1) $\frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right)+c$ (2) $\frac{1}{n}\log\left(\frac{x^n+1}{x^n}\right)+c$

(3) $\log\left(\frac{x^n}{x^n+1}\right)+c$ (4) none of these

14. $\int \frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}dx,\ \alpha \neq n\pi,\ n \in Z$ is equal to

(1) $-2\operatorname{cosec}\alpha(\cos\alpha - \tan x\sin\alpha)^{1/2} + C$

(2) $-2(\cos\alpha + \cot x\sin\alpha)^{1/2} + C$

(3) $-2\operatorname{cosec}\alpha(\cos\alpha + \cot x\sin\alpha)^{1/2} + C$

(4) $-2\operatorname{cosec}\alpha(\sin\alpha + \cot x\cos\alpha)^{1/2} + C$

15. $\int \frac{px^{p+2q-1} - qx^{q-1}}{x^{2p+2q} + 2x^{p+q} + 1}dx$ is equal to

(1) $-\frac{x^p}{x^{p+q}+1}+C$ (2) $\frac{x^q}{x^{p+q}+1}+C$

(3) $-\frac{x^q}{x^{p+q}+1}+C$ (4) $\frac{x^p}{x^{p+q}+1}+C$

16. If $y = \int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$ and $y = 0$ when $x = 0$, the value of y when $x = 1$ is

(1) $\frac{1}{\sqrt{2}}$ (2) $\sqrt{2}$

(3) $2\sqrt{2}$ (4) none of these

17. $\int \sqrt{x}\,(1 + x^{1/3})^4\,dx$ is equal to

(1) $2\left\{x^{2/3} + \frac{4}{11}x^{11/6} + \frac{6}{13}x^{13/6} + \frac{4}{15}x^{5/2} + \frac{1}{17}x^{17/6}\right\} + c$

(2) $6\left\{x^{2/3} - \frac{4}{11}x^{11/6} + \frac{6}{13}x^{13/6} - \frac{4}{15}x^{5/2} + \frac{1}{17}x^{17/6}\right\} + c$

(3) $6\left\{\frac{1}{9}x^{3/2} + \frac{4}{11}x^{11/6} + \frac{6}{13}x^{13/6} + \frac{4}{15}x^{5/2} + \frac{1}{17}x^{17/6}\right\} + c$

(4) none of these

18. $\int \frac{\ln(\tan x)}{\sin x\cos x}dx$ is equal to

(1) $\frac{1}{2}\ln(\tan x)+c$ (2) $\frac{1}{2}\ln(\tan^2 x)+c$

(3) $\frac{1}{2}(\ln(\tan x))^2+c$ (4) none of these

19. $\int \frac{x^{5m-1}+2x^{4m-1}}{(x^{2m}+x^m+1)^3}dx$ is equal to where $m \in R - \{0\}$

(1) $\frac{x^{6m}+x^{2m}}{2m(x^{2m}+x^m+1)^2}+c$

(2) $\frac{x^{4m}}{2m(x^{2m}+x^m+1)^2}+c$

(3) $\frac{2mx^{4m}}{(x^{2m}+x^m+1)^2}+c$

(4) $\frac{mx^{5m}}{2(x^{2m}+x^m+1)^2}+c$

20. If $l^r(x)$ means $\log\log\log \ldots x$, the log being repeated r times, then $\int [x\,l(x)l^2(x)l^3(x)\cdots l^r(x)]^{-1}\,dx$ is equal to

(1) $l^{r+1}(x) + C$ (2) $\frac{l^{r+1}(x)}{r+1}+C$

(3) $l^r(x) + C$ (4) none of these

21. If $I = \int\left(\sqrt{\cot x} - \sqrt{\tan x}\right)dx$, then I equals

(1) $\sqrt{2}\log\left(\sqrt{\tan x} - \sqrt{\cot x}\right) + C$

(2) $\sqrt{2}\log|\sin x + \cos x + \sqrt{\sin 2x}| + C$

(3) $\sqrt{2}\log|\sin x - \cos x + \sqrt{2}\sin x\cos x| + C$

(4) $\sqrt{2}\log|\sin(x + \pi/4) + \sqrt{2}\sin x\cos x| + C$

22. If $I = \int \frac{\sin 2x}{(3+4\cos x)^3}dx$, then I equals

(1) $\frac{3\cos x+8}{(3+4\cos x)^2}+C$ (2) $\frac{3+8\cos x}{16(3+4\cos x)^2}+C$

(3) $\frac{3+\cos x}{(3+4\cos x)^2}+C$ (4) $\frac{3-8\cos x}{16(3+4\cos x)^2}+C$

23. $\int \frac{\ln\left(\frac{x-1}{x+1}\right)}{x^2-1}dx$ is equal to

(1) $\frac{1}{2}\left(\ln\left(\frac{x-1}{x+1}\right)\right)^2+C$ (2) $\frac{1}{2}\left(\ln\left(\frac{x+1}{x-1}\right)\right)^2+C$

(3) $\frac{1}{4}\left(\ln\left(\frac{x-1}{x+1}\right)\right)^2+C$ (4) $\frac{1}{4}\left(\ln\left(\frac{x+1}{x-1}\right)\right)$

24. $\int \sqrt{e^x - 1}\,dx$ is equal to

(1) $2\left[\sqrt{e^x-1} - \tan^{-1}\sqrt{e^x-1}\right] + c$

(2) $\sqrt{e^x-1} - \tan^{-1}\sqrt{e^x-1} + c$

(3) $\sqrt{e^x-1} + \tan^{-1}\sqrt{e^x-1} + c$

(4) $2\left[\sqrt{e^x-1} + \tan^{-1}\sqrt{e^x-1}\right] + c$

25. $\int \sqrt{\frac{x^2+1}{x^2(1-x^2)}}\,dx =$

(1) $\frac{1}{4}\log_e\left|\frac{1-\sqrt{1-x^4}}{1+\sqrt{1-x^4}}\right|+\frac{1}{2}\sin^{-1}(x^2)+C$

(2) $\frac{1}{2}\log_e\left|\frac{1-\sqrt{1-x^4}}{1+\sqrt{1-x^4}}\right|+\frac{1}{2}\cos^{-1}(x^2)+C$

(3) $\frac{1}{2}\log_e\left|\frac{1-\sqrt{1-x^4}}{1+\sqrt{1-x^4}}\right|+\sin^{-1}(x^2)+c$

(4) $\log_e\left|\frac{1-\sqrt{1-x^4}}{1+\sqrt{1-x^4}}\right|+\frac{1}{2}\cos^{-1}(x^2)+C$

26. $\int \frac{\sqrt{x^2+10x+24}}{x+5}\,dx$ is equal to

(1) $\sqrt{x^2+10x+24}+\sec^{-1}(x+5)+c$

(2) $\sqrt{x^2+10x+24}-\operatorname{cosec}^{-1}(x+5)+c$

(3) $\sec^{-1}(x+5)-\sqrt{x^2+10x+24}+c$

(4) $\sqrt{x^2+10x+24}-\sec^{-1}(x+5)+c$

27. The value of $\int \frac{1+\log x}{\sqrt{(x^x)^2-1}}\,dx$ is

(1) $\sec^{-1}(x^x)+c$

(2) $\tan^{-1}(x^x)+c$

(3) $\log(x^x+\sqrt{x^{2x}-1})+c$

(4) $\cot^{-1}(x^x)+c$

28. If $\int x^5(1+x^3)^{2/3}\,dx = A(1+x^3)^{8/3}+B(1+x^3)^{5/3}+c$, then

(1) $A=\frac{1}{4}, B=\frac{1}{5}$ (2) $A=\frac{1}{8}, B=-\frac{1}{5}$

(3) $A=-\frac{1}{8}, B=\frac{1}{5}$ (4) none of these

29. $\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$ is equal to

(1) $\cot^{-1}(\tan^2 x)+c$ (2) $\tan^{-1}(\tan^2 x)+c$

(3) $\cot^{-1}(\cot^2 x)+c$ (4) $\tan^{-1}(\cot^2 x)+c$

30. $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$ is equal to

(1) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)$

(2) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)$

(3) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{x+1}}\right)$

(4) none of these

31. $\int \frac{\sec x\,dx}{\sqrt{\sin(2x+A)+\sin A}}$ is equal to

(1) $\frac{\sec A}{\sqrt{2}}\sqrt{\tan x\cos A-\sin A}+c$

(2) $\sqrt{2}\sec A\sqrt{\tan x\cos A-\sin A}+c$

(3) $\sqrt{2}\sec A\sqrt{\tan x\cos A+\sin A}+c$

(4) none of these

32. $\int \frac{\cos 2x}{(e^{-x}+\cos x)\sqrt{1+\sin 2x}}\,dx, x\in\left(0,\frac{\pi}{2}\right)$ is equal to

(1) $\log_e|1+e^x\sin x|+c$

(2) $\log_e|e^x+\cos x|+c$

(3) $\log_e|1+e^x\cos x|-x+c$

(4) $\log_e|1+e^x\cos x|+c$

33. $\int \frac{\cos 4x-1}{\cot x-\tan x}\,dx$ is equal to

(1) $\frac{1}{2}\ln|\sec 2x|-\frac{1}{4}\cos^2 2x+c$

(2) $\frac{1}{2}\ln|\sec 2x|+\frac{1}{4}\cos^2 x+c$

(3) $\frac{1}{2}\ln|\cos 2x|-\frac{1}{4}\cos^2 2x+c$

(4) $\frac{1}{2}\ln|\cos 2x|+\frac{1}{4}\cos^2 x+c$

34. If $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}} = -[f(x)]^{1/n}+c$, then $f(x)$ is

(1) $(1+x^n)$ (2) $1+x^{-n}$

(3) x^n+x^{-n} (4) none of these

35. $\int \sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}\,dx$ is equal to

(1) $\frac{2}{3}\sin^{-1}(\cos^{3/2}x)+C$ (2) $\frac{3}{2}\sin^{-1}(\cos^{3/2}x)+C$

(3) $\frac{2}{3}\cos^{-1}(\cos^{3/2}x)+C$ (4) none of these

36. $\int x\left(\frac{\ln a^{a^{x/2}}}{3a^{5x/2}b^{3x}}+\frac{\ln b^{b^x}}{2a^{2x}b^{4x}}\right)dx$ (where $a, b\in R^+$) is equal to

(1) $\frac{1}{6\ln a^2b^3}a^{2x}b^{3x}\ln\frac{a^{2x}b^{3x}}{e}+k$

(2) $\frac{1}{6\ln a^2b^3}\frac{1}{a^{2x}b^{3x}}\ln\frac{1}{ea^{2x}b^{3x}}+k$

(3) $\frac{1}{6\ln a^2b^3}\frac{1}{a^{2x}b^{3x}}\ln(a^{2x}b^{3x})+k$

(4) $-\frac{1}{6\ln a^2b^3}\frac{1}{a^{2x}b^{3x}}\ln(a^{2x}b^{3x})+k$

37. $\int \frac{3+2\cos x}{(2+3\cos x)^2} dx$ is equal to

(1) $\left(\frac{\sin x}{3\cos x+2}\right)+c$ (2) $\left(\frac{2\cos x}{3\sin x+2}\right)+c$

(3) $\left(\frac{2\cos x}{3\cos x+2}\right)+c$ (4) $\left(\frac{2\sin x}{3\sin x+2}\right)+c$

38. If $\frac{d}{dx}f(x)=f'(x)$, then $\int \frac{xf'(x)-2f(x)}{\sqrt{x^4 f(x)}} dx$ is equal to

(1) $\frac{x^2}{f(x)}+c$ (2) $|x|f'(x)+c$

(3) $\frac{2\sqrt{f(x)}}{|x|}+c$ (4) $|x|\sqrt{f(x)}+c$

39. The value of the integral $\int \frac{(1-\cos\theta)^{2/7}}{(1+\cos\theta)^{9/7}} d\theta$ is

(1) $\frac{7}{11}\left(\tan\frac{\theta}{2}\right)^{\frac{11}{7}}+C$ (2) $\frac{7}{11}\left(\cos\frac{\theta}{2}\right)^{\frac{11}{7}}+C$

(3) $\frac{7}{11}\left(\sin\frac{\theta}{2}\right)^{\frac{11}{7}}+C$ (4) none of these

40. If $\int \frac{dx}{\sqrt{\sin^3 x\cos^5 x}} = a\sqrt{\cot x}+b\sqrt{\tan^3 x}+c$, then

(1) $a=-1, b=1/3$ (2) $a=-3, b=2/3$

(3) $a=-2, b=4/3$ (4) $a=-2, b=2/3$

41. If $\int \frac{dx}{\cos^3 x\sqrt{\sin 2x}} = a(\tan^2 x+b)\sqrt{\tan x}+c$, then

(1) $a=\frac{\sqrt{2}}{5}, b=\frac{1}{\sqrt{5}}$ (2) $a=\frac{\sqrt{2}}{5}, b=5$

(3) $a=\frac{\sqrt{2}}{5}, b=-\frac{1}{\sqrt{5}}$ (4) $a=\frac{\sqrt{2}}{5}, b=\sqrt{5}$

42. If $\int \frac{dx}{(x+2)(x^2+1)} = a\ln(1+x^2)+b\tan^{-1}x$ $+\frac{1}{5}\ln|x+2|+C$. Then

(1) $a=-\frac{1}{10}, b=-\frac{2}{5}$ (2) $a=\frac{1}{10}, b=-\frac{2}{5}$

(3) $a=-\frac{1}{10}, b=\frac{2}{5}$ (4) $a=\frac{1}{10}, b=\frac{2}{5}$

43. If $\int \frac{3e^x-5e^{-x}}{4e^x+5e^{-x}} dx = ax+b\ln(4e^x+5e^{-x})+C$, then

(1) $a=-\frac{1}{8}, b=\frac{7}{8}$ (2) $a=\frac{1}{8}, b=\frac{7}{8}$

(3) $a=-\frac{1}{8}, b=-\frac{7}{8}$ (4) $a=\frac{1}{8}, b=-\frac{7}{8}$

44. If $\int f(x)\sin x\cos x\, dx = \frac{1}{2(b^2-a^2)}\ln f(x)+c$, then $f(x)$ is equal to

(1) $\frac{1}{a^2\sin^2 x+b^2\cos^2 x}$ (2) $\frac{1}{a^2\sin^2 x-b^2\cos^2 x}$

(3) $\frac{1}{a^2\cos^2 x+b^2\sin^2 x}$ (4) $\frac{1}{a^2\cos^2 x-b^2\sin^2 x}$

45. $\int \frac{x^9\, dx}{(4x^2+1)^6}$ is equal to

(1) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+c$ (2) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+c$

(3) $\frac{1}{10}(1+4x^2)^{-5}+c$ (4) $\frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5}+c$

46. If $\int \frac{1}{x\sqrt{1-x^3}} dx = a\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+b$, then a is equal to

(1) 1/3 (2) 2/3 (3) −1/3 (4) −2/3

47. The value of the integral $\int (x^2+x)(x^{-8}+2x^{-9})^{1/10} dx$ is

(1) $\frac{5}{11}(x^2+2x)^{11/10}+c$ (2) $\frac{5}{6}(x+1)^{11/10}+c$

(3) $\frac{6}{7}(x+1)^{11/10}+c$ (4) none of these

48. $\int \frac{x^3 dx}{\sqrt{1+x^2}}$ is equal to

(1) $\frac{1}{3}\sqrt{1+x^2}(2+x^2)+C$ (2) $\frac{1}{3}\sqrt{1+x^2}(x^2-1)+C$

(3) $\frac{1}{3}(1+x^2)^{3/2}+C$ (4) $\frac{1}{3}\sqrt{1+x^2}(x^2-2)+C$

49. If $I=\int \frac{dx}{(a^2-b^2x^2)^{3/2}}$, then I equals

(1) $\frac{x}{\sqrt{a^2-b^2x^2}}+C$ (2) $\frac{x}{a^2\sqrt{a^2-b^2x^2}}+C$

(3) $\frac{ax}{\sqrt{a^2-b^2x^2}}+C$ (4) none of these

50. $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}} dx =$

(1) $\sqrt{x^2+\frac{1}{x^2}+1}+C$ (2) $\frac{\sqrt{x^4+x^2+1}}{x^2}+C$

(3) $\frac{\sqrt{x^4+x^2+1}}{x}+C$ (4) none of these

51. $\int \frac{\sqrt{x^2+1}}{x^4} dx =$

(1) $-\frac{1}{3}\frac{(x^2+1)^{3/2}}{x^3}+C$ (2) $x^3(x^2+1)^{-1/2}+C$

(3) $\frac{\sqrt{x^2+1}}{x^2}+C$ (4) $-\frac{1}{3}\frac{(x^2+1)^{3/2}}{x^2}+C$

52. $\int \frac{1+2x^6}{(1-x^6)^{3/2}} dx$ is equal to

(1) $\frac{2x}{\sqrt{1+x^6}}+c$ (2) $\frac{1}{\sqrt{\frac{1}{x^2}-x^4}}+c$

(3) $\frac{x}{\sqrt{\frac{1}{x^2}-x^4}}+c$ (4) $\frac{x^2+x}{\sqrt{1-x^6}}+c$

53. $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3} dx$ is equal to

(1) $\frac{x^{10}}{2(1+x^3+x^5)^4}+c$ (2) $\frac{x^2+2x}{(x^5+x^3+1)^4}+c$

(3) $\frac{x^{10}}{2\left(x^5+x^3+1\right)^2}+c$ (4) $\frac{2x^{10}}{\left(x^5+x^3+1\right)^3}+c$

54. If $I_n = \int (\ln x)^n\, dx$, then $I_n + nI_{n-1} =$

(1) $\frac{(\ln x)^n}{x}+C$ (2) $x(\ln x)^{n-1}+C$

(3) $x(\ln x)^n + C$ (4) none of these

55. Let $\int e^x\{f(x)-f'(x)\}dx = \phi(x)$. Then $\int e^x f(x)dx$ is

(1) $\phi(x) = e^x f(x)$ (2) $\phi(x) - e^x f(x)$

(3) $\frac{1}{2}\{\phi(x)+e^x f(x)\}$ (4) $\frac{1}{2}\{\phi(x)+e^x f'(x)\}$

56. $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$ is equal to

(1) $x\tan^{-1}x - \ln|\sec(\tan^{-1}x)| + c$
(2) $x\tan^{-1}x + \ln|\sec(\tan^{-1}x)| + c$
(3) $x\tan^{-1}x - \ln|\cos(\tan^{-1}x)| + c$
(4) none of these

57. If $\int x\frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx = a\sqrt{1+x^2}\ln(x+\sqrt{1+x^2})+bx+c$,

then

(1) $a=1, b=-1$ (2) $a=1, b=1$

(3) $a=-1, b=1$ (4) $a=-1, b=-1$

58. If $\int x\log(1+1/x)dx = f(x)\log(x+1)+g(x)\,x^2 + Ax + ($

then

(1) $f(x)=\frac{1}{2}x^2$ (2) $g(x)=\log x$

(3) $A=1$ (4) none of these

59. If $I=\int e^{-x}\log(e^x+1)\,dx$, then I equals

(1) $x+(e^{-x}+1)\log(e^x+1)+C$
(2) $x+(e^x+1)\log(e^x+1)+C$
(3) $x-(e^{-x}+1)\log(e^x+1)+C$
(4) none of these

60. If $\int xe^x\cos x\,dx = ae^x(b(1-x)\sin x + cx\cos x)+d$, then

(1) $a=1, b=1, c=-1$
(2) $a=\frac{1}{2}, b=-1, c=1$
(3) $a=1, b=-1, c=1$
(4) $a=\frac{1}{2}, b=1, c=-1$

61. $\int x\sin x\sec^3 x\,dx$ is equal to

(1) $\frac{1}{2}\left[\sec^2 x-\tan x\right]+c$

(2) $\frac{1}{2}\left[x\sec^2 x-\tan x\right]+c$

(3) $\frac{1}{2}[x\sec^2 x+\tan x]+c$

(4) $\frac{1}{2}[\sec^2 x+\tan x]+c$

62. $\int e^{\tan^{-1}x}(1+x+x^2)\,d(\cot^{-1}x)$ is equal to

(1) $-e^{\tan^{-1}x}+c$ (2) $e^{\tan^{-1}x}+c$

(3) $-x\,e^{\tan^{-1}x}+c$ (4) $x\,e^{\tan^{-1}x}+c$

63. $\int e^x\left(\frac{2\tan x}{1+\tan x}+\cot^2\left(x+\frac{\pi}{4}\right)\right)dx$ is equal to

(1) $e^x\tan\left(\frac{\pi}{4}-x\right)+c$ (2) $e^x\tan\left(x-\frac{\pi}{4}\right)+c$

(3) $e^x\tan\left(\frac{3\pi}{4}-x\right)+c$ (4) none of these

64. $\int e^{x^4}(x+x^3+2x^5)e^{x^2}dx$ is equal to

(1) $\frac{1}{2}xe^{x^2}e^{x^4}+c$ (2) $\frac{1}{2}x^2e^{x^4}+c$

(3) $\frac{1}{2}e^{x^2}e^{x^4}+c$ (4) $\frac{1}{2}x^2e^{x^2}e^{x^4}+c$

65. The value of integral $\int e^x\left(\frac{1}{\sqrt{1+x^2}}+\frac{1-2x^2}{\sqrt{(1+x^2)^5}}\right)dx$ is equal to

(1) $e^x\left(\frac{1}{\sqrt{1+x^2}}+\frac{x}{\sqrt{(1+x^2)^3}}\right)+c$

(2) $e^x\left(\frac{1}{\sqrt{1+x^2}}-\frac{x}{\sqrt{(1+x^2)^3}}\right)+c$

(3) $e^x\left(\frac{1}{\sqrt{1+x^2}}+\frac{x}{\sqrt{(1+x^2)^5}}\right)+c$

(4) none of these

66. $\int e^x\frac{(x^2+1)}{(x+1)^2}dx$ is equal to

(1) $\left(\frac{x-1}{x+1}\right)e^x+c$ (2) $e^x\left(\frac{x+1}{x-1}\right)+c$

(3) $e^x(x+1)(x-1)+c$ (4) none of these

67. $\int\left(\frac{x+2}{x+4}\right)^2 e^x\,dx$ is equal to

(1) $e^x\left(\frac{x}{x+4}\right)+c$ (2) $e^x\left(\frac{x+2}{x+4}\right)+c$

(3) $e^x\left(\frac{x-2}{x+4}\right)+c$ (4) $\left(\frac{2xe^2}{x+4}\right)+c$

68. $\int e^{\tan x}(\sin x-\sec x)dx$ is equal to

(1) $e^{\tan x}\cos x+C$ (2) $e^{\tan x}\sin x+C$

(3) $-e^{\tan x}\cos x+C$ (4) $e^{\tan x}\sec x+C$

69. $\int\frac{\operatorname{cosec}^2 x-2005}{\cos^{2005}x}dx$ is equal to

(1) $\frac{\cot x}{(\cos x)^{2005}}+c$ (2) $\frac{\tan x}{(\cos x)^{2005}}+c$

(3) $\frac{-(\tan x)}{(\cos x)^{2005}}+c$ (4) none of these

70. $\int\left(1+2x^2+\frac{1}{x}\right)e^{x^2-\frac{1}{x}}dx$ is equal to

(1) $-xe^{x^2-\frac{1}{x}}+c$ (2) $xe^{x^2-\frac{1}{x}}+c$

(3) $(2x-1)e^{x^2-\frac{1}{x}}+c$ (4) $(2x+1)e^{x^2-\frac{1}{x}}+c$

71. $\int e^{\sin^{-1}x}\left(\frac{\log_e x}{\sqrt{1-x^2}}+\frac{1}{x}\right)dx$ is equal to

(1) $\log_e x\cdot e^{\sin^{-1}x}+c$ (2) $\frac{e^{\sin^{-1}x}}{x}+c$

(3) $-\log_e x\cdot e^{\sin^{-1}x}+c$ (4) $e^{\sin^{-1}x}\left(\log_e x+\frac{1}{x}\right)+c$

72. If $xf(x)=3f^2(x)+2$, then $\int\frac{2x^2-12xf(x)+f(x)}{(6f(x)-x)(x^2-f(x))^2}dx$ equals

(1) $\frac{1}{x^2-f(x)}+c$ (2) $\frac{1}{x^2+f(x)}+c$

(3) $\frac{1}{x-f(x)}+c$ (4) $\frac{1}{x+f(x)}+c$

73. The value of $\int\frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$ is equal to

(1) $\frac{1}{c}\sin^{-1}\left(ax+\frac{b}{x}\right)+k$ (2) $c\sin^{-1}\left(a+\frac{b}{x}\right)+k$

(3) $\sin^{-1}\left(\frac{ax+\frac{b}{x}}{c}\right)+k$ (4) none of these

74. $\int\frac{dx}{(1+\sqrt{x})\sqrt{(x-x^2)}}$ is equal to

(1) $\frac{1+\sqrt{x}}{(1-x)^2}+c$ (2) $\frac{1+\sqrt{x}}{(1+x)^2}+c$

(3) $\frac{1-\sqrt{x}}{(1-x)^2}+c$ (4) $\frac{2(\sqrt{x}-1)}{\sqrt{(1-x)}}+c$

75. $\int\frac{2\sin x}{(3+\sin 2x)}dx$ is equal to

(1) $\frac{1}{2}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right|-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)+c$

(2) $\frac{1}{2}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right|-\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)+c$

(3) $\frac{1}{4}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right|-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)+c$

(4) none of these

76. $4\int\frac{\sqrt{a^6+x^8}}{x}dx$ is equal to

(1) $\sqrt{a^6+x^8}+\frac{a^3}{2}\ln\left|\frac{\sqrt{a^6+x^8}+a^3}{\sqrt{a^6+x^8}-a^3}\right|+c$

(2) $a^6\ln\left|\frac{\sqrt{a^6+x^8}-a^3}{\sqrt{a^6+x^8}+a^3}\right|+c$

(3) $\sqrt{a^6+x^8}+\frac{a^3}{2}\ln\left|\frac{\sqrt{a^6+x^8}-a^3}{\sqrt{a^6+x^8}+a^3}\right|+c$

(4) $a^6\ln\left|\frac{\sqrt{a^6+x^8}+a^3}{\sqrt{a^6+x^8}-a^3}\right|+c$

77. If $I_{m,n} = \int \cos^m x \sin nx\, dx$, then $7\, I_{4,3} - 4\, I_{3,2}$ is equal to

(1) constant (2) $-\cos^2 x + C$

(3) $-\cos^4 x \cos 3x + C$ (4) $\cos 7x - \cos 4x + C$

Multiple Correct Answers Type

1. $\int \frac{dx}{\sqrt{2e^x - 1}} =$

(1) $2\sec^{-1}\sqrt{2e^x} + c$ (2) $-2\tan^{-1}\frac{1}{\sqrt{2e^x-1}} + c$

(3) $2\sec^{-1}(\sqrt{2}e^x) + c$ (4) $2\tan^{-1}\sqrt{2e^x-1} + c$

2. If $\int \sin x\, d(\sec x) = f(x) - g(x) + c$, then

(1) $f(x) = \sec x$ (2) $f(x) = \tan x$

(3) $g(x) = 2x$ (4) $g(x) = x$

3. $\int \sqrt{1 + \operatorname{cosec} x}\, dx$ equals

(1) $2\sin^{-1}\sqrt{\sin x} + c$ (2) $\sqrt{2}\cos^{-1}\sqrt{\cos x} + c$

(3) $c - 2\sin^{-1}(1 - 2\sin x)$ (4) $\cos^{-1}(1 - 2\sin x) + c$

4. If $I = \int \sec^2 x \operatorname{cosec}^4 x\, dx = A\cot^3 x + B\tan x + C\cot x + D$, then

(1) $A = -\frac{1}{3}$ (2) $B = 2$

(3) $C = -2$ (4) none of these

5. A curve $g(x) = \int x^{27}(1 + x + x^2)^6 (6x^2 + 5x + 4)dx$ is passing through origin. Then

(1) $g(1) = \frac{3^7}{7}$ (2) $g(1) = \frac{2^7}{7}$

(3) $g(-1) = \frac{1}{7}$ (4) $g(-1) = \frac{3^7}{14}$

6. If $\int \sqrt{\operatorname{cosec} x + 1}\, dx = k\, fog(x) + c$, where k is a real constant, then

(1) $k = -2, f(x) = \cot^{-1} x, g(x) = \sqrt{\operatorname{cosec} x - 1}$

(2) $k = -2, f(x) = \tan^{-1} x, g(x) = \sqrt{\operatorname{cosec} x - 1}$

(3) $k = 2, f(x) = \tan^{-1} x, g(x) = \frac{\cot x}{\sqrt{\operatorname{cosec} x - 1}}$

(4) $k = 2, f(x) = \cot^{-1} x, g(x) = \frac{\cot x}{\sqrt{\operatorname{cosec} x + 1}}$

7. If $I = \int \frac{\sin x + \sin^3 x}{\cos 2x} dx = P\cos x + Q\log|f(x)| + R$, then

(1) $P = 1/2, Q = -\frac{3}{4\sqrt{2}}$ (2) $P = 1/4, Q = -\frac{1}{\sqrt{2}}$

(3) $f(x) = \frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x - 1}$ (4) $f(x) = \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}$

8. If $\int \frac{e^{x-1}}{(x^2 - 5x + 4)} 2x\, dx = AF(x-1) + BF(x-4) + C$

and $F(x) = \int \frac{e^x}{x} dx$, then

(1) $A = -2/3$ (2) $B = (4/3)e^3$

(3) $A = 2/3$ (4) $B = (8/3)e^3$

9. If $\int x^2 e^{-2x} dx = e^{-2x}(ax^2 + bx + c) + d$, then

(1) $a = -\frac{1}{2}$ (2) $b = \frac{1}{2}$

(3) $c = -\frac{1}{4}$ (4) $d \in R$

10. If $\int \frac{x^4 + 1}{x^6 + 1} dx = \tan^{-1} f(x) - \frac{2}{3}\tan^{-1} g(x) + C$, then

(1) both $f(x)$ and $g(x)$ are odd functions

(2) $f(x)$ is one-one function

(3) $f(x) = g(x)$ has no real roots

(4) $\int \frac{f(x)}{g(x)} dx = -\frac{1}{x} + \frac{3}{x^3} + c$

11. If $\int \frac{x^2 - x + 1}{(x^2 + 1)^{\frac{3}{2}}} e^x dx = e^x f(x) + c$, then

(1) $f(x)$ is an even function

(2) $f(x)$ is a bounded function

(3) the range of $f(x)$ is (0, 1]

(4) $f(x)$ has two points of extrema

12. If $\int \frac{\cos 4x + 1}{\cot x - \tan x} dx = A f(x) + B$, then

(1) $A = -\frac{1}{8}$

(2) $B = \frac{1}{2}$

(3) $f(x)$ has fundamental period $\frac{\pi}{2}$

(4) $f(x)$ is an odd function

13. If $\int \sin^{-1} x \cos^{-1} x\, dx = f^{-1}(x)\left[Ax - x f^{-1}(x) - 2\sqrt{1-x^2}\right] + \frac{\pi}{2}\sqrt{1 - x^2} + 2x + C$, then

(1) $f(x) = \sin x$ (2) $f(x) = \cos x$

(3) $A = \frac{\pi}{4}$ (4) $A = \frac{\pi}{2}$

14. If $f(x) = \int \frac{x^8 + 4}{x^4 - 2x^2 + 2} dx$ and $f(0) = 0$, then

(1) $f(x)$ is an odd function

(2) $f(x)$ has range R

(3) $f(x)$ has at least one real root

(4) $f(x)$ is a monotonic function

15. If $\int \frac{dx}{x^2+ax+1} = f(g(x)) + c$, then

(1) $f(x)$ is inverse trigonometric function for $|a| < 2$
(2) $f(x)$ is logarithmic function for $|a| > 2$
(3) $g(x)$ is quadratic function for $|a| < 2$
(4) $g(x)$ is rational function for $|a| > 2$

16. If $\int \frac{1-x^7}{x(1+x^7)} dx = a \log_e |x| + b \log_e |x^7 + 1| + c$, then

(1) $a = 1$ (2) $a = -1$ (3) $b = \frac{2}{7}$ (4) $b = -\frac{2}{7}$

17. If $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} dx = ax + b \log_e |2 \sin x + 3 \cos x| + c$ then

(1) $a = -\frac{12}{13}$ (2) $b = \frac{6}{13}$
(3) $a = \frac{12}{13}$ (4) $b = -\frac{15}{39}$

Linked Comprehension Type

For Problems 1–3

$y = f(x)$ is a polynomial function passing through point (0, 1) and which increases in the intervals (1, 2) and (3, ∞) and decreases in the intervals (– ∞, 1) and (2, 3).

1. If $f(1) = -8$, then the value of $f(2)$ is
(1) –3 (2) –6 (3) –20 (4) –7

2. If $f(1) = -8$, then the range of $f(x)$ is
(1) [3, ∞) (2) [–8, ∞) (3) [–7, ∞) (4) (– ∞, 6]

3. If $f(x) = 0$ has four real roots, then the range of values of leading coefficient of polynomial is
(1) [4/9, 1/2] (2) [4/9, 1]
(3) [1/3, 1/2] (4) none of these

For Problems 4–6

If A is square matrix and e^A is defined as

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \frac{1}{2}\begin{bmatrix} f(x) & g(x) \\ g(x) & f(x) \end{bmatrix}, \text{ where}$$

$A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ and $0 < x < 1$, I is an identity matrix.

4. $\int \frac{g(x)}{f(x)} dx$ is equal to
(1) $\log(e^x + e^{-x}) + c$ (2) $\log|e^x - e^{-x}| + c$
(3) $\log|e^{2x} - 1| + c$ (4) none of these

5. $\int (g(x)+1) \sin x dx$ is equal to
(1) $\frac{e^x}{2}(\sin x - \cos x)$ (2) $\frac{e^{2x}}{5}(2\sin x - \cos x)$
(3) $\frac{e^x}{5}(\sin 2x - \cos 2x)$ (4) none of these

6. $\int \frac{f(x)}{\sqrt{g(x)}} dx$ is equal to
(1) $\frac{1}{2\sqrt{e^x - 1}} - \operatorname{cosec}^{-1}(e^x) + c$
(2) $\frac{2}{\sqrt{e^x - e^{-x}}} - \sec^{-1}(e^x) + c$
(3) $\frac{1}{2\sqrt{e^{2x} - 1}} + \sec^{-1}(e^x) + c$
(4) none of these

For Problems 7–9

Euler's substitution:

Integrals of the form $\int R(x, \sqrt{ax^2 + bx + c})\ dx$ are calculated with the aid of one of the following three Euler substitutions:

i. $\sqrt{ax^2 + bx + c} = t \pm x\sqrt{a}$ if $a > 0$
ii. $\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c}$ if $c > 0$
iii. $\sqrt{ax^2 + bx + c} = (x - a)\, t$ if $ax^2 + bx + c = a(x - a)(x - b)$
i.e., if α is a real root of $ax^2 + bx + c = 0$

7. Which of the following functions does not appear in the primitive of $\frac{1}{1 + \sqrt{x^2 + 2x + 2}}$ if t is a function of x?
(1) $\log_e|t + 1|$ (2) $\log_e|t + 2|$
(3) $\frac{1}{t+2}$ (4) None of these

8. Which of the following functions does not appear in the primitive of $\frac{dx}{x + \sqrt{x^2 - x + 1}}$ if t is a function of x?
(1) $\log_e|t|$ (2) $\log_e|t - 2|$
(3) $\log_e|t - 1|$ (4) $\log|t + 1|$

9. $\int \frac{x dx}{\left(\sqrt{7x - 10 - x^2}\right)^3}$ can be evaluated by substituting for x as
(1) $x = \frac{5+2t^2}{t^2+1}$ (2) $x = \frac{5-t^2}{t^2+2}$
(3) $x = \frac{2t^2-5}{3t^2-1}$ (4) none of these

For Problems 10 and 11

Let $f(x) = \int \frac{x^2 dx}{(1 + x^2)(1 + \sqrt{1 + x^2})}$ and $f(0) = 0$.

10. The value of $f(1)$ is
(1) $\log(1 + \sqrt{2})$ (2) $\log(1 + \sqrt{2}) - \frac{\pi}{4}$
(3) $\log(1 + \sqrt{2}) + \frac{\pi}{2}$ (4) none of these

11. $f(x)$ is

(1) an increasing function

(2) a decreasing function

(3) a non-monotonic function

(4) can't say anything

For Problems 12 and 13

If $\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}}\,dx$

$= \frac{1}{2}\log_e\left|\frac{\sqrt{f(x)}-1}{\sqrt{f(x)}+1}\right| - \tan^{-1}\sqrt{f(x)} + C$, then

12. The value of $f(1)$ is

(1) 1 (2) 2 (3) 3 (4) 4

13. The value of $\lim_{x\to\infty} \tan^{-1}\sqrt{f(x)}$ is

(1) $\pi/2$ (2) $\pi/4$ (3) π (4) 2π

For Problems 14 and 15

If a function satisfies the relation $f(x)f''(x) - f(x)f'(x) = (f'(x))^2$ $\forall x \in R$ and $f(0) = f'(0) = 1$, then

14. The value of $\lim_{x\to-\infty} f(x)$ is

(1) $\frac{1}{\sqrt{e}}$ (2) $\frac{1}{e}$ (3) $\sqrt{e}$ (4) e

15. Number of roots of the equation $f(x) = e^x$ is

(1) 0 (2) 1 (3) 2 (4) infinite

For Problems 16 and 17

Consider two differentiable functions $f(x)$, $g(x)$ satisfying

$6\int f(x)g(x)dx = x^6 + 3x^4 + 3x^2 + c$ and $2\int \frac{g(x)dx}{f(x)} = x^2 + c$,

where $f(x) > 0\ \forall x \in R$.

16. $\int (g(x) - f(x))\,dx$ is equal to

(1) $\frac{x^4}{4} - \frac{x^2}{2} + x + c$ (2) $\frac{x^4}{4} + \frac{x^2}{2} - \frac{x^3}{3} - x + c$

(3) $\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + c$ (4) $\frac{x^4}{4} + \frac{x^3}{3} + c$

17. $\lim_{x\to 0} \frac{\log(f(x))}{g(x)} =$

(1) e (2) 2 (3) 1 (4) 0

Matrix Match Type

1. Match the following lists:

List I	List II
a. If $\int \frac{2^x}{\sqrt{1-4^x}}dx = k\sin^{-1}(f(x)) + C$, then k is greater than	**p.** 0
b. If $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7 + x^6}dx = a\ln\frac{x^k}{x^k+1} + c$, then ak is less than	**q.** 1
c. If $\int \frac{x^4+1}{x(x^2+1)^2}dx = k\ln\lvert x\rvert + \frac{m}{1+x^2} + n$, where n is the constant of integration, then mk is greater than	**r.** 3
d. If $\int \frac{dx}{5+4\cos x} = k\tan^{-1}\left(m\tan\frac{x}{2}\right) + C$, then k/m is greater than	**s.** 4

2. Match the following lists:

List I	List II
a. $\int \frac{e^{2x}-1}{e^{2x}+1}dx$ is equal to	**p.** $x - \log\left[1+\sqrt{1-e^{2x}}\right] + c$
b. $\int \frac{1}{(e^x+e^{-x})^2}dx$ is equal to	**q.** $\log(e^x+1) - x - e^{-x} + c$
c. $\int \frac{e^{-x}}{1+e^x}dx$ is equal to	**r.** $\log(e^{2x}+1) - x + c$
d. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$ is equal to	**s.** $-\frac{1}{2(e^{2x}+1)} + c$

3. Match the following lists:

List I	List II (which of the following functions appear in integration of function in List I)
a. $\int \frac{x^2-x+1}{x^3-4x^2+4x}dx$	**p.** $\log\lvert x\rvert$
b. $\int \frac{x^2-1}{x(x-2)^3}dx$	**q.** $\log\lvert x-2\rvert$
c. $\int \frac{x^3+1}{x(x-2)^2}dx$	**r.** $\frac{1}{(x-2)}$
d. $\int \frac{x^5+1}{x(x-2)^3}dx$	**s.** x

4. Consider

$$\int \frac{x^3+3x^2+2x+1}{\sqrt{x^2+x+1}}dx$$

$$= (ax^2+bx+c)\sqrt{x^2+x+1} + \lambda\int \frac{dx}{\sqrt{x^2+x+1}}.$$

Now, match the following lists and then choose the correct code.

List I	List II
a. The value of a	**p.** $-\frac{7}{24}$
b. The value of b	**q.** $\frac{1}{3}$
c. The value of c	**r.** $\frac{1}{16}$
d. The value of λ	**s.** $\frac{13}{12}$

Codes:

	a	b	c	d
(1)	q	p	s	r
(2)	s	p	q	r
(3)	r	q	p	s
(4)	q	s	p	r

Numerical Value Type

1. Let $f(x) = \int x^{\sin x}(1 + x\cos x \cdot \ln x + \sin x)\,dx$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$. Then the value of $\cos f(\pi)$ is _______.

2. Let $g(x) = \int \frac{1 + 2\cos x}{(\cos x + 2)^2}dx$ and $g(0) = 0$. Then the value of $g(\pi/2)$ is_______.

3. Let $k(x) = \int \frac{(x^2 + 1)\,dx}{\sqrt[3]{x^3 + 3x + 6}}$ and $k(-1) = \frac{1}{\sqrt[3]{2}}$. Then the value of $k(-2)$ is _______.

4. If $= \int \frac{1 - \cot^{2008} x}{\tan x + \cot^{2009} x}dx = \frac{1}{k}\log\left|\sin^k x + \cos^k x\right| + C$, then the value of k is _______.

5. If $f(x) = \int \frac{3x^2 + 1}{(x^2 - 1)^3}dx$ and $f(0) = 0$, then the value of $\frac{2}{f(2)}$ is _______.

6. If $f(x) = \sqrt{x}$, $g(x) = e^x - 1$, and $\int fog(x)\,dx = A\,fog(x) + B\tan^{-1}(fog(x)) + C$, then $A + B$ is equal to _______.

7. If $\int \frac{2\cos x - \sin x + \lambda}{\cos x + \sin x - 2}dx = A\ln|\cos x + \sin x - 2| + Bx + C$, then the value of $A + B + |\lambda|$ is _______.

8. If $\int e^{x^3 + x^2 - 1}(3x^4 + 2x^3 + 2x)dx = f(x) + C$, then the value of $f(1) \times f(-1)$ is _______.

9. If $f(x) = \int \frac{dx}{x^{1/3} + 2}$ and $f(0) = 12\log_e 2$, then the value of $f(-1)$ is_______.

10. If $I = \int \frac{dx}{\sqrt[3]{\sin^{11} x \cos x}} = -A(\tan x)^{-\frac{8}{3}} + B(\tan x)^{-\frac{2}{3}} + c$, then the value of $4A + B$ is _______.

Archives

JEE MAIN

Single Correct Answer Type

1. If $\int \frac{5\tan x}{\tan x - 2}dx = x + a\ln(\sin x - 2\cos x) + k$ then $a =$
   (1) -1 (2) -2 (3) 1 (4) 2
   (AIEEE 2012)

2. If $\int f(x)\,dx = \Psi(x)$, then $\int x^5 f(x^3)\,dx$ is equal to
   (1) $\frac{1}{3}[x^3\Psi(x^3) - \int x^2\,\Psi(x^3)dx] + c$
   (2) $\frac{1}{3}x^3\Psi(x^3) - 3\int x^3\,\Psi(x^3)dx + c$
   (3) $\frac{1}{3}x^3\Psi(x^3) - \int x^2\,\Psi(x^3)dx + c$
   (4) $\frac{1}{3}[x^3\Psi(x^3) - \int x^3\,\Psi(x^3)dx] + c$
   (JEE Main 2013)

3. The integral $\int\left(1 + x - \frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx$ is equal to
   (1) $(x-1)\,e^{x+\frac{1}{x}} + c$ (2) $xe^{x+\frac{1}{x}} + c$
   (3) $(x+1)e^{x+\frac{1}{x}} + c$ (4) $-xe^{x+\frac{1}{x}} + c$
   (JEE Main 2014)

4. The integral $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ equals
   (1) $\left(\frac{x^4+1}{x^4}\right)^{1/4} + c$ (2) $(x^4+1)^{1/4} + c$
   (3) $-(x^4+1)^{1/4} + c$ (4) $-\left(\frac{x^4+1}{x^4}\right)^{1/4} + c$
   (JEE Main 2015)

5. The integral $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3}dx$ is equal to
   (1) $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$ (2) $\frac{x^5}{2(x^5 + x^3 + 1)^2} + C$
   (3) $\frac{-x^{10}}{2(x^5 + x^3 + 1)^2}$ (4) $\frac{-x^5}{(x^5 + x^3 + 1)^2} + C$
   (JEE Main 2016)

6. Let $I_n = \int \tan^n x\,dx, (n > 1)$. $I_4 + I_6 = a\tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to
   (1) $\left(-\frac{1}{5}, 0\right)$ (2) $\left(-\frac{1}{5}, 1\right)$
   (3) $\left(\frac{1}{5}, 0\right)$ (4) $\left(\frac{1}{5}, -1\right)$
   (JEE Main 2017)

7. The integral

$$\int \frac{\sin^2 x \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2} dx$$

is equal to

(1) $\dfrac{-1}{1+\cot^3 x} + C$

(2) $\dfrac{1}{3(1+\tan^3 x)} + C$

(3) $\dfrac{-1}{3(1+\tan^3 x)} + C$

(4) $\dfrac{1}{1+\cot^3 x} + C$ **(JEE Main 2018)**

JEE ADVANCED

Single Correct Answer Type

1. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals (for some arbitrary constant K)

(1) $-\dfrac{1}{(\sec x + \tan x)^{11/2}} \left\{ \dfrac{1}{11} - \dfrac{1}{7}(\sec x + \tan x)^2 \right\} + K$

(2) $\dfrac{1}{(\sec x + \tan x)^{1/11}} \left\{ \dfrac{1}{11} - \dfrac{1}{7}(\sec x + \tan x)^2 \right\} + K$

(3) $-\dfrac{1}{(\sec x + \tan x)^{11/2}} \left\{ \dfrac{1}{11} + \dfrac{1}{7}(\sec x + \tan x)^2 \right\} + K$

(4) $\dfrac{1}{(\sec x + \tan x)^{11/2}} \left\{ \dfrac{1}{11} + \dfrac{1}{7}(\sec x + \tan x)^2 \right\} + K$

(IIT-JEE, 2012)

Answers Key

EXERCISES

Single Correct Answer Type

1. (2)	**2.** (2)	**3.** (2)	**4.** (1)	**5.** (2)
6. (1)	**7.** (4)	**8.** (4)	**9.** (4)	**10.** (3)
11. (2)	**12.** (3)	**13.** (1)	**14.** (3)	**15.** (3)
16. (1)	**17.** (3)	**18.** (3)	**19.** (2)	**20.** (1)
21. (2)	**22.** (2)	**23.** (3)	**24.** (1)	**25.** (1)
26. (4)	**27.** (1)	**28.** (2)	**29.** (2)	**30.** (2)
31. (3)	**32.** (4)	**33.** (3)	**34.** (2)	**35.** (3)
36. (2)	**37.** (1)	**38.** (3)	**39.** (1)	**40.** (4)
41. (2)	**42.** (3)	**43.** (1)	**44.** (1)	**45.** (4)
46. (1)	**47.** (1)	**48.** (4)	**49.** (2)	**50.** (1)
51. (1)	**52.** (2)	**53.** (3)	**54.** (3)	**55.** (3)
56. (4)	**57.** (1)	**58.** (4)	**59.** (3)	**60.** (2)
61. (2)	**62.** (3)	**63.** (2)	**64.** (4)	**65.** (1)
66. (1)	**67.** (1)	**68.** (3)	**69.** (4)	**70.** (2)
71. (1)	**72.** (1)	**73.** (3)	**74.** (4)	**75.** (3)
76. (3)	**77.** (3)			

Multiple Correct Answers Type

1. (1), (2), (4)	**2.** (2), (4)
3. (1), (4)	**4.** (1), (3)
5. (1), (3)	**6.** (2), (4)
7. (1), (3)	**8.** (1), (4)
9. (1), (3), (4)	**10.** (1), (3), (4)
11. (1), (2), (3)	**12.** (1), (3)
13. (1), (4)	**14.** (1), (2), (3), (4)
15. (1), (2), (4)	**16.** (1), (4)
17. (3), (4)	

Linked Comprehension Type

1. (4)	**2.** (2)	**3.** (1)	**4.** (1)	**5.** (2)
6. (3)	**7.** (4)	**8.** (2)	**9.** (1)	**10.** (2)
11. (1)	**12.** (3)	**13.** (1)	**14.** (2)	**15.** (2)
16. (2)	**17.** (4)			

Matrix Match Type

1. a → p, q; b → r, s; c → p; d → p, q
2. a → r; b → s; c → q; d → p
3. a → p, q, r; b → p, q, r; c → p, q, r, s; d → p, q, r, s
4. (4)

Numerical Value Type

1. (–1)	**2.** (0.5)	**3.** (2)	**4.** (2010)	**5.** (–9)
6. (0)	**7.** (3)	**8.** (1)	**9.** (7.5)	**10.** (–3)

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (4)	**2.** (3)	**3.** (2)	**4.** (4)	**5.** (1)
6. (3)	**7.** (3)			

JEE ADVANCED

Single Correct Answer Type

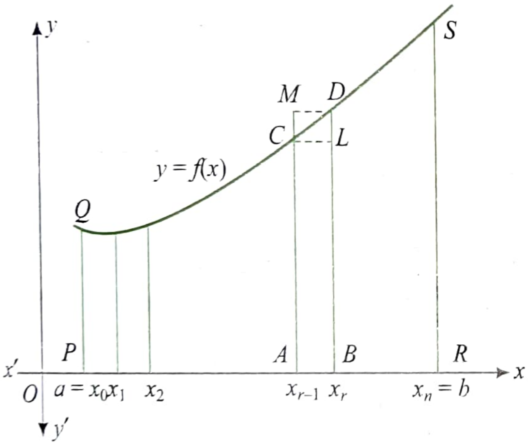
1. (3)

8 Definite Integration

DEFINITE INTEGRATION AS LIMIT OF SUM

Let f be a continuous function defined on a close interval $[a, b]$. Assume that all the values taken by the function are non-negative. So, the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\,dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. To evaluate this area, consider the region $PRSQP$ between the curve, x-axis, and the ordinates $x = a$, $x = b$.



Divide the interval $[a, b]$ into n equal sub-intervals denoted by $[x_0, x_1]$, $[x_1, x_2]$ ... $[x_{r-1}, x_r]$ ... $[x_{n-1}, x_n]$, where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ..., $x_r = a + rh$ and $x_n = b = a + nh$ or $n = \dfrac{b-a}{h}$.

We note that as $n \to \infty$, $h \to 0$.

The region $PRSQP$ under consideration is the sum of n sub-regions, where each sub-region is defined on sub-intervals $[x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots, n$.

From the figure, we have area of the rectangle $ABLC <$ area of the region $ABDCA <$ area of the rectangle $ABDM$. (1)

Evidently, as $x_r - x_{r-1} \to 0$, i.e., $h \to 0$, all the three areas shown in the figure become nearly equal to each other.

Now, we form the following sums:

$$s_n = h[f(x_0) + \cdots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \quad (2)$$

and $$S_n = h[f(x_1) + f(x_2) + \cdots + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \quad (3)$$

Here, s_n and S_n denote the sum of area of all lower and upper rectangles raised over sub-intervals $[x_{r-1}, x_r]$, for $r = 1, 2, 3, \ldots, n$, respectively.

In view of the inequality (1) for an arbitrary sub-interval $[x_{r-1}, x_r]$, we have

$$s_n < \text{area of the region } PRSQP < S_n \quad (4)$$

As $n \to \infty$, the strips become narrower. It is assumed that the limiting values of equations (2) and (3) are same in both the cases and the common limiting value is the required area under the curve. Symbolically, we can write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{area of the region } PRSQP$$

$$= \int_a^b f(x)\,dx \quad (5)$$

or $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=0}^{n-1} hf(a+rh) = \lim_{n\to\infty} \sum_{r=1}^{n} hf(a+rh)$$

ILLUSTRATION 8.1

Evaluate the following definite integrals as limit of sum $\int_2^1 x^2 dx$.

Sol. Here, $f(x) = x^2$, $a = 1$, $b = 2$.

$$\therefore \quad \int_1^2 f(x)\,dx = \lim_{n\to\infty} h \cdot \sum_{r=0}^{n-1} (1 + rh)^2$$

$$\therefore \quad \int_1^2 x^2 dx = \lim_{n\to\infty} h[1^2 + (1 + h)^2 + (1 + 2h)^2 + \cdots + \{1 + (n-1)h\}^2], \text{ where } nh = 2 - 1 = 1$$

$$= \lim_{n\to\infty} h[n1^2 + 2h\{1 + 2 + \cdots + (n-1)\} + h^2\{1^2 + 2^2 + \cdots + (n-1)^2\}]$$

$$= \lim_{n\to\infty} \left[nh + 2h^2 \sum_{r=1}^{n-1} r + h^3 \sum_{r=1}^{n-1} r^2\right]$$

$$= \lim_{n\to\infty} [nh + 2h^2 \cdot \frac{1}{2}(n-1)n + h^3 \cdot (1/6)(n-1)n(2n-1)]$$

$$= \lim_{n\to\infty} [(nh) + (nh - h)(nh) + (1/6)(nh - h)(nh)(2nh - h)]$$

$$= \lim_{h\to 0} [1 + (1 - h)(1) + (1/6)(1 - h)(1)(2 - h)]$$

$$= 1 + 1 + 1/3$$

$$= 7/3$$

ILLUSTRATION 8.2

Evaluate $\int_a^b e^x dx$ using limit of sum.

Sol. Here, $f(x) = e^x$.

$$\therefore \int_a^b e^x\,dx = \lim_{n\to\infty} h\cdot\sum_{r=0}^{n-1} e^{a+rh}$$

$$= \lim_{n\to\infty} [e^a + e^{a+h} + e^{a+2h} + \dots + e^{a+(n-1)h}]$$

where $b - a = nh$

$$= \lim_{n\to\infty} he^a\left[1 + e^h + e^{2h} + \dots + e^{(n-1)h}\right]$$

$$= \lim_{n\to\infty} he^a\left[\{(e^h)^n - 1\}/(e^h - 1)\right] \quad \text{(sum of G.P.)}$$

$$= \lim_{h\to 0} e^a(e^{nh} - 1)\cdot[h/(e^h - 1)]$$

$[\because$ as $n \to \infty$, $h \to 0$, and $nh = b - a]$

$$= e^a(e^{b-a} - 1)1 = e^b - e^a$$

ILLUSTRATION 8.3

Evaluate $\int_a^b \sin x\,dx$ using limit of sum.

Sol. We have

$$\int_a^b \sin x\,dx = \lim_{n\to\infty} h[\sin a + \sin(a+h) + \sin(a+2h) + \dots + \sin\{a + (n-1)h\}] \text{ where } nh = b - a$$

$$= \lim_{n\to\infty} h\left[\frac{\sin\left\{a + \frac{1}{2}(n-1)h\right\}\sin\left(\frac{1}{2}nh\right)}{\sin\left(\frac{1}{2}h\right)}\right]$$

$$= \lim_{h\to 0}\frac{2\sin\left\{a + \frac{1}{2}(nh - h)\right\}\sin\left(\frac{1}{2}nh\right)\cdot\left(\frac{1}{2}h\right)}{\sin\left(\frac{1}{2}h\right)}$$

$$= 2\sin\left\{a + \frac{1}{2}(b - a - 0)\right\}\cdot\sin\frac{1}{2}(b-a)\cdot 1$$

$[\because$ as $n \to \infty$, $h \to 0$; and $nh = b - a]$

$$= 2\sin\left\{\frac{1}{2}(b+a)\right\}\sin\left\{\frac{1}{2}(b-a)\right\}$$

$$= \cos a - \cos b$$

ILLUSTRATION 8.4

Evaluate $\int_a^b \frac{dx}{\sqrt{x}}$, where $a, b > 0$.

Sol. $I = \int_a^b \frac{dx}{\sqrt{x}}$, where $a, b > 0$

$$= h\lim_{n\to\infty}\left[\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a+h}} + \frac{1}{\sqrt{a+2h}} + \dots + \frac{1}{\sqrt{a+(n-1)h}}\right]$$

We know that $\sqrt{r} + \sqrt{r-h} < 2\sqrt{r} < \sqrt{r+h} + \sqrt{r}$ (for sufficiently small $h > 0$). Thus,

$$\frac{1}{\sqrt{r+h}+\sqrt{r}} < \frac{1}{2\sqrt{r}} < \frac{1}{\sqrt{r-h}+\sqrt{r}}$$

or

$$\frac{\sqrt{r+h}-\sqrt{r}}{h} < \frac{1}{2\sqrt{r}} < \frac{\sqrt{r}-\sqrt{r-h}}{h}$$

Let put $r = a, a+h, a+2h, \dots, a+(n-1)h$

$$\therefore \frac{\sqrt{a+h}-\sqrt{a}}{h} < \frac{1}{2\sqrt{a}} < \frac{\sqrt{a}-\sqrt{a-h}}{h}$$

$$\frac{\sqrt{a+2h}-\sqrt{a+h}}{h} < \frac{1}{2\sqrt{a+h}} < \frac{\sqrt{a+h}-\sqrt{a}}{h}$$

$$\frac{\sqrt{a+3h}-\sqrt{a+2h}}{h} < \frac{1}{2\sqrt{a+2h}} < \frac{\sqrt{a+2h}-\sqrt{a+h}}{h}$$

$$\vdots$$

$$\frac{\sqrt{a+nh}-\sqrt{a+(n-1)h}}{h} < \frac{1}{2\sqrt{a+(n-1)h}} < \frac{\sqrt{a+(n-1)h}-\sqrt{a+(n-2)h}}{h}$$

Adding, we get

$$\frac{\sqrt{a+nh}-\sqrt{a}}{h} < \sum_{r=0}^{n-1}\frac{1}{2\sqrt{a+rh}} < \frac{\sqrt{a+(n-1)h}-\sqrt{a-h}}{h}$$

or $$2\left(\sqrt{a+b-a}-\sqrt{a}\right) < h\sum_{r=0}^{n-1}\frac{1}{\sqrt{a+rh}} < 2\left(\sqrt{a+b-a-h}-\sqrt{a-h}\right) \quad (\text{Put } nh = b - a)$$

or $$\lim_{h\to 0} 2\left(\sqrt{a+b-a}-\sqrt{a}\right) < \lim_{h\to 0} h\sum_{r=0}^{n-1}\frac{1}{\sqrt{a+rh}} < \lim_{h\to 0} 2\left(\sqrt{a+b-a-h}-\sqrt{a-h}\right)$$

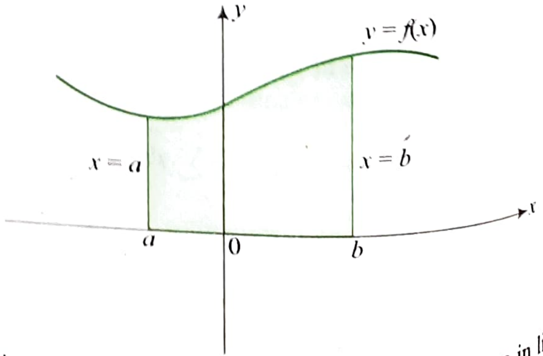
or $$2\left(\sqrt{b}-\sqrt{a}\right) < \lim_{h\to 0} h\sum_{r=0}^{n-1}\frac{1}{\sqrt{a+rh}} < 2\left(\sqrt{b}-\sqrt{a}\right)$$

or $$2\left(\sqrt{b}-\sqrt{a}\right) < \int_a^b \frac{1}{\sqrt{x}}dx < 2\left(\sqrt{b}-\sqrt{a}\right)$$

or $$\int_a^b \frac{1}{\sqrt{x}}dx = 2\left(\sqrt{b}-\sqrt{a}\right)$$

DEFINITE INTEGRATION AS AREA FUNCTION

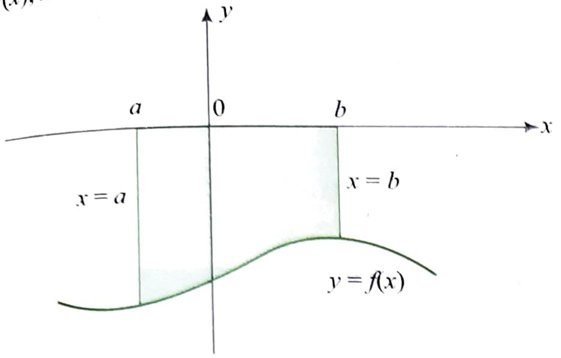
In the previous section we learned that if $f(x) \geq 0\ \forall\ x \in [a, b]$, the integral $\int_a^b f(x)dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ as shown in the following figure.



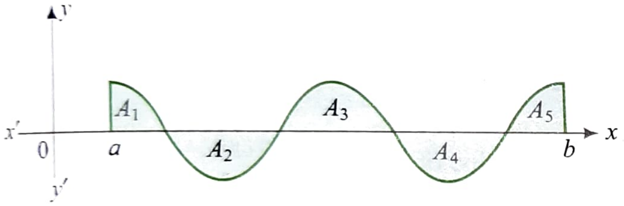
If we divide above area into rectangles like we done in limit of sum, all the $f(a + rh)$ values are positive as graph is lying above

x-axis. So, the value $\int_a^b f(x)dx$ is positive which is exactly equal to the area.

If the graph of the function $f(x)$ entirely lies below x-axis then, all the $f(a + rh)$ values will be negative. So, the value of $\int_a^b f(x)dx$ will be negative, which is negative of the area bounded by $y = f(x)$, x-axis and lines $x = a$, $x = b$.



If the graph of the function $f(x)$ between lines $x = a$ and $x = b$ intersect x-axis several times then the value $\int_a^b f(x)dx$ is not equal to exact area bounded by $y = f(x)$, x-axis and lines $x = a$, $x = b$.



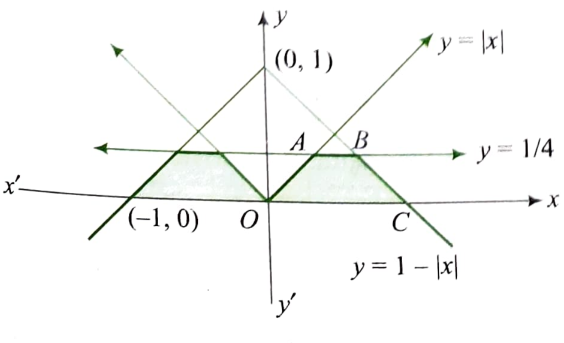
In the above graph, $\int_a^b f(x)dx = A_1 + A_2 + A_3 + A_4 + A_5$; where $A_1, A_3, A_5 > 0$ and $A_2, A_4 < 0$.

Note:
From the above discussion we understand that if $f(x)$ is continuous function and $\int_a^b f(x)dx = 0$ then the equation $f(x) = 0$ has at least one root lying in (a, b).

ILLUSTRATION 8.5

If $f(x) = \min\left(|x|, 1-|x|, \dfrac{1}{4}\right) \forall x \in R$, then find the value of $\int_{-1}^{1} f(x)\, dx$.

Sol. $f(x) = \min\left(|x|, 1-|x|, \dfrac{1}{4}\right)$



Now, from the figure,

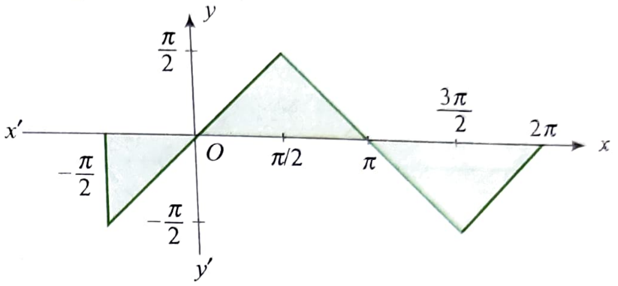
$$\int_{-1}^{1} f(x)dx = 2(\text{Area of trapezium } OABC)$$

$$= 2\left(\frac{1}{2}\left(1+\frac{1}{2}\right)\frac{1}{4}\right) = \frac{3}{8}$$

ILLUSTRATION 8.6

Evaluate: $\int_{-\pi/2}^{2\pi} \sin^{-1}(\sin x)dx$.

Sol. The graph of $f(x) = \sin^{-1}(\sin x)$ is shown as in the following figure.



$$\int_{-\pi/2}^{2\pi} \sin^{-1}(\sin x)dx$$

$$= \underbrace{-\left(\frac{1}{2}\times\frac{\pi}{2}\times\frac{\pi}{2}\right)}_{\text{for } -\frac{\pi}{2}<x<0} + \underbrace{\left(\frac{1}{2}\times\pi\times\frac{\pi}{2}\right)}_{\text{for } 0<x<\pi} - \underbrace{\left(\frac{1}{2}\times\pi\times\frac{\pi}{2}\right)}_{\text{for } \pi<x<2\pi}$$

$$= -\frac{\pi^2}{8}$$

ILLUSTRATION 8.7

Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^2}} \sin^{-1}(2x\sqrt{1-x^2})dx$.

Sol. $I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \sin^{-1}(2x\sqrt{1-x^2})dx$

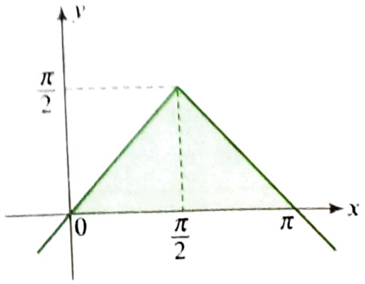
Put $x = \sin\theta$. When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{2}$.

$$\therefore \quad I = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2\theta}} \sin^{-1}(2\sin\theta\cos\theta)\cos\theta d\theta$$

$$= \int_0^{\pi/2} \sin^{-1}(\sin 2\theta)d\theta$$

Put $2\theta = t$

$$\therefore \quad I = \frac{1}{2}\int_0^{\pi} \sin^{-1}(\sin t)dt$$



From the graph, $I = \frac{1}{2}$ (area of triangle)

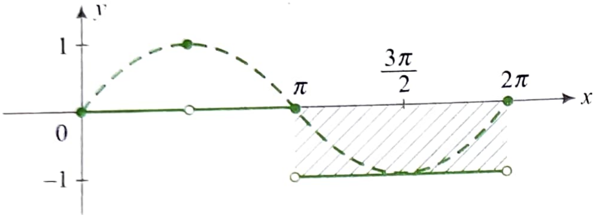
$$= \frac{1}{2} \times \frac{1}{2}\frac{\pi}{2}\pi$$

$$= \frac{\pi^2}{8}$$

ILLUSTRATION 8.8

Evaluate: $\int_0^{2\pi} [\sin x]dx$, where [.] denotes the greatest integer function.

Sol. The graph of the function $y = [\sin x]$ is as shown in the following figure.



Since for $x \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$, $[\sin x] = 0$, $\int_0^{\pi} [\sin x]dx = 0$.

For $x \in (\pi, 2\pi)$, $[\sin x] = -1$

$\therefore \quad \int_{\pi}^{2\pi} [\sin x]dx = -(\pi \times 1) = -\pi$

ILLUSTRATION 8.9

Prove that $\frac{1+\sqrt{2}}{2} < \int_0^{\pi/2} \frac{\sin x}{x}dx < \frac{\pi + 2\sqrt{2}}{4}$.

Sol. $f(x) = \frac{\sin x}{x}$

$\therefore \quad f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2}$

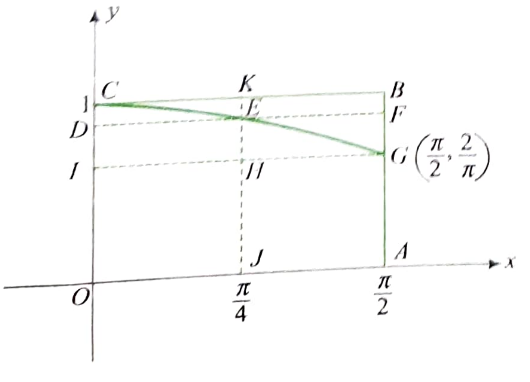
For $x \in (0, \pi/2)$, $x < \tan x$.

$\therefore \quad f'(x) < 0$.

So, $f(x)$ is decreasing function.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \to \pi/2} \frac{\sin x}{x} = \frac{2}{\pi}$$

So, the graph of the function is as shown in the following figure.



From the figure,

Area of rectangle $OJED$ + Area of rectangle $JAGH$

$$< \int_0^{\pi/2} \frac{\sin x}{x}dx$$

< Area of $OJKC$ + Area of $JAFE$

$$\therefore \quad \frac{\pi}{4}\cdot\frac{4}{\pi\sqrt{2}} + \frac{\pi}{4}\cdot\frac{2}{\pi} < \int_0^{\pi/2} \frac{\sin x}{x}dx < \frac{\pi}{4}\cdot 1 + \frac{\pi}{4}\cdot\frac{4}{\pi\sqrt{2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} + \frac{1}{2} < \int_0^{\pi/2} \frac{\sin x}{x}dx < \frac{\pi}{4} + \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{1+\sqrt{2}}{2} < \int_0^{\pi/2} \frac{\sin x}{x}dx < \frac{\pi + 2\sqrt{2}}{4}$$

CONCEPT APPLICATION EXERCISE 8.1

1. Evaluate the following integrals using limit of sum.

   (a) $\int_a^b \cos x dx$ **(b)** $\int_a^b x^3 dx$

2. Find the value of $\int_0^4 [x]dx$, where $[\cdot]$ represents the greatest integer function.

3. If $f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$, and $g(x) = f(x-1) + f(x+1)$, find the value of $\int_{-3}^{5} g(x)dx$.

ANSWERS

1. (a) $\sin b - \sin a$ (b) $\frac{1}{4}(b^4 - a^4)$ 2. 6 3. 2

THEOREMS OF DEFINITE INTEGRATION

Theorem 1:

Let f be a continuous function on the closed interval $[a, b]$ and let $F(x)$ be the area function. Then $F'(x) = f(x)$, for all $x \in [a, b]$.

Theorem 2:

This theorem enables us to evaluate definite integrals by making use of antiderivative.

The theorem is as follows:

Let f be continuous function defined on the closed interval $[a, b]$ and F be an antiderivative of f. Then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

= [Value of anti-derivative F of f at the upper limit b] – [value of the same anti derivative at the lower limit a]

In $\int_a^b f(x)dx$, function f needs to be well defined and continuous in $[a, b]$. For instance, the consideration of definite integral

$\int_{-2}^{3} x(x^2-1)^{\frac{1}{2}}dx$ is erroneous since the function f expressed by $f(x)=x(x^2-1)^{\frac{1}{2}}$ is not defined in a portion $-1 < x < 1$ of the closed interval $[-2, 3]$.

ILLUSTRATION 8.10

Evaluate $\int_{-1}^{0} \frac{dx}{x^2+2x+2}$.

Sol. $\int_{-1}^{0} \frac{dx}{x^2+2x+2} = \int_{-1}^{0} \frac{dx}{(x+1)^2+1}$

$= |\tan^{-1}(x+1)|_{-1}^{0}$

$= \tan^{-1} 1 - \tan^{-1} 0$

$= \frac{\pi}{4}$

ILLUSTRATION 8.11

Let $P(x)$ be a polynomial of least degree whose graph has three points of inflection $(-1, -1)$, $(1, 1)$ and a point with abscissa 0 at which the curve is inclined to the axis of abscissa at an angle of 60°. Then find the value of $\int_{0}^{1} P(x)dx$.

Sol. Polynomial $P(x)$ has point of inflection at $x=-1$, 1, and 0.

$\therefore \quad P''(x) = ax(x-1)(x+1) = a(x^3-x)$

Also, given that $P'(0) = \tan 60° = \sqrt{3}$

$$P'(x) = \int_{0}^{x} P''(x)\,dx + \sqrt{3} = a\left(\frac{x^4}{4}-\frac{x^2}{2}\right)+\sqrt{3}$$

$P(1) = 1$

$$\therefore \quad P(x) = \int_{1}^{x} P'(x)\,dx + 1$$

$$= a\left(\frac{x^5}{20}-\frac{x^3}{6}+\frac{7}{60}\right)+\sqrt{3}(x-1)+1$$

$P(-1) = -1$,

$$\therefore \quad a = \frac{60(\sqrt{3}-1)}{7}$$

$$\therefore \quad \int_{0}^{1} P(x)\,dx = \int_{0}^{1}\left[\left(\frac{\sqrt{3}-1}{7}(3x^5-10x^3)\right)+x\sqrt{3}\right]dx$$

$$= \left[\frac{\sqrt{3}-1}{7}\left(\frac{x^6}{2}-\frac{5}{2}x^4\right)+\frac{\sqrt{3}x^2}{2}\right]_0^1$$

$$= \frac{\sqrt{3}-1}{7}\left(\frac{1}{2}-\frac{5}{2}\right)+\frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{14}+\frac{2}{7}$$

ILLUSTRATION 8.12

Let f be a continuous function on $[a, b]$. Prove that there exists a number $x \in [a, b]$ such that $\int_{a}^{x} f(t)\,dt = \int_{x}^{b} f(t)\,dt$.

Sol. Let $g(x) = \int_{a}^{x} f(t)\,dt - \int_{x}^{b} f(t)\,dt,\ x\in[a,b]$

We have $g(a) = -\int_{a}^{b} f(t)\,dt$ and $g(b) = \int_{a}^{b} f(t)\,dt$

$$\therefore \quad g(a)\cdot g(b) = -\left[\int_{a}^{b} f(t)\,dt\right]^2 \le 0$$

Clearly, $g(x)$ is continuous in $[a, b]$ and $g(a)\cdot g(b) \le 0$

It implies that $g(x)$ will become zero at least once in $[a, b]$.

Hence, $\int_{a}^{x} f(t)\,dt = \int_{x}^{b} f(t)\,dt$ for at least one value of $x \in [a, b]$.

ILLUSTRATION 8.13

Evaluate $\int_{0}^{1} \frac{dx}{e^x+e^{-x}}$.

Sol. Let $I = \int_{0}^{1} \frac{dx}{e^x+e^{-x}} = \int_{0}^{1} \frac{e^x}{1+e^{2x}}dx$

Put $\quad e^x = t$

$\Rightarrow \quad e^x\,dx = dt$

When $\quad x \to 0$, then $t \to 1$

and $\quad x \to 1$, then $t \to e$

$$\therefore \quad I = \int_{1}^{e} \frac{dt}{1+t^2} = [\tan^{-1} t]_1^e$$

$= \tan^{-1}e - \tan^{-1}1$

$= \tan^{-1} e - \frac{\pi}{4}$

ILLUSTRATION 8.14

Evaluate $\int_{0}^{\frac{\pi}{2}} \frac{\tan x\,dx}{1+m^2\tan^2 x}$.

Sol. Let $I = \int_{0}^{\pi/2} \frac{\tan x\,dx}{1+m^2\tan^2 x}$

$$= \int_{0}^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1+m^2\cdot\frac{\sin^2 x}{\cos^2 x}}dx$$

$$= \int_{0}^{\pi/2} \frac{\sin x\cos x\,dx}{1-\sin^2 x+m^2\sin^2 x}dx$$

$$= \int_{0}^{\pi/2} \frac{\sin x\cos x}{1+(m^2-1)\sin^2 x}dx$$

Put $\sin^2 x = t$

$\Rightarrow$ $2 \sin x \cos x \, dx = dt$

When $x \to 0$, then $t \to 0$

and $x \to \pi/2$, then $t \to 1$

$$\therefore \quad I = \frac{1}{2}\int_0^1 \frac{dt}{1+(m^2-1)t}$$

$$= \frac{1}{2}\left[\frac{1}{m^2-1}\log(1+(m^2-1)t)\right]_0^1$$

$$= \frac{1}{2(m^2-1)}\left[\log(1+(m^2-1))-\log 1\right]$$

$$= \frac{\log m^2}{2(m^2-1)}$$

$$= \frac{2\log|m|}{2(m^2-1)}$$

$$= \frac{\log|m|}{m^2-1}$$

ILLUSTRATION 8.15

Find the mistake in the following evaluation of the integral

$$I = \int_0^\pi \frac{dx}{1+2\sin^2 x}$$

$$I = \int_0^\pi \frac{dx}{\cos^2 x + 3\sin^2 x}$$

$$= \int_0^\pi \frac{\sec^2 x dx}{1+3\tan^2 x} = \frac{1}{\sqrt{3}}[\tan^{-1}(\sqrt{3}\tan x)]_0^\pi = 0.$$

Sol. Here, the anti-derivative

$$\frac{1}{\sqrt{3}}[\tan^{-1}(\sqrt{3}\tan x)] = F(x)$$

is discontinuous at $x = \pi/2$ in the interval $[0, \pi]$.

Since $F\left(\frac{\pi}{2}^+\right) = \lim_{h\to 0} F\left(\frac{\pi}{2}+h\right)$

$$= \lim_{h\to 0}\left(\frac{1}{\sqrt{3}}\right)\tan^{-1}\left\{\sqrt{3}\tan\left(\frac{1}{2}\pi + h\right)\right\}$$

$$= \lim_{h\to 0}(1/\sqrt{3})\tan^{-1}\{-\sqrt{3}\cot h\}$$

$$= \left(\frac{1}{\sqrt{3}}\right)\tan^{-1}(-\infty) = -\pi/(2\sqrt{3})$$

and $F\left(\frac{1}{2}\pi - 0\right) = \pi/(2\sqrt{3}) \neq F\left(\frac{1}{2}\pi + 0\right)$

the second fundamental theorem of integral calculus is not applicable.

ILLUSTRATION 8.16

Let $\frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}, x > 0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x}dx = F(k) - F(1)$ then find the possible value of k.

Sol. $I = \int_1^4 \frac{2e^{\sin x^2}}{x}dx = F(k) - F(1) = [F(x)]_1^k$

Now put $x^2 = t$

$\therefore$ $2x\,dx = dt$

$\therefore$ $I = \int_1^{16}\frac{e^{\sin t}}{t}dt = [F(t)]_1^{16}$

$\therefore$ $I = F(16) - F(1)$

$\therefore$ $k = 16$

ILLUSTRATION 8.17

If $\int_a^b (f(x)-3x)dx = a^2 - b^2$, then find the value of $f\left(\frac{\pi}{6}\right)$.

Sol. $\int_a^b (f(x)-3x)dx = a^2 - b^2$

$\Rightarrow$ $\int_a^b f(x)dx - \frac{3}{2}[x^2]_a^b = a^2 - b^2$

$\Rightarrow$ $\int_a^b f(x)dx - \frac{3}{2}[b^2 - a^2] = a^2 - b^2$

$\Rightarrow$ $\int_a^b f(x)dx = \frac{b^2}{2} - \frac{a^2}{2}$

$\Rightarrow$ $f(x) = x$

$\Rightarrow$ $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$

DEFINITE INTEGRATION BY PARTS

If u and v are functions of x, then

$$\int_a^b uv dx = \left[u\int v dx\right]_a^b - \int_a^b\left(u'\int v dx\right)dx$$

or

$$\int_a^b uv dx = \left[u\int v dx - \int\left(u'\int v dx\right)dx\right]_a^b$$

ILLUSTRATION 8.18

If $f(0) = 1, f(2) = 3, f'(2) = 5$, then find the value of $\int_0^1 x f''(2x)dx$.

Sol. $I_1 = \int_0^1 x f''(2x)dx$. Putting $t = 2x$, i.e.,

$dx = \frac{dt}{2}$, we get

$$I_1 = \frac{1}{4}\int_0^2 t f''(t)dt$$

$$= \frac{1}{4}\left[t\, f'(t)\Big|_0^2 - \int_0^2 f'(t)dt\right] \quad \text{(Integrating by parts)}$$

$$= \frac{1}{4}\left[t\, f'(t)\Big|_0^2 - f(t)\Big|_0^2\right]$$

$$= \frac{1}{4}(2f'(2) - f(2) + f(0))$$

$$= \frac{1}{4}(10 - 3 + 1) = 2$$

ILLUSTRATION 8.19

Find the value of $\int_0^1 \log x dx$.

Sol. $I = \int_0^1 \log x dx = x \log x|_0^1 - \int_0^1 1 dx$

$$= 1 \times \log 1 - \left(\lim_{x\to 0} x \log x\right) - 1$$

$$= 0 - \lim_{x\to 0} \frac{\log x}{\frac{1}{x}} - 1$$

$$= -\lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} - 1 \quad \text{(Using L'Hopital's rule)}$$

$$= \lim_{x\to 0} x - 1 = -1$$

ILLUSTRATION 8.20

Evaluate $\int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$.

Sol. Let $I = \int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Put $x = \sin\theta$ and $dx = \cos\theta\, d\theta$

Also, when $x = 0$, $\theta = 0$, and when $x = 1/2$, $\theta = \pi/6$.

Thus, $I = \int_0^{\pi/6} \frac{\sin\theta \sin^{-1}(\sin\theta)}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta = \int_0^{\pi/6} \theta \sin\theta\, d\theta$

Integrating by parts, we get

$$I = [\theta(-\cos\theta)]_0^{\pi/6} + \int_0^{\pi/6} 1 \cos\theta\, d\theta$$

$$= [-\theta\cos\theta + \sin\theta]_0^{\pi/6}$$

$$= \frac{-\pi}{6}\frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{6 - \pi\sqrt{3}}{12}$$

ILLUSTRATION 8.21

If $\lambda = \int_0^1 \frac{e^t}{1+t}$, then find the value of $\int_0^1 e^t \log_e(1+t) dt$

Sol. $\int_0^1 e^t \log_e(1+t) dt$

$$= \left[e^t \log_e(1+t)\right]_0^1 - \int_0^1 \frac{1}{1+t} \cdot e^t dt$$

$$= e \log_e 2 - \lambda$$

ILLUSTRATION 8.22

If $\int_0^1 e^{-x^2} dx = a$, then find the value of $\int_0^1 x^2 e^{-x^2} dx$ in terms of a.

Sol. $I = \int_0^1 x^2 e^{-x^2} dx = \frac{-1}{2}\int_0^1 x(-2x)e^{-x^2} dx$

$$= -\frac{1}{2}\left(x e^{-x^2}\Big|_0^1 - \int_0^1 e^{-x^2} dx\right) \quad \text{(Integrating by parts)}$$

$$= -\frac{1}{2e} + \frac{1}{2}a$$

ILLUSTRATION 8.23

If $f(x) = x + \sin x$, then find the value of $\int_\pi^{2\pi} f^{-1}(x) dx$.

Sol. $I = \int_\pi^{2\pi} f^{-1}(x) dx$

Putting $x = f(t)$ and $dx = f'(t)\, dt$, we get

$$I = \int_\pi^{2\pi} t.f'(t)\, dt \qquad [\because f(\pi) = \pi \text{ and } f(2\pi) = 2\pi]$$

$$= |t \cdot f(t)|_\pi^{2\pi} - \int_\pi^{2\pi} 1 \cdot f(t) dt$$

$$= 2\pi f(2\pi) - \pi f(\pi) - \left|\frac{t^2}{2} - \cos t\right|_\pi^{2\pi}$$

$$= 4\pi^2 - \pi^2 - \frac{1}{2}(4\pi^2 - \pi^2) + (\cos 2\pi - \cos\pi)$$

$$= \frac{3\pi^2}{2} + 2$$

ILLUSTRATION 8.24

Find the value of $\int_0^{\pi/2} \cos^5 x \sin(7x) dx$

Sol. $I = \int_0^{\frac{\pi}{2}} \cos^5 x(\sin(6x + x)) dx$

$$I = \underbrace{\int_0^{\frac{\pi}{2}} \sin 6x \cos^6 x\, dx}_{I_1} + \underbrace{\int_0^{\frac{\pi}{2}} \underbrace{\cos 6x}_{I}\ \underbrace{\sin x \cos^5 x}_{II} dx}_{I_2}$$

$$I_2 = -\cos 6x \frac{(\cos x)^6}{6}\Bigg|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 6 \sin 6x \frac{\cos^6 x}{6} dx$$

$$\therefore \quad I = \frac{1}{6}$$

Sum of Infinite Series Using Integration

We know that $\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$

Now let $a = 0$ and $b = 1$ then we have

$$\int_0^1 f(x)dx = \lim_{h\to 0}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

More generally, $\int_0^k f(x)dx = \lim_{h\to 0}\frac{1}{n}\sum_{r=1}^{kn} f\left(\frac{r}{n}\right)$

Illustration 8.25

Evaluate $\lim_{n\to\infty} n\left[\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+4)}+\cdots+\frac{1}{6n^2}\right]$.

Sol. The given limit is

$$L = \lim_{n\to\infty}\sum_{r=1}^{n} n\cdot\frac{1}{(n+r)(n+2r)}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(1+r/n)(1+2r/n)}$$

$$= \int_0^1\frac{dx}{(1+x)(1+2x)}$$

$$= \int_0^1\left(\frac{-1}{1+x}+\frac{2}{1+2x}\right)dx$$

$$= \left[-\log(1+x)+\log(1+2x)\right]_0^1$$

$$= [(-\log 2+\log 3)-(-\log 1+\log 1)]$$

$$= \log(3/2)$$

Illustration 8.26

Evaluate $\lim_{n\to\infty}\left[\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+\cdots+\frac{1}{nb}\right]$.

Sol. The given limit is

$$L = \lim_{n\to\infty}\left[\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+\cdots+\frac{1}{na+n(b-a)}\right]$$

$$= \lim_{n\to\infty}\sum_{r=0}^{(b-a)n}\frac{1}{na+r}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{(b-a)n}\frac{1}{a+r/n}$$

$$= \int_0^{(b-a)}\frac{dx}{a+x} = \left[\log(a+x)\right]_0^{b-a}$$

$$= \log b - \log a = \log(b/a)$$

Illustration 8.27

Evaluate $\lim_{n\to\infty}\frac{[(n+1)(n+2)\cdots(n+n)]^{1/n}}{n}$.

Sol. Let $L = \lim_{n\to\infty}\frac{[(n+1)(n+2)\cdots(n+n)]^{1/n}}{n}$

$$= \lim_{n\to\infty}\left[\frac{(n+1)(n+2)\cdots(n+n)}{n^n}\right]^{1/n}$$

$$= \lim_{n\to\infty}\left[\frac{n+1}{n}\cdot\frac{n+2}{n}\cdots\frac{n+n}{n}\right]^{1/n}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

$$\therefore \log L = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\cdots+\log\left(1+\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(1+\frac{r}{n}\right) = \int_0^1\log(1+x)dx$$

$$= \left[x\log(1+x)\right]_0^1 - \int_0^1\frac{x}{1+x}dx$$

$$= \log 2 - \int_0^1\left[1-(1/(1+x))\right]dx$$

$$= \log 2 - [x-\log(1+x)]_0^1$$

$$= \log 2 - [(1-\log 2)-(0-\log 1)]$$

$$= 2\log 2 - 1 = \log(2^2/e)$$

$$\therefore \quad L = 2^2/e = 4/e$$

Illustration 8.28

Evaluate $\lim_{n\to\infty}\frac{(1^2+2^2+3^2+\cdots+n^2)(1^3+2^3+3^3+\cdots+n^3)}{1^6+2^6+3^6+\cdots+n^6}$.

Sol. The given limit is

$$\lim_{n\to\infty}\frac{\sum_{r=1}^{n} r^2\times\sum_{r=1}^{n} r^3}{\sum_{r=1}^{n} r^6} = \lim_{n\to\infty}\frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^2\times\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^3}{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^6}$$

$$= \frac{\int_0^1 x^2dx\int_0^1 x^3dx}{\int_0^1 x^6dx} = \frac{\left[\frac{x^3}{3}\right]_0^1\left[\frac{x^4}{4}\right]_0^1}{\left[\frac{x^7}{7}\right]_0^1} = \frac{\frac{1}{3}\times\frac{1}{4}}{\frac{1}{7}} = \frac{7}{12}$$

Concept Application Exercise 8.2

1. Consider the integral $I = \int_0^{2\pi}\frac{dx}{5-2\cos x}$.

   Making the substitution $\tan\frac{1}{2}x = t$, we have

   $$I = \int_0^{2\pi}\frac{dx}{5-2\cos x} = \int_0^0\frac{2dt}{(1+t^2)[5-2(1-t^2)/(1+t^2)]} = 0$$

   The result is obviously wrong, since the integrand is positive and consequently the integral of this function cannot be equal to zero. Find the mistake.

2. Evaluate the following

(a) $\displaystyle\int_0^{\pi} \frac{dx}{1+\sin x}$

(b) $\displaystyle\int_1^{\infty} (e^{x+1}+e^{3-x})^{-1}\,dx$

(c) $\displaystyle\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)\sqrt{1-x^2}}\,dx$

(d) $\displaystyle\int_0^{1} \frac{2-x^2}{(1+x)\sqrt{1-x^2}}\,dx$

(e) $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$

3. Evaluate $\displaystyle\int_{\pi/6}^{\pi/4} \frac{1+\cot x}{e^x \sin x}\,dx$.

4. Evaluate $\displaystyle\int_0^1 \frac{e^{-x}\,dx}{1+e^x}$.

5. Prove that $\displaystyle\int_0^{102} (x-1)(x-2)\ldots(x-100)$

$\displaystyle\times\left(\frac{1}{(x-1)}+\frac{1}{(x-2)}+\ldots+\frac{1}{(x-100)}\right)dx = 101! - 100!$

6. Show that $\displaystyle\int_0^1 \frac{\log x}{(1+x)}\,dx = -\int_0^1 \frac{\log(1+x)}{x}\,dx$.

7. If $\displaystyle\int_0^1 \frac{e^t}{1+t}\,dt = a$, then find the value of $\displaystyle\int_0^1 \frac{e^t}{(1+t)^2}\,dt$ in terms of a.

8. Let f be a one-to-one continuous function such that $f(2) = 3$ and $f(5) = 7$. Given $\displaystyle\int_2^5 f(x)\,dx = 17$, then find the value of $\displaystyle\int_3^7 f^{-1}(x)\,dx$.

9. Evaluate the following limits:

(a) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+\cdots+\frac{1}{\sqrt{3n^2}}\right)$

(b) $\displaystyle\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+\cdots+\frac{1}{n}\sec^2 1\right]$

(c) $\displaystyle\lim_{n\to\infty}\sum_{K=1}^{n}\frac{K}{n^2+K^2}$

(d) $\displaystyle\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{r}\sum_{r=1}^{n}\frac{1}{\sqrt{r}}}{\sum_{r=1}^{n} r}$

(e) $\displaystyle\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$

ANSWERS

2. (a) 2 (b) $\frac{\pi}{4e^2}$ (c) $\frac{\pi}{4}-\frac{1}{2}\log 2$ (d) $\frac{\pi}{2}$ (e) $\frac{\pi}{2ab}$

3. $2e^{\pi/6} - \sqrt{2}e^{-\pi/4}$ 4. $\log(1+e) - \frac{1}{e} - \log 2$

7. $a+1-\frac{e}{2}$ 8. 12

9. (a) $\frac{\pi}{6}$ (b) $\frac{1}{2}\tan 1$ (c) $\frac{1}{2}\log 2$ (d) $\frac{8}{3}$ (e) $\frac{1}{e}$

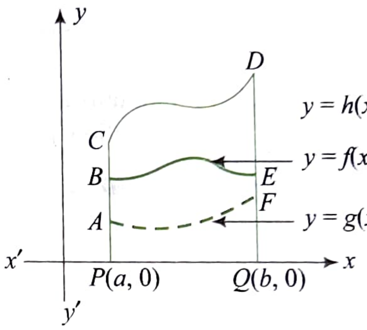
INEQUALITIES IN DEFINITE INTEGRATION

Property I:

If $g(x) \le f(x) \le h(x)$, $\forall\, x \in [a, b]$, then

$$\int_a^b g(x)dx \le \int_a^b f(x)dx \le \int_a^b h(x)dx$$

Proof:



From the figure,

Area of curvilinear trapezoid $PAFQ$

$\le$ Area of curvilinear trapezoid $PBEQ$

$\le$ Area of curvilinear trapezoid $PCDQ$

$$\therefore \quad \int_a^b g(x)dx \le \int_a^b f(x)dx \le \int_a^b h(x)dx$$

ILLUSTRATION 8.29

Prove that $0 < \displaystyle\int_0^1 \frac{x^7\,dx}{\sqrt[3]{(1+x^8)}} < \frac{1}{8}$.

Sol. Since $0 < \dfrac{x^7}{\sqrt[3]{(1+x^8)}} < x^7\ \forall\ 0 < x < 1$, we have

$$\int_0^1 0\,dx < \int_0^1 \frac{x^7}{\sqrt[3]{(1+x^8)}}\,dx < \int_0^1 x^7\,dx$$

Hence, $0 < \displaystyle\int_0^1 \frac{x^7 dx}{\sqrt[3]{(1+x^8)}} < \frac{1}{8}$.

ILLUSTRATION 8.30

Prove that $\dfrac{1}{2} \le \displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} \le \frac{\pi}{6}$ for $n \ge 1$.

Sol. For $n \ge 1$ and $-1 \le x \le 1$, we have

$$1 \ge \sqrt{1-x^{2n}} \ge \sqrt{1-x^2}$$

or $\displaystyle\int_0^{1/2} dx \le \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} \le \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1} x]_0^{1/2}$

or $\displaystyle\frac{1}{2} \le \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} \le \frac{\pi}{6}$

ILLUSTRATION 8.31

Let $\displaystyle I_1 = \int_{\pi/6}^{\pi/3} \frac{\sin x}{x} dx,\ I_2 = \int_{\pi/6}^{\pi/3} \frac{\sin(\sin x)}{\sin x} dx,\ I_3 = \int_{\pi/6}^{\pi/3} \frac{\sin(\tan x)}{\tan x} dx.$

Then arrange in the decreasing order in which values I_1, I_2, I_3 lie.

Sol. $f(x) = \dfrac{\sin x}{x}$ is a decreasing function and $\dfrac{\sin x}{x} > 0$ for all x in $(0, \pi)$.

Since $\sin x < x < \tan x$,

$$\frac{\sin(\sin x)}{\sin x} > \frac{\sin x}{x} > \frac{\sin(\tan x)}{\tan x} \text{ for } \frac{\pi}{6} < x < \frac{\pi}{3}$$

$\therefore \quad I_2 > I_1 > I_3$

Property II:

If m is the least value (global minimum) and M is the greatest value (global maximum) of the function $f(x)$ on the interval $[a, b]$, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$.

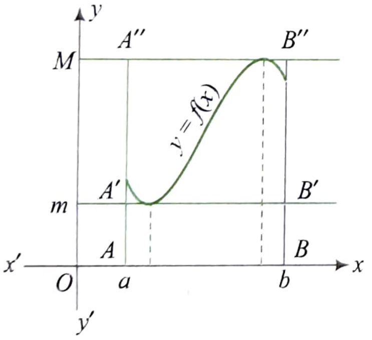
Proof:

It is given that $m \le f(x) \le M$, $\forall x \in [a, b]$

$\therefore \quad \displaystyle\int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx$

$\Rightarrow \quad \displaystyle m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

We can also establish this graphically,



From the figure, Area of $ABB'A' \le \int_a^b f(x)dx \le$ Area of $ABB''A''$

$\Rightarrow \quad \displaystyle m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

ILLUSTRATION 8.32

Prove that $\displaystyle 1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right) dx < \frac{6}{5}.$

Sol. Let $f(x) = \dfrac{5-x}{(9-x^2)}$.

$\therefore \quad f'(x) = -\dfrac{(x-9)(x-1)}{(9-x^2)^2}$

For $f'(x) = 0$, we have $x = 1$ as $x \in [0, 2]$.

Now $f(0) = 5/9, f(1) = 1/2, f(2) = 3/5$.

Therefore, the greatest and the least values of the integrand in the interval [0, 2] are, respectively, equal to

$f(2) = 3/5$ and $f(1) = 1/2$

Hence, $\displaystyle (2-0)\frac{1}{2} < \int_0^2 \left(\frac{5-x}{9-x^2}\right) dx < (2-0)\frac{3}{5},$

or $\quad \displaystyle 1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right) dx < \frac{6}{5}.$

Property III:

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Proof:

We know that $-|f(x)| \le f(x) \le |f(x)|$, $\forall x \in [a, b]$

$\therefore \quad \displaystyle -\int_a^b |f(x)|\,dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|\,dx$

$\Rightarrow \quad \displaystyle \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

ILLUSTRATION 8.33

Prove that $\displaystyle \left|\int_{10}^{19} \frac{\sin x}{1+x^8} dx\right| \le 10^{-17}.$

Sol. Since $|\sin x| \le 1$, then

$$\left|\frac{\sin x}{1+x^8}\right| \le \frac{1}{|1+x^8|} \quad (1)$$

But $10 \le x \le 19$. So, $1 + x^8 > x^8 \ge 10^8$

or $\quad \dfrac{1}{1+x^8} < \dfrac{1}{x^8} \le \dfrac{1}{10^8}$

or $\quad \dfrac{1}{|1+x^8|} \le \dfrac{1}{10^8} \quad (2)$

From (1) and (2), we get $\left|\dfrac{\sin x}{1+x^8}\right| \le 10^{-8}$

Then $\displaystyle \int_{10}^{19} \frac{\sin x}{1+x^8} dx \le (19-10) \times 10^{-8}$

$= 9 \times 10^{-8} = (10-1) \times 10^{-8}$

$= 10^{-7} - 10^{-8} < 10^{-7}$

Hence, $\displaystyle \left|\int_{10}^{19} \frac{\sin x dx}{1+x^8}\right| < 10^{-7}.$

Property IV:

If $f^2(x)$ and $g^2(x)$ are integrable on the interval $[a, b]$ then

$$\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)}.$$

Proof:

Let $F(x) = \{f(x) - \lambda g(x)\}^2$, where λ is real number.

Since $\{f(x) - \lambda g(x)\}^2 \ge 0$

$$\int_a^b \{f(x) - \lambda g(x)\}^2 dx \ge 0$$

$\Rightarrow \quad \int_a^b \{\lambda^2 (g(x))^2 - 2\lambda f(x) g(x) + f^2(x)\} dx \geq 0$

$\Rightarrow \quad \lambda^2 \int_a^b (g(x))^2 dx - 2\lambda \int_a^b f(x) g(x) dx + \int_a^b f^2(x) dx \geq 0$

$\therefore$ Discriminant is non-positive i.e., $B^2 - 4AC \leq 0$

$\Rightarrow \quad 4\left\{\int_a^b f(x) g(x) dx\right\}^2 \leq 4 \int_a^b f^2(x) dx \int_a^b g^2(x) dx$

$\Rightarrow \quad \left|\int_a^b f(x) g(x) dx\right| \leq \sqrt{\int_a^b f^2(x) dx . \int_a^b g^2(x) dx}$

ILLUSTRATION 8.34

Prove that $\int_0^1 \sqrt{(1+x)(1+x^3)}\, dx$ cannot exceed $\sqrt{15/8}$.

Sol. $\int_0^1 \sqrt{(1+x)(1+x^3)}\, dx \leq \sqrt{\left(\int_0^1 (1+x) dx\right)\left(\int_0^1 (1+x^3) dx\right)}$

$= \sqrt{\left(x + \frac{x^2}{2}\right)_0^1 \left(x + \frac{x^4}{4}\right)_0^1}$

$= \sqrt{\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)}$

$= \sqrt{\frac{15}{8}}$

CONCEPT APPLICATION EXERCISE 8.3

1. Prove that $4 \leq \int_1^3 \sqrt{3 + x^2} \leq 4\sqrt{3}$.

2. If $I_1 = \int_0^1 2^{x^2} dx$, $I_2 = \int_0^1 2^{x^3} dx$, $I_3 = \int_1^2 2^{x^2} dx$, $I_4 = \int_1^2 2^{x^3} dx$, then which of the following is/are true?

   (a) $I_1 > I_2$ **(b)** $I_2 > I_1$

   (c) $I_3 > I_4$ **(d)** $I_3 < I_4$

3. If $I_1 = \int_0^{\pi/2} \cos(\sin x) dx$, $I_2 = \int_0^{\pi/2} \sin(\cos x) dx$, and $I_3 = \int_0^{\pi/2} \cos x dx$, then find the order in which the values I_1, I_2, I_3 exist.

4. Prove that $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi}{4\sqrt{2}}$.

ANSWERS

2. a, d **3.** $I_1 > I_3 > I_2$

ELEMENTARY PROPERTIES OF DEFINITE INTEGRATION

Property I: Changing dummy variable

The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we chose to represent the independent variable. If the independent variable is denoted by y or t instead of x, we simply write the integral as $\int_a^b f(y) dy$ or $\int_a^b f(t) dt$ or instead of $\int_a^b f(x) dx$. Hence, variable of the integration is called a dummy variable.

ILLUSTRATION 8.35

Prove that $\int_a^b f(x) dx = (b - a) \int_0^1 f((b - a)x + a) dx$

Sol. $I = (b - a) \int_0^1 f((b - a)x + a) dx$

Let $(b - a)x + a = t$

$\Rightarrow \quad dx = \frac{dt}{(b - a)}$.

Also, when $x = 0$ then $t = a$ and $x = 1$ then $t = b$

$\therefore \quad I = (b - a) \int_a^b f(t) \frac{dt}{(b - a)}$

$= \int_a^b f(t) dt$

$= \int_a^b f(x) dx$ (Change of dummy variable)

Property II: Interchanging the limits

If $\frac{d}{dx} F(x) = f(x)$, then $\int_a^b f(x) dx = [F(x)]_a^b$

$= F(b) - F(a)$

$= -(F(a)) - F(b))$

$= -\int_b^a f(x) dx$

Property III: Splitting limits

$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, where $c \in R$.

This property is useful when function is in the form of piecewise definition, or is discontinuous or is non–differentiable at $x = c$ in (a, b)

Proof:

Let $\frac{d}{dx} F(x) = f(x)$

$\therefore \quad \int_a^c f(x) dx + \int_c^b f(x) dx = F(x)|_a^c + F(x)|_c^b$

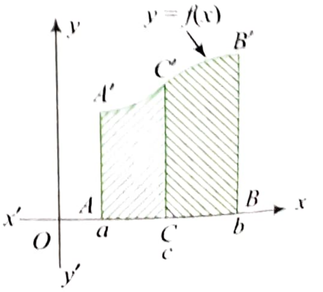
$= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$

$= \int_a^b f(x) dx$

Graphical Method:

Case I: If $a < c < b$

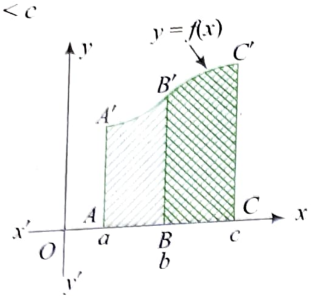


From the graph

Area of $ABB'A'A$ = Area of $(ACC'A'A)$ + Area of $(CBB'C'C)$

$$\therefore \quad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Case II: If $a < b < c$



From the graph

Area of $(ABB'A'A)$ = Area of $(ACC'A'A)$ – Area of $(BCC'B'B)$

$$\therefore \quad \int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx$$

$$= \int_a^c f(x)dx + \int_c^b f(x)dx \qquad \text{(By property II)}$$

Similarly, for $c < a < b$, we get the same result.

In general, $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \ldots + \int_{c_n}^{b} f(x)dx$

ILLUSTRATION 8.36

Evaluate $\int_{-1}^{2} |x^3 - x| dx$.

Sol. $f(x) = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$

Sign scheme of $f(x)$ is as shown in the following figure.

$$\underset{-}{} \quad -1 \quad \underset{+}{} \quad 0 \quad \underset{-}{} \quad 1 \quad \underset{+}{}$$

From the sign scheme, we have

$$\int_{-1}^{2} |x^3 - x| dx = \int_{-1}^{0} (x^3 - x)dx + \int_{0}^{1} -(x^3 - x)dx + \int_{1}^{2} (x^3 - x)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + (4-2) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$$

ILLUSTRATION 8.37

Find the value of $\int_{-1}^{3/2} |x \sin \pi x| dx$.

Sol. $|x \sin \pi x| = \begin{cases} (-x)(-\sin \pi x) \text{ if } -1 \le x < 0 \\ x \sin \pi x \text{ if } 0 \le x \le 1 \\ x(-\sin \pi x) \text{ if } 1 < x \le 3/2 \end{cases}$

$$\therefore \int_{-1}^{3/2} |x \sin \pi x| dx$$

$$= \int_{-1}^{0} x \sin \pi x \, dx + \int_{-1}^{1} x \sin \pi x \, dx + \int_{1}^{3/2} (-x \sin \pi x) dx$$

$$= \int_{-1}^{1} x \sin \pi x \, dx - \int_{1}^{3/2} x \sin \pi x \, dx$$

$$= 2\int_{0}^{1} x \sin \pi x \, dx - \int_{1}^{3/2} x \sin \pi x \, dx$$

$$= 2\left[\left\{x\left(\frac{-1}{\pi}\right)\cos \pi x\right\}_0^1 - \int_0^1 1\left(\frac{-1}{\pi}\right)\cos \pi x \, dx\right]$$

$$- \left\{x\left(\frac{-1}{\pi}\right)\cos \pi x\right\}_1^{3/2} + \int_1^{3/2} 1\left(\frac{-1}{\pi}\right)\cos \pi x \, dx$$

$$= \left(\frac{2}{\pi}\right) + \left(\frac{2}{\pi^2}\right)[\sin \pi x]_0^1 + \left\{\frac{3}{(2\pi)}\right\}\cos\frac{3}{2}\pi + \left(\frac{1}{\pi}\right) - \left(\frac{1}{\pi^2}\right)[\sin \pi x]_1^{3/2}$$

$$= (2/\pi) + 0 + 0 + (1/\pi) + (1/\pi^2)$$

$$= (3\pi + 1)/\pi^2$$

ILLUSTRATION 8.38

Show that $\int_a^b \frac{|x|}{x} dx = |b| - |a|$.

Sol. **Case I:** If $0 \le a < b$, then $|x| / x = 1$

$$\therefore \quad I = \int_a^b 1 \, dx = b - a = |b| - |a|$$

Case II: If $a < b \le 0$, then $|x| = -x$

$$\therefore \quad I = \int_a^b \frac{-x}{x} dx = \int_a^b (-1) \, dx$$

$$= [-x]_a^b = -b - (-a) = |b| - |a|$$

Case III: If $a < 0 < b$

then $|x| = -x$, when $a < x < 0$

and $|x| = x$, when $0 < x < b$

$$\therefore \quad I = \int_a^b \frac{|x|}{x} dx = \int_a^0 \frac{|x|}{x} dx + \int_0^b \frac{|x|}{x} dx$$

$$= \int_a^0 \frac{-x}{x} dx + \int_0^b \frac{x}{x} dx$$

$$= \int_a^0 (-1) dx + \int_0^b 1 dx$$

$$= [-x]_a^0 + [x]_0^b = a + b = b - (-a) = |b| - |a|$$

Hence, in all the cases, $I = \int_a^b \frac{|x|}{x} dx = |b| - |a|$.

ILLUSTRATION 8.39

If $f(n) = \int_0^{2015} \frac{e^x}{1 + x^n} dx$, then find the value of $\lim_{n \to \infty} f(n)$

Sol. $\lim_{n \to \infty} \int_0^{2015} \frac{e^x}{1 + x^n} dx$

$$= \int_0^1 \lim_{n\to\infty}\left(\frac{e^x}{1+x^n}\right)dx + \int_1^{2015} \lim_{n\to\infty}\left(\frac{e^x}{1+x^n}\right)dx$$

$$= \int_0^1 e^x dx + 0 = e - 1$$

ILLUSTRATION 8.40

Let $f(x) = \int_0^x |2t-3|\,dt$. Then find $f(x)$ and discuss continuity and differentiability of $f(x)$ at $x = \frac{3}{2}$.

Sol. $f(x) = \int_0^x |2t-3|\,dt$

$$= \begin{cases} \int_0^x (3-2t)\,dt, & x < \frac{3}{2} \\ \int_0^{3/2} (3-2t)\,dt + \int_{3/2}^x (2t-3)\,dt, & x \ge \frac{3}{2} \end{cases}$$

$$= \begin{cases} 3x - x^2, & x < \frac{3}{2} \\ x^2 - 3x + \frac{9}{2}, & x \ge \frac{3}{2} \end{cases}$$

Clearly, $f(x)$ is continuous at $x = \frac{3}{2}$ as $f\left(\frac{3^-}{2}\right) = f\left(\frac{3^+}{2}\right) = \frac{9}{4}$.

$$f'(x) = \begin{cases} 3 - 2x, & x < \frac{3}{2} \\ 2x - 3, & x > \frac{3}{2} \end{cases}$$

$$\therefore \quad f'\left(\frac{3^-}{2}\right) = f'\left(\frac{3^+}{2}\right) = 0$$

Hence, $f(x)$ is differentiable at $x = \frac{3}{2}$.

ILLUSTRATION 8.41

A continuous real function f satisfies $f(2x) = 3f(x)\ \forall\, x \in R$. If $\int_0^1 f(x)dx = 1$, then find the value of $\int_1^2 f(x)dx$.

Sol. We have $f(2x) = 3f(x)$ and $\int_0^1 f(x)dx = 1$

$$\therefore \quad \frac{1}{3}\int_0^1 f(2x)dx = 1$$

Put $\quad 2x = t$

$$\therefore \quad \frac{1}{6}\int_0^2 f(t)dt = 1$$

or $\quad \int_0^2 f(t)dt = 6$

or $\quad \int_0^1 f(t)dt + \int_1^2 f(t)dt = 6$

Hence, $\int_1^2 f(t)dt = 6 - \int_0^1 f(t)dt = 6 - 1 = 5$

ILLUSTRATION 8.42

Let $g(x) = \int_0^x f(t)dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$, for $t \in [0, 1]$ and $0 \le f(t) \le \frac{1}{2}$, for $t \in [1, 2]$. Then prove that $\frac{1}{2} \le g(2) \le \frac{3}{2}$.

Sol. $g(x) = \int_0^x f(t)dt$

$$\therefore \quad g(2) = \int_0^2 f(t)dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)dt$$

Now, $\frac{1}{2} \le f(t) \le 1$ for $t \in [0, 1]$

$$\Rightarrow \quad \int_0^1 \frac{1}{2}dt \le \int_0^1 f(t)\,dt \le \int_0^1 1\,dt$$

$$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)\,dt \le 1 \qquad \ldots(1)$$

Also, $\quad 0 \le f(t) \le \frac{1}{2}$ *for* $t \in [1, 2]$

$$\Rightarrow \quad \int_1^2 0dt \le \int_1^2 f(t) \le \int_1^2 \frac{1}{2}dt$$

$$\Rightarrow \quad 0 \le \int_1^2 f(t)\,dt \le \frac{1}{2} \qquad \ldots(2)$$

Adding (1) and (2), we get

$$\frac{1}{2} \le \int_0^1 f(t) + \int_1^2 f(t)\,dt \le \frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{2} \le \int_0^2 f(t)dt \le \frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{2} \le g(2) \le \frac{3}{2}$$

ILLUSTRATION 8.43

If $[x]$ denotes the greatest integer less than or equal to x, then find the value of the integral $\int_0^2 x^2\,[x]\,dx$.

Sol. $\int_0^2 x^2[x]\,dx = \int_0^1 x^2[x]\,dx + \int_1^2 x^2[x]dx$

$$= \int_0^1 x^2\,(0)dx + \int_1^2 x^2\,(1)dx$$

$$= 0 + \int_1^2 x^2\,dx = \left|\frac{x^3}{3}\right|_1^2$$

$$= \frac{8-1}{3} = \frac{7}{3}$$

ILLUSTRATION 8.44

Evaluate $\int_0^{5\pi/12} [\tan x]\, dx$, where [.] denotes the greatest integer function.

Sol. Let $I = \int_0^{5\pi/12} [\tan x]\, dx$.

Here, $y = \tan x$ is a monotonically increasing function.

Also, when $x = 0$, $\tan x = 0$ and when $x = \frac{5\pi}{12}$, $\tan x = 2 + \sqrt{3}$.

Hence, $[\tan x]$ is discontinuous when $\tan x = 1$, $\tan x = 2$, $\tan x = 3$.

Thus, $x = \tan^{-1} 1$, $x = \tan^{-1} 2$, $x = \tan^{-1} 3$

$$\therefore \quad I = \int_0^{\tan^{-1}1} [\tan x] dx + \int_{\tan^{-1}1}^{\tan^{-1}2} [\tan x] dx + \int_{\tan^{-1}2}^{\tan^{-1}3} [\tan x]\, dx + \int_{\tan^{-1}3}^{5\pi/12} [\tan x]\, dx$$

$$= \int_0^{\tan^{-1}1} 0\, dx + \int_{\tan^{-1}1}^{\tan^{-1}2} 1\, dx + \int_{\tan^{-1}2}^{\tan^{-1}3} 2\, dx + \int_{\tan^{-1}3}^{5\pi/12} 3\, dx$$

$$= 0 + (\tan^{-1} 2 - \tan^{-1} 1) + 2(\tan^{-1} 3 - \tan^{-1} 2) + 3\left(\frac{5\pi}{12} - \tan^{-1} 3\right)$$

$$= \frac{5\pi}{4} - \frac{\pi}{4} - \tan^{-1} 3 - \tan^{-1} 2$$

$$= \pi - \left[\pi + \tan^{-1}\left(\frac{3+2}{1-(3)(2)}\right)\right]$$

$$= \pi - [\pi + \tan^{-1}(-1)] = \pi/4$$

ILLUSTRATION 8.45

Evaluate $\int_0^{10\pi} [\tan^{-1} x] dx$, where $[x]$ represents greatest integer function.

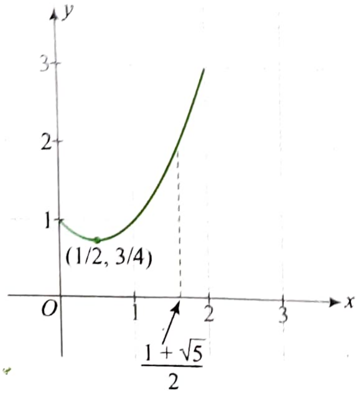
Sol. We have $\begin{cases} 0 < \tan^{-1} x < 1; & \text{when } 0 < x < \tan 1 \\ 1 < \tan^{-1} x < \dfrac{\pi}{2}; & \text{when } \tan 1 < x < 10\pi \end{cases}$

$$\therefore \quad I = \int_0^{10\pi} [\tan^{-1} x] dx = \int_0^{\tan 1} 0\, dx + \int_{\tan 1}^{10\pi} 1 dx = 10\pi - \tan 1$$

ILLUSTRATION 8.46

Evaluate $\int_0^2 [x^2 - x + 1] dx$, where $[x]$ denotes the greatest integer function.

Sol. Let first draw the graph of $y = x^2 - x + 1$ for $0 < x < 2$



From the graph $(x^2 - x + 1) \in \begin{cases} (1/2, 1), & \text{if } 0 < x < 1 \\ (1, 2), & \text{if } 1 < x < \dfrac{\sqrt{5}+1}{2} \\ (2, 3), & \text{if } \dfrac{\sqrt{5}+1}{2} < x < 2 \end{cases}$.

$$\therefore \quad \int_0^2 [x^2 - x + 1] dx = \int_0^1 0 dx + \int_1^{\frac{1+\sqrt{5}}{2}} 1 dx + \int_{\frac{1+\sqrt{5}}{2}}^2 2dx$$

$$= 0 + \left(\frac{1+\sqrt{5}}{2} - 1\right) + 2\left(2 - \frac{1+\sqrt{5}}{2}\right)$$

$$= \frac{5-\sqrt{5}}{2}$$

ILLUSTRATION 8.47

Prove that $\int_0^{\infty} [ne^{-x}] dx = \ln\left(\frac{n^n}{n!}\right)$, where n is a natural number greater than 1 and [.] denotes the greatest integer function.

Sol. $\forall x \in [0, \infty)$, $ne^{-x} \in (0, n]$

If $0 < ne^{-x} < 1$, $x \in (\ln n, \infty)$.

If $1 \le ne^{-x} < 2$, $x \in (\ln n/2, \ln n]$.

If $2 \le ne^{-x} < 3$, $x \in (\ln n/3, \ln n/2]$.

…

If $n - 1 \le ne^{-x} < n$, $x \in \left(0, \ln \frac{n}{n-1}\right]$.

$$\therefore \quad \int_0^{\infty} [ne^{-x}] dx = \int_0^{\ln\frac{n}{n-1}} (n-1) dx + \int_{\ln\frac{n}{n-1}}^{\ln\frac{n}{n-2}} (n-2) dx + \cdots + \int_{\ln\frac{n}{2}}^{\ln n} 1\, dx + \int_{\ln n}^{\infty} 0\, dx$$

$$= (n-1)\left(\ln \frac{n}{n-1}\right) + (n-2)\left[\ln\left(\frac{n}{n-2}\right) - \ln\left(\frac{n}{n-1}\right)\right] + \cdots + 1\left[\ln n - \ln\frac{n}{2}\right]$$

$$= \ln\left(\frac{n^n}{n!}\right).$$

ILLUSTRATION 8.48

Evaluate $\int_0^{\sqrt{3}} \frac{1}{1+x^2} \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$.

Sol. For $0 < x < 1$, $\sin^{-1} \frac{2x}{1+x^2} = 2\tan^{-1} x$

For $x > 1$, $\sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\frac{\frac{2}{x}}{1+\frac{1}{x^2}} = 2\tan^{-1}(1/x) = 2\cot^{-1}x$

Hence, the given integral is

$$\int_0^{\sqrt{3}} \frac{\sin^{-1}\left(\frac{2x}{1+x^2}\right)}{(1+x^2)}dx = \int_0^1 \frac{2}{1+x^2}(\tan^{-1}x)dx + \int_1^{\sqrt{3}} \frac{2\cot^{-1}x}{1+x^2}dx$$

$$= \left[(\tan^{-1}x)^2\right]_0^1 - \left[(\cot^{-1}x)^2\right]_1^{\sqrt{3}} = \frac{\pi^2}{16} - \left(\frac{\pi^2}{36} - \frac{\pi^2}{16}\right)$$

$$= \frac{\pi^2}{8} - \frac{\pi^2}{36} = \frac{7\pi^2}{72}$$

CONCEPT APPLICATION EXERCISE 8.4

1. Evaluate $\int_0^{\pi/2} |\sin x - \cos x| dx$.
2. If $\int_{-1}^4 f(x)dx = 4$ and $\int_2^4 (3 - f(x))dx = 7$, then find the value of $\int_2^{-1} f(x)dx$.
3. Evaluate $\int_1^5 \sqrt{x - 2\sqrt{x-1}}\,dx$.
4. Evaluate $\int_{-1}^3 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$.
5. Evaluate $\int_1^a x \cdot a^{-[\log_a x]} dx$, $(a > 1)$. Here, [.] represents the greatest integer function.
6. Evaluate $\int_1^{e^6} \left[\frac{\log x}{3}\right]dx$, where [.] denotes the greatest integer function.
7. Find the value of $\int_{-1}^1 [x^2 + \{x\}]dx$, where [.] and {·} denote the greatest function and fractional parts of x, respectively.
8. Evaluate $\int_{\frac{\pi}{2}}^{2\pi} [\cot^{-1} x]dx$, where [.] denotes the greatest integer function.
9. Prove that $\int_0^x [t]\,dt = \frac{[x]([x]-1)}{2} + [x](x - [x])$, where [.] denotes the greatest integer function.
10. Evaluate $\int_0^\infty [2e^{-x}]dx$, where $[x]$ represents greatest integer function.

ANSWERS

1. $2(\sqrt{2} - 1)$ 2. -5 3. 2 4. 2π
5. $\frac{1}{2}(a^2 - 1)$ 6. $(e^6 - e^3)$ 7. $\frac{3-\sqrt{5}}{2}$
8. $\pi + \cot 1 + \cot 2$ 10. ln 2

REPLACING DUMMY VARIABLE WITH SUM OF THE LIMITS MINUS DUMMY VARIABLE

This is one of the most important properties of definite integration. Application of this property changes the integrand from non-integrable to integrable or it simplifies the integrand to simple function for integration.

Property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Proof:

In R.H.S. put $a + b - x = t$

$\therefore \quad dx = -dt$

When $x = a$, $t = b$ and when $x = b$, $t = a$

$$\therefore \quad \text{R.H.S.} = \int_b^a f(t)(-dt)$$

$$= -\int_b^a f(t)dt$$

$$= \int_a^b f(t)dt$$

$$= \int_a^b f(x)dx$$

$= \text{L.H.S.}$

ILLUSTRATION 8.49

Evaluate $\int_{\pi/6}^{\pi/3} \frac{\sqrt{(\sin x)}\,dx}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}$.

Sol. We have $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{(\sin x)}\,dx}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}$ (1)

or $\quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{(\cos x)}\,dx}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}$ $\left(\text{Replacing } x \text{ by } \frac{\pi}{2} - x\right)$ (2)

Adding (1) and (2), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{(\sin x)} + \sqrt{(\cos x)}}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}dx$$

$$= \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3} = \pi/3 - \pi/6 = \pi/6$$

Hence, $I = \pi/12$

ILLUSTRATION 8.50

Evaluate $\int_0^a \frac{dx}{x + \sqrt{(a^2 - x^2)}}$ or $\int_0^{\pi/2} \frac{d\theta}{1 + \tan\theta}$.

Sol. Putting $x = a\sin\theta$, we get $dx = a\cos\theta\, d\theta$.

When $x = 0 = a\sin\theta$, $\theta = 0$.

When $x = a = a\sin\theta$, $\sin\theta = 1$. Therefore, $\theta = \pi/2$.

The given integral

$$I = \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sin\theta + \cos\theta} \quad (1)$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\cos\left(\frac{1}{2}\pi - \theta\right) d\theta}{\sin\left(\frac{1}{2}\pi - \theta\right) + \cos\left(\frac{1}{2}\pi - \theta\right)}$$

$$= \int_0^{\pi/2} \frac{\sin\theta \, d\theta}{\cos\theta + \sin\theta} \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta} d\theta = \int_0^{\pi/2} d\theta$$

$$= [\theta]_0^{\pi/2} = \pi/2$$

$$\therefore \quad I = \pi/4$$

ILLUSTRATION 8.51

Evaluate $\int_0^{\pi} \frac{\sin 6x}{\sin x} dx$.

Sol. $I = \int_0^{\pi} \frac{\sin 6x}{\sin x} dx$

$$= \int_0^{\pi} \frac{\sin(6\pi - 6x)}{\sin(\pi - x)}$$

$$= -\int_0^{\pi} \frac{\sin 6x}{\sin x} dx$$

$$= -I$$

$$\therefore \quad 2I = 0$$

$$\therefore \quad I = 0$$

ILLUSTRATION 8.52

Evaluate $\int_0^{\frac{\pi}{2}} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx$.

Sol. Let $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx \quad (1)$

$$= \int_0^{\frac{\pi}{2}} \log\left[\frac{4 + 3\sin\left(\frac{\pi}{2} - x\right)}{4 + 3\cos\left(\frac{\pi}{2} - x\right)}\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{4 + 3\cos x}{4 + 3\cos x}\right) dx \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left\{\log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) + \log\left(\frac{4 + 3\cos x}{4 + 3\sin x}\right)\right\} dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x} \times \frac{4 + 3\cos x}{4 + 3\sin x}\right) dx$$

$$= \int_0^{\pi/2} \log 1 \, dx = 0$$

or $\quad I = 0$

ILLUSTRATION 8.53

Evaluate $\int_{-\pi}^{3\pi} \log(\sec\theta - \tan\theta) d\theta$.

Sol. Let $I = \int_{-\pi}^{3\pi} \log(\sec\theta - \tan\theta)\, d\theta \quad (1)$

Using the property $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$, we get

$$I = \int_{-\pi}^{3\pi} \log\,[\sec(2\pi - \theta) - \tan(2\pi - \theta)]\, d\theta$$

$$= \int_{-\pi}^{3\pi} \log[\sec\theta + \tan\theta]\, d\theta \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_{-\pi}^{3\pi} \log\,[(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)]\, d\theta$$

$$= \int_{-\pi}^{3\pi} \log(1)\, d\theta = \int_{-\pi}^{3\pi} 0.\, d\theta = 0 \text{ or } I = 0$$

ILLUSTRATION 8.54

Prove that $\int_0^{2a} f(x)dx = \int_0^a [f(a-x) + f(a+x)]dx$

Sol. $I = \int_0^a [f(a-x) + f(a+x)]dx$

$$= \int_0^a f(a-x)dx + \int_0^a f(a+x)dx$$

$$= \int_0^a f(a-(a-x))dx + \underbrace{\int_a^{2a} f(t)dt}_{x+a=t}$$

$$= \int_0^a f(x)dx + \underbrace{\int_a^{2a} f(x)dx}_{\text{Change of dummy variable}}$$

$$= \int_0^{2a} f(x)dx$$

ILLUSTRATION 8.55

Evaluate $\int_0^{\pi/4} \ln(1 + \tan x)dx$.

Sol. Let $I = \int_0^{\pi/4} \ln(1 + \tan x)dx \quad (1)$

$$= \int_0^{\pi/4} \ln\left(1 + \tan(\pi/4 - x)\right)dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/4} \ln\left[1 + \frac{1 - \tan x}{1 + \tan x}\right] dx$$

$$= \int_0^{\pi/4} \ln\left(\frac{2}{1 + \tan x}\right) dx$$

$$I = \int_0^{\pi/4} [\ln 2 - \ln(1 + \tan x)]dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/4} \ln 2\, dx$$

$$= \ln 2\,[x]_0^{\pi/4} = \ln 2\left[\frac{\pi}{4}\right]$$

or $\quad I = \frac{\pi}{8}\ln 2$

ILLUSTRATION 8.56

Evaluate: $\int_{-5}^{5} x^2\left[x+\frac{1}{2}\right]dx$ (where [.] denotes the greatest integer function).

Sol. Let $I = \int_{-5}^{5} x^2\left[x+\frac{1}{2}\right]dx$

$$= \int_{-5}^{5} x^2\left[-x+\frac{1}{2}\right]dx \quad \text{(Replacing } x \text{ by } -x)$$

$$= \int_{-5}^{5} x^2\left[-x-\frac{1}{2}+1\right]dx$$

$$= \int_{-5}^{5} x^2\left(\left[-x-\frac{1}{2}\right]+1\right)dx$$

$$= -\int_{-5}^{5} x^2\left[x+\frac{1}{2}\right]dx$$

$$= -I$$

$\therefore \quad I = 0$

ILLUSTRATION 8.57

Evaluate $\int_{-\pi}^{\pi} \frac{x\sin x dx}{e^x+1}$.

Sol. Let $I = \int_{-\pi}^{\pi} \frac{x\sin x dx}{e^x+1}$ (1)

Replacing x by $0-x$ or $-x$, we get

$$I = \int_{-\pi}^{\pi} \frac{(-x)\sin(-x)dx}{e^{-x}+1} = \int_{-\pi}^{\pi} \frac{e^x\, x\sin x dx}{e^x+1} \quad (2)$$

Adding (1) and (2), we get $2I = \int_{-\pi}^{\pi} x\sin x dx$

or $\quad I = \int_0^{\pi} x\sin x dx$

$$= \int_0^{\pi} (\pi-x)\sin(\pi-x)dx = \int_0^{\pi} \pi\sin x dx - I \text{ or } I = \pi$$

ILLUSTRATION 8.58

Evaluate $\int_0^{\pi/2} \frac{\sin^2 x dx}{\sin x+\cos x}$.

Sol. Let $I = \int \frac{\sin^2 x\, dx}{\sin x+\cos x}$ (1)

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{1}{2}\pi - x\right)dx}{\sin\left(\frac{1}{2}\pi - x\right)+\cos\left(\frac{1}{2}\pi - x\right)}$$

$$= \int_0^{\pi/2} \frac{\cos^2 x\, dx}{\cos x + \sin x} \quad (2)$$

Now adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)dx}{\sin x + \cos x}$$

or $\quad I = \frac{1}{2}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\sin\left(x+\frac{1}{4}\pi\right)}$$

$$= \left(\frac{1}{2\sqrt{2}}\right)\int_0^{\pi/2} \text{cosec}\left(x+\frac{1}{4}\pi\right)dx$$

$$= \left(\frac{1}{2\sqrt{2}}\right)\left[\log\left\{\text{cosec}\left(x+\frac{1}{4}\pi\right)-\cot\left(x+\frac{1}{4}\pi\right)\right\}\right]_0^{\pi/2}$$

$$= (1/2\sqrt{2})\left[\log\left\{\text{cosec}\left(\frac{1}{2}\pi+\frac{1}{4}\pi\right)-\cot\left(\frac{1}{2}\pi+\frac{1}{4}\pi\right)\right\} - \log\left\{\text{cosec}\left(\frac{1}{4}\pi\right)-\cot\left(\frac{1}{4}\pi\right)\right\}\right]$$

$$= (1/2\sqrt{2})\left[\log\left\{\sec\left(\frac{1}{4}\pi\right)+\tan\left(\frac{1}{4}\pi\right)\right\}-\log(\sqrt{2}-1)\right]$$

$$= (1/2\sqrt{2})\,[\log(\sqrt{2}+1) - \log(\sqrt{2}-1)]$$

$$= \left(\frac{1}{2\sqrt{2}}\right)\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

ILLUSTRATION 8.59

Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = 2\int_0^1 \tan^{-1} x dx$. Hence or otherwise, evaluate the integral $\int_0^1 \tan^{-1}(1-x+x^2)dx$.

Sol. $\int_0^1 \tan^{-1}\frac{1}{1-x+x^2}dx = \int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}dx$

$$= \int_0^1 [\tan^{-1} x + \tan^{-1}(1-x)]\,dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x)\,dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}[1-(1-x)]\,dx$$

$$= 2\int_0^1 \tan^{-1} x\, dx \quad (1)$$

Now, $I=\int_0^1 \tan^{-1}(1-x+x^2)\,dx$

$$=\int_0^1 \cot^{-1}\left(\frac{1}{1-x+x^2}\right)dx$$

$$=\int_0^1\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{1-x+x^2}\right)\right]dx$$

$$=\frac{\pi}{2}-2\int_0^1 \tan^{-1}x\,dx \qquad \text{[from equation (1)]}$$

$$=\frac{\pi}{2}-2\left\{x\tan^{-1}x-\frac{1}{2}\log(1+x^2)\right\}_0^1$$

$$=\log_e 2$$

ILLUSTRATION 8.60

Show that $\int_0^{\pi/2}\sqrt{(\sin 2\theta)}\sin\theta d\theta=\pi/4$.

Sol. Let $I=\int_0^{\pi/2}\sqrt{(\sin 2\theta)}\,\sin\theta\,d\theta$ (1)

$$\therefore \quad I=\int_0^{\pi/2}\sqrt{\sin(2(\pi/2)-\theta)}\ \sin(\pi/2-\theta)d\theta$$

$$=\int_0^{\pi/2}\sqrt{(\sin 2\theta)}\ \cos\theta\,d\theta \qquad (2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\pi/2}\sqrt{(\sin 2\theta)}\ (\sin\theta+\cos\theta)\,d\theta$$

or

$$I=\frac{1}{2}\int_0^{\pi/2}\sqrt{1-(\sin\theta-\cos\theta)^2}\ (\sin\theta+\cos\theta)\,d\theta$$

$$=\frac{1}{2}\int_{-1}^{1}\sqrt{1-t^2}\ dt \qquad \text{[Let }\sin\theta-\cos\theta=t\text{]}$$

$$=\frac{1}{2}\left[\frac{1}{2}t\sqrt{(1-t^2)}+\frac{1}{2}\sin^{-1}t\right]_{-1}^{1}=\frac{\pi}{4}$$

ILLUSTRATION 8.61

For $\theta\in\left(0,\frac{\pi}{2}\right)$, prove that $\int_0^\theta \log(1+\tan\theta\tan x)\,dx=\theta\log(\sec\theta)$.

Sol. $I=\int_0^\theta \log(1+\tan\theta\tan x)\,dx$

$$=\int_0^\theta \log(1+\tan\theta\tan(\theta-x))\,dx$$

$$=\int_0^\theta \log\left(1+\frac{\tan\theta(\tan\theta-\tan x)}{1+\tan\theta\tan x}\right)dx$$

$$=\int_0^\theta \log\left(\frac{1+\tan^2\theta}{1+\tan\theta\tan x}\right)dx$$

$$=\int_0^\theta \log(1+\tan^2\theta)\,dx-\int_0^\theta \log(1+\tan\theta\tan x)\,dx$$

$$=2\theta\log\sec\theta-I$$

or $\quad 2I=2\theta\log\sec\theta$

or $\quad I=\theta\log\sec\theta$

ILLUSTRATION 8.62

Evaluate $\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$.

Sol. Let $I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

$$=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{(-x)^4}{1-(-x)^4}\cos^{-1}\left(\frac{2(-x)}{1+(-x)^2}\right)dx$$

$$=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\left[\pi-\cos^{-1}\left(\frac{2x}{1+x^2}\right)\right]dx$$

$$=\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx-I$$

$$\therefore \quad 2I=2\pi\int_0^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx$$

$$\therefore \quad I=\pi\int_0^{1/\sqrt{3}}\left[-1+\frac{1}{1-x^4}\right]dx$$

$$=-\pi\int_0^{1/\sqrt{3}}dx+\frac{\pi}{2}\int_0^{1/\sqrt{3}}\left[\frac{1}{1-x^2}+\frac{1}{1+x^2}\right]dx$$

$$=-\frac{\pi}{\sqrt{3}}+\frac{\pi}{2}\left[\left(-\frac{1}{2}\log_e\left|\frac{1-x}{1+x}\right|+\tan^{-1}x\right)\right]_0^{1/\sqrt{3}}$$

$$=-\frac{\pi}{\sqrt{3}}+\frac{\pi}{2}\left(\frac{1}{2}\log_e\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right|+\frac{\pi}{6}\right)$$

ILLUSTRATION 8.63

Evaluate $\int_0^1\frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$.

Sol. Let $I=\int_0^1\frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$ (1)

Also, $I=\int_0^1\frac{dx}{[5+2(1-x)-2(1-x)^2][1+e^{2-4(1-x)}]}$

$$=\int_0^1\frac{dx}{(5+2x-2x^2)(1+e^{-2+4x})}$$

$$=\int_0^1\frac{e^{2-4x}\,dx}{(5+2x-2x^2)(e^{2-4x}+1)} \qquad (2)$$

Adding (1) and (2), we get

$$2I=\int_0^1\frac{(1+e^{2-4x})dx}{(5+2x-2x^2)(1+e^{2-4x})}$$

$$= \int_0^1 \frac{dx}{5-2(x^2-x)} = \int_0^1 \frac{dx}{\frac{1}{2}+5-2\left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\int_0^1 \frac{dx}{\frac{11}{4}-\left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{4\sqrt{11/2}}\left|\log \frac{\sqrt{11/2}+x-\frac{1}{2}}{\sqrt{11/2}-\left(x-\frac{1}{2}\right)}\right|_0^1$$

$$= \frac{1}{2\sqrt{11}}\left[\log \frac{\frac{\sqrt{11}}{2}+\frac{1}{2}}{\frac{\sqrt{11}}{2}-\frac{1}{2}} - \log \frac{\frac{\sqrt{11}}{2}-\frac{1}{2}}{\frac{\sqrt{11}}{2}+\frac{1}{2}}\right]$$

$$= \frac{1}{2\sqrt{11}}\left[2\log\left(\frac{\sqrt{11}+1}{\sqrt{11}-1}\right)\right]$$

$$= \frac{1}{\sqrt{11}}\log\left(\frac{\sqrt{11}+1}{\sqrt{11}-1}\right)$$

$$= \frac{1}{\sqrt{11}}\log\frac{\sqrt{11}+1}{\sqrt{11}-1}\,\frac{\sqrt{11}+1}{\sqrt{11}+1}$$

$$= \frac{1}{\sqrt{11}}\log\frac{(\sqrt{11}+1)^2}{10}$$

$$\therefore \quad I = \frac{1}{2\sqrt{11}}\log\frac{(\sqrt{11}+1)^2}{10}$$

CONCEPT APPLICATION EXERCISE 8.5

1. If $f(a+b-x) = f(x)$, then prove that $\int_a^b x\, f(x)\, dx = \frac{a+b}{2}\int_a^b f(x)\, dx$.

2. Find the value of the integral $\int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}dx$.

3. Find the value of $\int_0^1 \sqrt[3]{2x^3-3x^2-x+1}\, dx$.

4. Show that $\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$.

5. Find the value of $\int_0^1 x(1-x)^n dx$.

6. If a continuous function f on $[0, a]$ satisfies $f(x)f(a-x) = 1$, $a > 0$, then find the value of $\int_0^a \frac{dx}{1+f(x)}$.

7. If f and g are continuous functions on $[0, a]$ satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$, then show that $\int_0^a f(x)g(x)dx = \int_0^a f(x)\, dx$.

8. Find the value of $\int_0^{\pi/2} \sin 2x \log \tan x\, dx$.

9. Find the value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}dx$, $a > 0$.

10. Find the value of $\int_0^\pi \frac{x\sin x dx}{1+\cos^2 x}$.

11. Evaluate $\int_0^\pi \frac{x\, dx}{1+\cos\alpha \sin x}$, where $0 < \alpha < \pi$.

ANSWERS

2. 3/2	3. 0	5. $\frac{1}{(n+1)(n+2)}$	6. $a/2$
8. 0	9. $\pi/2$	10. $\pi^2/4$	11. $\frac{\pi\alpha}{\sin\alpha}$

HALVING THE UPPER LIMIT

Property:

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$= \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

Proof:

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \underbrace{\int_a^{2a} f(x)dx}_{\text{Put } x=2a-t}$$

$$= \int_0^a f(x)dx + \int_a^0 f(2a-t)(-dt)$$

$$= \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$= \begin{cases} \int_0^a f(x)dx - \int_0^a f(x)dx = 0, & \text{if } f(2a-x) = -f(x) \\ \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \end{cases}$$

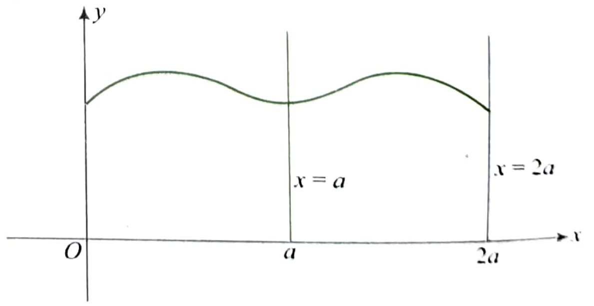
This can also be established graphically.

Let $\quad f(2a-x) = f(x)$.

Replacing x by $(a-x)$ we get

$\quad f(a+x) = f(a-x)$.

So, graph of the function is symmetrical about the line $x = a$.

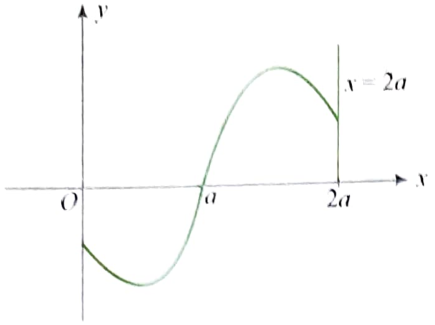


$$\therefore \quad \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$$

$$= 2\int_0^a f(x)dx$$

If $f(2a-x)=f(x)$, then we have $f(a+x)=-f(a-x)$.

So, graph of the function is symmetrical about the point $(a, 0)$.



Now if $\int_0^a f(x)dx=\lambda$ then $\int_a^{2a} f(x)dx=-\lambda$.

$\therefore \quad \int_0^{2a} f(x)dx=0$

ILLUSTRATION 8.64

Evaluate $\int_0^{2\pi} \frac{dx}{1+3\cos^2 x}$.

Sol. $I=\int_0^{2\pi} \frac{dx}{1+3\cos^2 x}$

$$=\int_0^{2\pi} \frac{\sec^2 x dx}{\sec^2 x+3}$$

$$=\int_0^{2\pi} \frac{\sec^2 x dx}{4+\tan^2 x}$$

Here we cannot substitute $\tan x=t$ as $\tan x$ is discontinuous in $(0, 2\pi)$

Let $\quad f(x)=\frac{\sec^2 x}{4+\tan^2 x}$

$\therefore \quad f(2\pi-x)=f(x)$

$\therefore \quad I=2\int_0^{\pi} \frac{\sec^2 x dx}{4+\tan^2 x}$

Also $\quad f(\pi-x)=f(x)$

$\therefore \quad I=4\int_0^{\pi/2} \frac{\sec^2 x dx}{4+\tan^2 x}$

$$=4\int_0^{\infty} \frac{dt}{4+t^2}$$ (Putting $\tan x=t$ as $\tan x$ is continuous in $(0, \pi/2)$)

$$=\frac{4}{2}\left[\tan^{-1}\frac{t}{2}\right]_0^{\infty}$$

$$=2[\tan^{-1}\infty-\tan^{-1}0]$$

$$=\pi$$

ILLUSTRATION 8.65

Evaluate $\int_0^{2\pi} \frac{x\cos^{2n} x}{\cos^{2n} x+\sin^{2n} x}dx$

Sol. $I=\int_0^{2\pi} \frac{x\cos^{2n} x}{\cos^{2n} x+\sin^{2n} x}dx$ ...(1)

$\therefore \quad I=\int_0^{2\pi} \frac{(2\pi-x)\cos^{2n} x}{\cos^{2n} x+\sin^{2n} x}dx$ ...(2)

$\left[\text{Using } \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$

Adding (1) and (2), we get

$$2I=2\pi\int_0^{2\pi} \underbrace{\frac{\cos^{2n} x}{\cos^{2n} x+\sin^{2n} x}}_{f(x)}dx$$

$$=4\pi\int_0^{\pi} \frac{\cos^{2n} x}{\cos^{2n} x+\sin^{2n} x}dx \qquad [\because f(2\pi-x)=f(x)]$$

$\therefore \quad 2I=8\pi\int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x+\sin^{2n} x}dx$ ...(3) $\quad [\because f(\pi-x)=f(x)]$

$$=8\pi\int_0^{\pi/2} \frac{\cos^{2n}\left(\frac{\pi}{2}-x\right)}{\cos^{2n}\left(\frac{\pi}{2}-x\right)+\sin^{2n}\left(\frac{\pi}{2}-x\right)}dx$$

$\left[\text{Using } \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$

$\therefore \quad 2I=8\pi\int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x}dx$ ...(4)

Adding (3) and (4), we have $4I=8\pi\int_0^{\pi/2} 1dx$

$\therefore \quad I=\pi^2$

ILLUSTRATION 8.66

Evaluate $\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$

Sol. $I=\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$

$$=\int_0^{\pi} e^{|\cos x|}\,2\sin\left(\frac{1}{2}\cos x\right)\sin x\,dx$$

$$+\int_0^{\pi} e^{|\cos x|}\,3\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$$

$$=I_1+I_2$$

Now, using the property that

$$\int_0^{2a} f(x)dx=\begin{cases}0 & \text{if } f(2a-x)=-f(x)\\ 2\int_0^a f(x)dx & \text{if } f(2a-x)=f(x)\end{cases}$$

we get $I_1=0$ and

$$I_2=2\int_0^{\pi/2} e^{|\cos x|}\,3\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$$

$$=6\int_0^{\pi/2} e^{\cos x}\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$$

Put $\cos x = t$ or $-\sin x\, dx = dt$

$\therefore \quad I_2 = 6\int_0^1 e^t \cos(t/2)\, dt$

$$= 6\left[\frac{e^t}{1+\frac{1}{4}}\left(\frac{1}{2}\sin\frac{x}{2}+\cos\frac{x}{2}\right)\right]_0^1$$

$$\left[\text{Using } \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx)\right]$$

$$= \frac{24}{5}\left[e\cos\left(\frac{1}{2}\right)+\frac{1}{2}e\sin\left(\frac{1}{2}\right)-1\right]$$

IMPORTANT RESULT

Result: $\int_0^{\pi/2} \log_e \sin x dx = \int_0^{\pi/2} \log_e \cos x dx = \frac{\pi}{2}\log_e\frac{1}{2}$

Proof:

Let $\quad I = \int_0^{\pi/2} \log_e \sin x dx \quad \ldots(1)$

$\therefore \quad I = \int_0^{\pi/2} \log_e \sin\left(\frac{\pi}{2}-x\right)dx$

or $\quad I = \int_0^{\pi/2} \log_e \cos x dx \quad \ldots(2)$

Adding (1) and (2), we have

$$2I = \int_0^{\pi/2} \log_e \sin x\cos x dx$$

$$= \int_0^{\pi/2} \log_e \frac{\sin 2x}{2} dx$$

$$= \underbrace{\int_0^{\pi/2} \log_e \sin 2x dx}_{\text{Put } 2x=t} - \int_0^{\pi/2} \log_e 2 dx$$

$$= \frac{1}{2}\int_0^{\pi} \underbrace{\log_e \sin t}_{f(t)} dt - \frac{\pi}{2}\log_e 2$$

$$= \int_0^{\pi/2} \log_e \sin t dt - \frac{\pi}{2}\log_e 2 \quad (\text{as } f(\pi - t) = f(t))$$

$$= I + \frac{\pi}{2}\log_e\frac{1}{2}$$

$\therefore \quad I = \frac{\pi}{2}\log_e\frac{1}{2}$

ILLUSTRATION 8.67

Evaluate $\int_0^{\pi} x\log \sin x\, dx$.

Sol. Let $I = \int_0^{\pi} x\log \sin x\, dx \quad (1)$

$\therefore \quad I = \int_0^{\pi}(\pi - x)\log\sin(\pi - x)\, dx$

or $\quad I = \int_0^{\pi}(\pi - x)\log \sin x\, dx \quad (2)$

Adding (1) and (2), we get

$$2I = \pi\int_0^{\pi}\log\sin x\, dx$$

or $\quad 2I = 2\pi\int_0^{\pi/2}\log \sin x\, dx$

or $\quad I = \pi\int_0^{\pi/2}\log\sin x$

$$= \pi\left\{\frac{1}{2}\pi\log(1/2)\right\} = \frac{1}{2}\pi^2\log(1/2)$$

ILLUSTRATION 8.68

Evaluate $\int_{-\pi/4}^{\pi/4}\log(\sin x + \cos x)dx$.

Sol. Let $I = \int_{-\pi/4}^{\pi/4}\log\left\{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right\}dx$

Putting $x + \frac{\pi}{4} = \theta$, $dx = d\theta$, we get

$$I = \int_0^{\pi/2}\log\left(\sqrt{2}\sin\theta\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\log 2\, d\theta + \int_0^{\pi/2}\log\sin\theta\, d\theta$$

$$= \left(\frac{1}{4}\pi\log 2\right) - \frac{1}{2}\pi\log 2$$

$$= -\frac{1}{4}\pi\log 2$$

ILLUSTRATION 8.69

Evaluate $\int_0^{\pi/2} x\cot x dx$.

Sol. Integrating by parts, taking $\cot x$ as second function, given integral becomes

$$I = [x\log\sin x]_0^{\pi/2} - \int_0^{\pi/2}\log\sin x\, dx$$

$$= 0 - \lim_{x\to 0}(x\log\sin x) - \int_0^{\pi/2}\log\sin x\, dx = \frac{1}{2}\pi\log 2$$

as $\quad \lim_{x\to 0} x\log\sin x = \lim_{x\to 0}\left(\frac{\log\sin x}{1/x}\right)$

$$= \lim_{x\to 0}\left(\frac{\cot x}{-1/x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{-x^2}{\tan x}\right)$$

$$= \lim_{x\to 0}\left(-x\times\frac{x}{\tan x}\right)$$

$$= 0\times 1 = 0$$

ILLUSTRATION 8.70

Evaluate $\int_0^{\infty} \log(x+1/x)\,\dfrac{dx}{1+x^2}$.

Sol. Putting $x=\tan\theta$, $dx=\sec^2\theta\,d\theta$, given integral becomes

$$I=\int_0^{\pi/2}\frac{\log(\tan\theta+\cot\theta)}{1+\tan^2\theta}\sec^2\theta\,d\theta$$

$$=\int_0^{\pi/2}\log(\sin\theta/\cos\theta+\cos\theta/\sin\theta)d\theta$$

$$=\int_0^{\pi/2}\log\{1/(\sin\theta\cos\theta)\}d\theta$$

$$=-\int_0^{\pi/2}\log\sin\theta d\theta-\int_0^{\pi/2}\log\cos\theta d\theta$$

$$=-2\left(-\frac{1}{2}\pi\log 2\right)=\pi\log 2$$

CONCEPT APPLICATION EXERCISE 8.6

1. Find the value of $\displaystyle\int_0^{2\pi}\frac{1}{1+\tan^4 x}dx$

2. Evaluate $\displaystyle\int_0^{2\pi}\sin^{100}x\cos^{99}x\,dx$.

3. If $U_n=\displaystyle\int_0^1 x^n(2-x)^n dx$ and $V_n=\displaystyle\int_0^1 x^n(1-x)^n dx$ $n\in N$.
   Then prove that $U_n=2^{2n}.V_n$

4. Find the value of the integral $\int_0^{\pi}\log(1+\cos x)dx$.

5. Find the value of $\int_0^1\{(\sin^{-1}x)/x\}dx$.

6. Evaluate $\displaystyle\int_{-\infty}^{0}\frac{te^t}{\sqrt{1-e^{2t}}}dt$

7. If $I_1=\int_0^{\pi}xf(\sin^3x+\cos^2x)dx$ and
   $I_2=\int_0^{\pi/2}f(\sin^3x+\cos^2x)dx$, then relate I_1 and I_2.

ANSWERS

1. π 2. 0 4. $-\pi\log_e 2$ 5. $\dfrac{\pi}{2}\log_e 2$

6. $-\dfrac{\pi}{2}\log_e 2$ 7. $I_1=\pi I_2$

DEFINITE INTEGRATION OF ODD AND EVEN FUNCTIONS

We have following properties for definite integration of odd and even functions.

Property I:

$$\int_{-a}^{a}f(x)dx=\begin{cases}0, & \text{if } f(x)\text{ is odd i.e., } f(-x)=-f(x)\\ 2\int_0^a f(x)dx, & \text{if } f(x)\text{ is even i.e., } f(-x)=f(x)\end{cases}$$

Proof:

$$\int_{-a}^{a}f(x)dx=\underbrace{\int_{-a}^{0}f(x)dx}_{x=-t}+\int_0^a f(x)dx$$

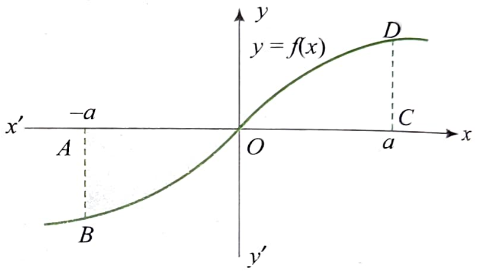
$$=\int_a^0 f(-t)(-dt)+\int_0^a f(x)dx$$

$$=\int_0^a f(-x)dx+\int_0^a f(x)dx$$

$$=\begin{cases}-\int_0^a f(x)dx+\int_0^a f(x)dx, & \text{if } f(x)\text{ is odd}\\ \int_0^a f(x)dx+\int_0^a f(x)dx, & \text{if } f(x)\text{ is even}\end{cases}$$

$$=\begin{cases}0, & \text{if } f(x)\text{ is odd}\\ 2\int_0^a f(x)dx, & \text{if } f(x)\text{ is even}\end{cases}$$

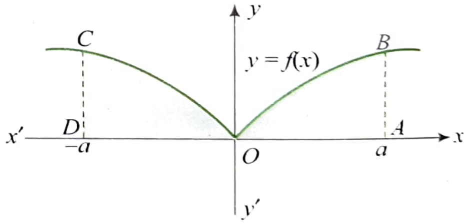
Graphical method:



The graph of odd function is symmetrical about origin as shown in the above figure.

So, if $\int_0^a f(x)dx=\lambda$ then $\int_{-a}^{0}f(x)dx=-\lambda$

$\therefore\quad \int_{-a}^{a}f(x)dx=0$



The graph of even function is symmetrical about y-axis as shown in the above figure.

So, $\quad\int_{-a}^{0}f(x)dx=\int_0^a f(x)dx=\mu$

$\therefore\quad\int_{-a}^{a}f(x)dx=2\mu=\int_0^a f(x)dx$

Property II:

If $f(t)$ is an odd function then $\phi(x)=\int_a^x f(t)dt$ is an even function

Proof:

We have $\phi(x)=\int_a^x f(t)dt$

$\Rightarrow\quad \phi(-x)=\int_a^{-x}f(t)dt=\int_a^x f(t)dt+\int_x^{-x}f(t)dt$

$=\int_a^x f(t)dt+0\quad$ (as $f(x)$ is an odd function)

$=\phi(x)$

Thus, $\phi(x)$ is an even function.

ILLUSTRATION 8.71

Evaluate $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\, dx$.

Sol. $I = \int_{-\pi/2}^{\pi/2} \underbrace{\sqrt{\cos x - \cos^3 x}}_{f(x)}\, dx$

$= 2\int_0^{\pi/2} \sqrt{\cos x - \cos^3 x}\, dx$ (as $f(x)$ is an even function)

$= 2\int_0^{\pi/2} \sqrt{\cos x \sin^2 x}\, dx$

$= 2\int_0^{\pi/2} \sqrt{\cos x}\, \sin x dx$

$= 2\left[-\frac{(\cos x)^{3/2}}{3/2}\right]_0^{\pi/2}$

$= 2 \times \frac{2}{3}$

$= \frac{4}{3}$

ILLUSTRATION 8.72

Evaluate $\int_{-\pi/2}^{\pi/2} \log\left(\frac{a - \sin\theta}{a + \sin\theta}\right) d\theta, a > 0.$

Sol. $f(\theta) = \log\left(\frac{a - \sin\theta}{a + \sin\theta}\right)$

$\therefore \quad f(-\theta) = \log\left(\frac{a + \sin\theta}{a - \sin\theta}\right)$

$= -\log\left(\frac{a - \sin\theta}{a + \sin\theta}\right)$

$= -f(\theta)$

Hence, the integrand is an odd function.

So, the given integral is zero.

ILLUSTRATION 8.73

Evaluate $\int_{-\pi/2}^{\pi/2} \log\left\{\frac{ax^2 + bx + c}{ax^2 - bx + c}(a + b)\,|\sin x|\right\} dx, (a + b) > 0.$

Sol. $I = \int_{-\pi/2}^{\pi/2} \log\left\{\frac{ax^2 + bx + c}{ax^2 - bx + c}(a + b)\,|\sin x|\right\} dx$

$= \int_{-\pi/2}^{\pi/2} \log\left(\frac{ax^2 + bx + c}{ax^2 - bx + c}\right) dx + \int_{-\pi/2}^{\pi/2} \log(a + b)\, dx$

$+ \int_{-\pi/2}^{\pi/2} \log|\sin x|\, dx$

$= I_1 + I_2 + I_3$ (1)

Now, let $f(x) = \log\left(\frac{ax^2 + bx + c}{ax^2 - bx + c}\right)$

or $f(-x) = \log\left(\frac{ax^2 - bx + c}{ax^2 + bx + c}\right)$

$= -f(x)$

$\therefore \quad I_1 = \int_{-\pi/2}^{\pi/2} f(x)\, dx = 0$

$I_2 = \log(a + b)\, [x]_{-\pi/2}^{\pi/2}$

$= \pi \log(a + b)$

$I_3 = \int_{-\pi/2}^{\pi/2} \log|\sin x| dx$

$= 2\int_0^{\pi/2} \log|\sin x|\, dx$

$= 2\int_0^{\pi/2} \log \sin x\, dx$

$= 2\left(-\frac{1}{2}\pi \log 2\right)$

Hence, from (1), we have

$I = 0 + \pi \log(a + b) - \pi \log 2$

$= \pi \log\{(a + b)/2\}$

ILLUSTRATION 8.74

Evaluate $\int_{-\pi/4}^{\pi/4} \frac{x^9 - 3x^5 + 7x^3 - x + 1}{\cos^2 x} dx.$

Sol. $f(x) = \frac{x^9 - 3x^5 + 7x^3 - x}{\cos^2 x} + \sec^2 x$

$= \sec^2 x\, (x^9 - 3x^5 + 7x^3 - x) + \sec^2 x$

$\therefore \quad \int_{-\pi/4}^{\pi/4} f(x) dx = \int_{-\pi/4}^{\pi/4} \sec^2 x dx$

[$\because \sec^2 x(x^9 - 3x^5 + 7x^3 - x)$ is an odd function]

$= 2\int_0^{\pi/4} \sec^2 x dx$

$= 2\,[\tan x]_0^{\pi/4} = 2$

ILLUSTRATION 8.75

If f is an odd function, then evaluate

$I = \int_{-a}^{a} \frac{f(\sin x)}{f(\cos x) + f(\sin^2 x)} dx.$

Sol. Let $\phi(x) = \frac{f(\sin x)}{f(\cos x) + f(\sin^2 x)}$

$\therefore \quad \phi(-x) = \frac{f(\sin(-x))}{f(\cos(-x)) + f(\sin^2(-x))}$

$= \frac{f(-\sin x)}{f(\cos x) + f(\sin^2 x)}$

$= \frac{-f(\sin x)}{f(\cos x) + f(\sin^2 x)} = -\phi(x)$

($\because f$ is an odd function)

$\therefore \quad I = \int_{-a}^{a} \frac{f(\sin x)}{f(\cos x) + f(\sin^2 x)} dx = 0$

ILLUSTRATION 8.76

Evaluate $\int_{-1/2}^{1/2}\left[\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2\right]^{1/2}dx.$

Sol. Given integral

$$\int_{-1/2}^{1/2}\left[\left\{\frac{x+1}{x-1}-\frac{x-1}{x+1}\right\}^2\right]^{1/2}dx$$

$$=\int_{-1/2}^{1/2}\left|\frac{x+1}{x-1}-\frac{x-1}{x+1}\right|dx$$

$$=\int_{-1/2}^{1/2}\left|\frac{4x}{x^2-1}\right|dx=2\int_0^{1/2}\left|\frac{4x}{x^2-1}\right|dx$$

$$=2\int_0^{1/2}\frac{4x}{1-x^2}dx$$

$$\left[\because\left|\frac{4x}{x^2-1}\right|=-\frac{4x}{x^2-1}\text{ when } 0\le x\le\frac{1}{2}\right]$$

$$=-4\left[\log\left(1-x^2\right)\right]_0^{1/2}$$

$$=-4\log(3/4)$$

$$=4\log(4/3)$$

ILLUSTRATION 8.77

Find the value of $\int_{-2}^{2}\frac{\sin^{-1}(\sin x)+\cos^{-1}(\cos x)}{(1+x^2)\left(1+\left[\frac{x^2}{5}\right]\right)}dx$, where [.] represents the greatest integer function.

Sol. $I=\int_{-2}^{2}\frac{\sin^{-1}(\sin x)+\cos^{-1}(\cos x)}{(1+x^2)\left(1+\left[\frac{x^2}{5}\right]\right)}dx$

$\sin^{-1}(\sin x)$ is an odd function.

Also, for $-2<x<2$, $0\le x^2<4$.

$$\therefore\quad\left[\frac{x^2}{5}\right]=0$$

$$\therefore\quad I=\int_{-2}^{2}\frac{\cos^{-1}(\cos x)}{1+x^2}dx$$

$$=\int_0^2\frac{2\cos^{-1}(\cos x)}{1+x^2}dx$$

$$=\int_0^2\frac{2x}{1+x^2}dx$$

$$=\left[\log_e(1+x^2)\right]_0^2$$

$$=\log_e 4$$

ILLUSTRATION 8.78

Determine the value of $\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx.$

Sol. $I=\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx$

$$=\int_{-\pi}^{\pi}\frac{2x}{1+\cos^2 x}dx+2\int_{-\pi}^{\pi}\frac{x\sin x}{1+\cos^2 x}dx \quad (1)$$

$$=0+4\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$$

$$=4\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$=4\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$$

$$=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx-4\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$$

or $\quad 2I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

or $\quad I=2\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

Put $\cos x=t$ so that $-\sin x\,dx=dt$.
When $x=0$, $t=1$; when $x=\pi$, $t=-1$.

$$\therefore\quad I=2\pi\int_1^{-1}\frac{-dt}{1+t^2}=4\pi\int_0^1\frac{dt}{1+t^2}$$

$$=4\pi[\tan^{-1}t]_0^1$$

$$=4\pi\frac{\pi}{4}=\pi^2$$

CONCEPT APPLICATION EXERCISE 8.7

Evaluate the following:

1. $\int_{-\pi/2}^{\pi/2}\sin^2 x\cos^2 x(\sin x+\cos x)dx$
2. $\int_{-1}^{1}\frac{x^3+|x|+1}{x^2+2|x|+1}dx$
3. $\int_{-\pi}^{\pi}\left(1-x^2\right)\sin x\cos^2 x\,dx$
4. $\int_{-1}^{1}\frac{\sin x-x^2}{3-|x|}dx$
5. $\int_{-1/2}^{1/2}\cos x\log\frac{1-x}{1+x}dx$
6. $\int_{-3\pi/2}^{-\pi/2}[(x+\pi)^3+\cos^2(x+3\pi)]\,dx$

ANSWERS

1. 4/15 **2.** $2\log_e 2$ **3.** 0 **4.** $7+18\log_e(2/3)$
5. 0 **6.** $\pi/2$

DEFINITE INTEGRATION OF PERIODIC FUNCTIONS

We have following properties for the definite integration of periodic functions.

Property I:

$\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$; where '$T$' is the period of the function and $n \in N$

Proof:

$$\int_0^{nT} f(x)dx$$

$$= \int_0^T f(x)dx + \int_T^{2T} f(x)dx + \int_{2T}^{3T} f(x)dx + \ldots + \int_{(n-1)T}^{nT} f(x)dx$$

$$= \int_0^T f(x)dx + \int_0^T f(x+T)dx + \int_0^T f(x+2T)dx + \ldots$$
$$+ \int_0^T f(x+(n-1)T)dx$$

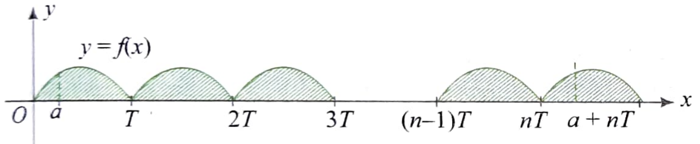
$$= \underbrace{\int_0^T f(x)dx + \int_0^T f(x)dx + \int_0^T f(x)dx + \ldots + \int_0^T f(x)dx}_{n \text{ times}}$$

$\{\because f(x) = f(x+T) = f(x+2T) = \cdots = f(x+(n-1)T)\}$

$$= n\int_0^T f(x)dx$$

Graphical method:

Consider the following graph of the periodic function $f(x)$.



Since period of $f(x)$ is T, graph of $f(x)$ is same in each of the intervals $(0, T)$, $(T, 2T)$, $(2T, 3T)$, ..., $((n-1)T, nT)$

So, $\int_0^{nT} f(x)dx$ = total shaded area

= $n \times$ (area in the in the interval (0, T))

$$= n\int_0^T f(x)dx$$

Property II:

$\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx$, where $n \in N$, $a \in R$ and T is the period of the function.

Proof:

$$I = \int_a^{a+nT} f(x)dx$$

$$= \int_a^0 f(x)dx + \int_0^{nT} f(x)dx + \underbrace{\int_{nT}^{a+nT} f(x)dx}_{\text{Put } x = y+nT}$$

$$= \int_a^0 f(x)dx + \int_0^{nT} f(x)dx + \int_0^a f(y)dy$$

$$= n\int_0^{nT} f(x)dx$$

Property III:

$\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx$, where '$T$' is the period of the function and $m, n \in I$

Proof:

$$\int_{mT}^{nT} f(x)dx = \int_0^{(n-m)T} f(x+mT)dx$$
$$= \int_0^{(n-m)T} f(x)dx$$
$$= (n-m)\int_0^T f(x)dx$$

Property IV:

$$\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx,$$

where 'T' is the period of the function and $n \in I$

Proof:

$$\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x+nT)dx = \int_a^b f(x)dx$$

ILLUSTRATION 8.79

Evaluate $\int_0^{16\pi/3} |\sin x|\,dx$.

Sol. $\int_0^{16\pi/3} |\sin x|\,dx = \int_0^{5\pi} |\sin x|\,dx + \int_{5\pi}^{5\pi+\pi/3} |\sin x|\,dx$

$= 5\int_0^{\pi} |\sin x|\,dx + \int_0^{\pi/3} |\sin x|\,dx$ [$\because$ $|\sin x|$ is periodic with period π]

$$= 5\int_0^{\pi} \sin x\,dx + \int_0^{\pi/3} \sin x\,dx$$

$$= 5[-\cos x]_0^{\pi} + [-\cos x]_0^{\pi/3}$$

$$= 5\times 2 + \left(-\frac{1}{2}+1\right) = \frac{21}{2}$$

ILLUSTRATION 8.80

Evaluate $\dfrac{\int_0^n [x]dx}{\int_0^n \{x\}dx}$ (where $[x]$ and $\{x\}$ are integral and fractional parts of x respectively and $n \in N$).

Sol. $I = \dfrac{\int_0^n [x]dx}{\int_0^n \{x\}dx}$

$$= \frac{\int_0^n (x - \{x\})dx}{\int_0^n \{x\}dx}$$

$$= \frac{\int_0^n x\,dx}{\int_0^n \{x\}dx} - 1$$

$$= \frac{\left.\frac{x^2}{2}\right|_0^n}{n\int_0^1 \{x\}dx} - 1$$

$$= \frac{\frac{n^2}{2}}{n\int_0^1 x\,dx} - 1$$

$$= \frac{\frac{n^2}{2}}{n\frac{1}{2}} - 1 = n - 1$$

ILLUSTRATION 8.81

Let $f(x)$ be a continuous and periodic function such that $f(x) = f(x + T)$ for all $x \in R$, $T > 0$. If $\int_{-2T}^{a+5T} f(x)dx = 19\ (a > 0)$ and $\int_0^T f(x)dx = 2$, then find the value of $\int_0^a f(x)dx$.

Sol. Given that

$$\int_{-2T}^{a+5T} f(x)dx = 19$$

$$\Rightarrow \int_{-2T}^{5T} f(x)dx + \int_{5T}^{a+5T} f(x)dx = 19$$

$$\Rightarrow 7\int_0^T f(x)\,dx + \int_0^a f(x)dx = 19$$

$$\Rightarrow \int_0^a f(x)\,dx = 19 - 7\times 2 = 19 - 14 = 5$$

ILLUSTRATION 8.82

If $g(x) = \int_0^x \cos^4 t\, dt$, then prove that $g(x + \pi) = g(x) + g(\pi)$.

Sol. We have $g(x) = \int_0^x \cos^4 t dt$

$$\therefore \quad g(x+\pi) = \int_0^{x+\pi} \cos^4 t dt$$

$$= \int_0^x \cos^4 t dt + \int_x^{x+\pi} \cos^4 t dt$$

$$= g(x) + \int_0^{\pi} \cos^4 t dt \qquad [\because \text{period of } \cos^4 t \text{ is } \pi]$$

$$= g(x) + g(\pi)$$

ILLUSTRATION 8.83

Evaluate $\int_{-\pi/4}^{n\pi - \frac{\pi}{4}} |\sin x + \cos x|\, dx,\ n \in N$

Sol. $I = \int_{-\pi/4}^{n\pi-\pi/4} |\sin x + \cos x|\, dx$

$$= \int_{-\pi/4}^{n\pi - \frac{\pi}{4}} \sqrt{2}\,|\sin(x + \pi/4)|\, dx$$

(Multiplying and dividing by $\sqrt{2}$)

$$= n\int_0^{\pi} \sqrt{2}\,|\sin(x+\pi/4)|\, dx$$

[As $|\sin(x + \pi/4)|$ is periodic with period π]

$$= \sqrt{2}n\left[\int_0^{3\pi/4} \sin(x+\pi/4)dx + \int_{3\pi/4}^{\pi} -\sin(x+\pi/4)dx\right]$$

$$= 2\sqrt{2}n$$

ILLUSTRATION 8.84

Evaluate $\int_0^x [\cos t]\, dt$ where $x \in \left(2n\pi, (4n+1)\frac{\pi}{2}\right)$, $n \in N$, and [.] denotes the greatest integer function.

Sol. Let

$$I = \int_0^x [\cos t]\, dt$$

$$= \int_0^{2n\pi} [\cos t]dt + \int_{2n\pi}^x [\cos t]dt$$

$$= n\int_0^{2\pi} [\cos t]dt + \int_{2n\pi}^x [\cos t]dt$$

$$= -n\pi + \int_{2n\pi}^x 0 dt$$

$$= -n\pi$$

ILLUSTRATION 8.85

Let f be a real-valued function satisfying $f(x) + f(x+4) = f(x+2) + f(x+6)$.

Prove that $\int_x^{x+8} f(t)dt$ is a constant function.

Sol. Given that $f(x) + f(x+4) = f(x+2) + f(x+6)$ (1)

Replacing x by $x + 2$, we get

$$f(x+2) + f(x+6) = f(x+4) + f(x+8) \quad (2)$$

From equations (1) and (2), we get $f(x) = f(x + 8)$ (3)

or $$\int_x^{x+8} f(t)dt = \int_0^8 f(t)dt$$

Thus, g is a constant function.

ILLUSTRATION 8.86

A periodic function with period 1 is integrable over any finite interval. Also, for two real numbers a, b and for two unequal non-zero positive integers m and n, $\int_a^{a+n} f(x)dx = \int_b^{b+m} f(x)\,dx$.

Calculate the value of $\int_m^n f(x)dx$.

Sol. Given $f(1 + x) = f(x)$

$$\therefore \quad \int_a^{a+n} f(x)\, dx = n\int_0^1 f(x)\, dx \qquad [\because f(x) \text{ is periodic}]$$

Similarly, $\int_b^{b+m} f(x)\, dx = m\int_0^1 f(x)dx$

Given $\int_a^{a+n} f(x)\,dx = \int_b^{b+m} f(x)\,dx$

or $n\int_0^1 f(x)\,dx = m\int_0^1 f(x)dx$

or $(n-m)\int_0^1 f(x)\,dx = 0$

or $\int_0^1 f(x)\,dx = 0 \quad (\because n \neq m)$ (1)

$\therefore$ $\int_m^n f(x)\,dx = \int_0^{n-m} f(m+x)\,dx = \int_0^{n-m} f(x)\,dx$

($\because f$ is periodic)

$= (n-m)\int_0^1 f(x)\,dx$ (Assume $n > m$)

$= 0$ [From equation (1)]

CONCEPT APPLICATION EXERCISE 8.8

1. Evaluate $\int_0^{100}(x-[x])dx$ (where $[\cdot]$ represents the greatest integer function).
2. Evaluate $\int_0^{100\pi}\sqrt{(1-\cos 2x)}dx$.
3. If $\int_0^{n\pi} f(\cos^2 x)dx = k\int_0^{\pi} f(\cos^2 x)dx$, $n \in N$ then find the value k.
4. Evaluate $\int_0^{n\pi+t}(|\cos x|+|\sin x|)dx$, where $n \in N$ and $t \in [0, \pi/2]$.
5. Find the value of $\int_0^{10} e^{2x-[2x]}\, d(x-[x])$ (where [.] denotes the greatest integer function).
6. If $f(x)$ is a function satisfying $f(x+a)+f(x)=0$ for all $x \in R$ and positive constant a such that $\int_b^{c+b} f(x)dx$ is independent of b, then find the least positive value of c.
7. Show that $\int_0^{n\pi+v}|\sin x|dx = 2n+1-\cos v$, where n is a positive integer and $0 \le v < \pi$.

ANSWERS

1. 50	2. $200\sqrt{2}$	3. n
4. $4n+\sin t-\cos t+1$	5. $10(e-1)$	6. $2a$
7. $2n+1-\cos v$		

LEIBNIZ'S RULE (DIFFERENTIATION UNDER INTEGRAL SIGN)

If f is continuous function in $[a, b]$ and $u(x)$ and $v(x)$ are differentiable functions of x then

$$\frac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = f(v(x))\frac{dv(x)}{dx} - f(u(x))\frac{du(x)}{dx}$$

Proof:

Let $\dfrac{d}{dx}F(x) = f(x)$

$\Rightarrow \int_{u(x)}^{v(x)} f(t)\,dt = F(v(x)) - F(u(x))$

$\Rightarrow \dfrac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = \dfrac{d}{dx}(F(v(x)) - F(u(x)))$

$\Rightarrow \dfrac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = F'(v(x))\dfrac{d(v(x))}{dx} - F'(u(x))\dfrac{d(u(x))}{dx}$

$\Rightarrow \dfrac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = f(v(x))\dfrac{d(v(x))}{dx} - f(u(x))\dfrac{d(u(x))}{dx}$

This is called Leibniz's rule. This is a rule for differentiation under integral sign.

ILLUSTRATION 8.87

If $y = \int_{x^2}^{x^3}\dfrac{1}{\log t}dt\ (x > 0)$, then find $\dfrac{dy}{dx}$

Sol. $y = \int_{x^2}^{x^3}\dfrac{1}{\log t}dt$

$$\therefore \frac{dy}{dx} = \frac{1}{\log x^3}\frac{d}{dx}(x^3) - \frac{1}{\log x^2}\frac{d}{dx}(x^2)$$

$$= \frac{3x^2}{3\log x} - \frac{2x}{2\log x}$$

$$= \frac{x^2-x}{\log x}$$

ILLUSTRATION 8.88

If $\int_0^y \cos t^2 dt = \int_0^{x^2}\dfrac{\sin t}{t}dt$, then prove that $\dfrac{dy}{dx} = \dfrac{2\sin x^2}{x\cos y^2}$.

Sol. Given that $\int_0^y \cos t^2\, dt = \int_0^{x^2}\dfrac{\sin t}{t}dt$

Differentiating w.r.t. x, we get

$$\cos y^2\frac{dy}{dx} = \frac{\sin x^2}{x^2}2x$$

or $$\frac{dy}{dx} = \frac{2\sin x^2}{x\cos y^2}$$

ILLUSTRATION 8.89

If $x = \int_0^y \dfrac{dt}{\sqrt{1+9t^2}}$ and $\dfrac{d^2y}{dx^2} = ay$, then find a.

Sol. $x = \int_0^y \dfrac{dt}{\sqrt{1+9t^2}}$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{\sqrt{1+9y^2}}$$

or $$\frac{dy}{dx} = \sqrt{1+9y^2}$$

or $$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\sqrt{1+9y^2}\right)\frac{dy}{dx}$$

or $$\frac{d^2y}{dx^2} = \frac{18y}{2\sqrt{1+9y^2}}\sqrt{1+9y^2} = 9y$$

or $a = 9$

ILLUSTRATION 8.90

If $\int_{\sin x}^{1} t^2 f(t)dt = 1 - \sin x,\ x \in \left(0, \frac{\pi}{2}\right)$ then find the value of $f\left(\frac{1}{\sqrt{3}}\right)$.

Sol. We have $\int_{\sin x}^{1} t^2 f(t)dt = 1 - \sin x$

Differentiating both sides w.r.t. x, we get

$$1^2 \times f(1) \times 0 - \sin^2 x \times f(\sin x) \times \cos x = -\cos x$$

$$\Rightarrow \quad f(\sin x) = \frac{1}{\sin^2 x}$$

$$\therefore \quad f\left(\frac{1}{\sqrt{3}}\right) = 3.$$

ILLUSTRATION 8.91

Let $f: R \to R$ be a differentiable function having $f(2) = 6, f'(2) = \frac{1}{48}$. Then evaluate $\lim_{x\to 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt$.

Sol. $\lim_{x\to 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt = \lim_{x\to 2} \frac{\int_6^{f(x)} 4t^3\, dt}{x-2}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to 2} \frac{4\left(f(x)\right)^3 f'(x)}{1}$$

(applying L'Hopital rule)

$$= 4(f(2))^3 \times f'(2)$$

$$= 4(6)^3 \times \frac{1}{48}$$

$$= 18$$

ILLUSTRATION 8.92

Evaluate $\lim_{x\to\infty} \frac{\left(\int_0^x e^{x^2} dx\right)^2}{\int_0^x e^{2x^2} dx}$.

Sol. Since $e^{x^2} > 0$, $e^{2x^2} > 0$ in $[0, x]$, where $x > 0$,

$\int_0^x e^{x^2} dx$ and $\int_0^x e^{2x^2} dx \to \infty$ as $x \to \infty$

$L = \lim_{x\to\infty} \frac{\left(\int_0^x e^{x^2}\right)^2}{\int_0^x e^{2x^2}}$ is of the form $\frac{\infty}{\infty}$.

Therefore, using L' Hopital's rule

$$L = \lim_{x\to\infty} \frac{2e^{x^2} \int_0^x e^{x^2} dx}{e^{2x^2}}$$

$$= 2 \lim_{x\to\infty} \frac{\int_0^x e^{x^2} dx}{e^{x^2}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= 2 \lim_{x\to\infty} \frac{e^{x^2}}{2xe^{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0$$

ILLUSTRATION 8.93

Prove that $y = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}$, where $0 \le x \le \pi/2$, is the equation of a straight line parallel to the x-axis. Find its equation.

Sol. Here, we have to prove that y = constant or derivative of y w.r.t. x is zero.

$$y = \int_{\frac{1}{8}}^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt,\ dt + \int_{\frac{1}{8}}^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt \qquad (1)$$

$$\frac{dy}{dx} = \sin^{-1}\sqrt{\sin^2 x}\ 2 \sin x \cos x$$

$$+ \cos^{-1}\sqrt{\cos^2 x}\ (-2 \cos x \sin x)$$

$$= 2x \sin x \cos x - 2x \sin x \cos x$$

$$= 0 \text{ for all } x$$

Therefore, the curve in equation (1) is a straight line parallel to the x-axis.

Now, since y is constant, it is independent of x. So let us select $x = \pi/4$. Then

$$y = \int_{1/8}^{1/2} \sin^{-1}\sqrt{t}\, dt + \int_{1/8}^{1/2} \cos^{-1}\sqrt{t}\, dt$$

$$= \int_{1/8}^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\, dt$$

$$= \int_{1/8}^{1/2} \pi/2\, dt$$

$$= \frac{\pi}{2}\left[\frac{1}{2} - \frac{1}{8}\right]$$

$$= \frac{3\pi}{16}$$

Therefore, equation of the line is $y = \frac{3\pi}{16}$.

ILLUSTRATION 8.94

If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, then find the interval in which $f(x)$ increases.

Sol. We have $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$

Differentiating w.r.t. x, we get

$$\Rightarrow \quad f'(x) = e^{-(x^2+1)^2} . 2x - e^{-x^4} . 2x$$

$$= 2x\left[e^{-(x^2+1)^2} - e^{-x^4}\right]$$

Now, $(x^2 + 1)^2 > x^4$

$\therefore \quad -(x^2 + 1)^2 < -x^4$

$\Rightarrow \quad e^{-(x^2+1)^2} < e^{-x^4}$

$\Rightarrow \quad e^{-(x^2+1)^2} - e^{-x^4} < 0$

$\therefore \quad f'(x) \geq 0, \forall\, x \leq 0$

$\therefore \quad f(x)$ increases when $x \leq 0$

ILLUSTRATION 8.95

Let $f: (0, \infty) \to (0, \infty)$ be a differentiable function satisfying, $x\int_0^x (1-t)f(t)dt = \int_0^x tf(t)dt\ x \in R^+$ and $f(1) = 1$. Determine $f(x)$.

Sol. We have $x\int_0^x (1-t)f(t)dt = \int_0^x tf(t)dx$

Differentiating both sides w.r.t. x, we get

$$x(1-x)f(x) + \int_0^x (1-t)f(t)dt = xf(x)$$

or $$x^2 f(x) = \int_0^x (1-t)f(t)dt$$

Differentiating both sides w.r.t. x again, we get

$$x^2 f'(x) + 2x f(x) = (1-x)f(x)$$

or $$\frac{f'(x)}{f(x)} = \frac{1-3x}{x^2}$$

or $$\int \frac{f'(x)}{f(x)}dx = \int \frac{1-3x}{x^2}dx$$

or $$\log f(x) = -\frac{1}{x} - 3\log x + \log c$$

or $$\log\left[\frac{f(x)}{c}\right] = \frac{-1}{x} - 3\log x$$

Given $f(1) = 1$ or $\log\left(\frac{1}{c}\right) = -1$ or $c = e$

$\therefore$ $$\log\left(\frac{f(x)x^3}{e}\right) = -\frac{1}{x}$$

or $$f(x) = \frac{1}{x^3}e^{\left(1-\frac{1}{x}\right)}$$

LEIBNIZ'S RULE WHEN INTEGRAND IS $f(x, t)$

Consider $f(x) = \int_0^x e^{x-t}\,dt$. What is $f'(x)$?

Here $f'(x) = e^{x-x}$ is certainly wrong.

as $$f(x) = \int_0^x e^{x-t}\,dt$$

$$= e^x \int_0^x e^{-t}\,dt$$

$$= e^x\left[-e^{-t}\right]_0^x$$

$$= e^x\left[-e^{-x}+1\right]$$

$$= -1 + e^x.$$

$\therefore \quad f'(x) = e^x.$

But if integrand cannot be integrated then we use following method.

$$f(x) = e^x \int_0^x e^{-t}\,dt$$

$\therefore$ $$f'(x) = e^x \int_0^x e^{-t}\,dt + e^x \times e^{-x}$$

$$= e^x \int_0^x e^{-t}\,dt + 1 \text{ etc.}$$

Now consider $f(x) = \int_0^x e^t f(x-t)dt$.

Before differentiating we do the following simplification.

$$f(x) = \int_0^x e^{x-t} f(x-(x-t))dt$$

(Using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$)

$\therefore$ $$f(x) = e^x \int_0^x e^{-t} f(t)dt$$

Now we can differentiate w.r.t. x,

$\therefore$ $$f'(x) = e^x \int_0^x e^{-t} f(t)dt + e^x e^{-x} f(x)$$

or $$f'(x) = e^x \int_0^x e^{-t} f(t)dt + f(x) \text{ etc.}$$

ILLUSTRATION 8.96

Let $f: R \to (0, \infty)$ be a real valued function satisfying $\int_0^x tf(x-t)dt = e^{2x} - 1$, then find $f(x)$.

Sol. $$\int_0^x tf(x-t)dt = e^{2x} - 1$$

$\Rightarrow$ $$\int_0^x (x-t)\, f(t)\, dt = e^{2x} - 1$$

$\Rightarrow$ $$x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt = e^{2x} - 1$$

Differentiating both sides w.r.t. x, we get

$$xf(x) + \int_0^x f(t)\,dt - xf(x) = 2e^{2x}$$

$\Rightarrow$ $$\int_0^x f(t)dt = 2e^{2x}$$

$\Rightarrow$ $$f(x) = 4e^{2x}$$

ILLUSTRATION 8.97

Let $f: R \to R$ be a differentiable function satisfying

$f(x) = x^2 + 3\int_0^{x^{1/3}} e^{-t^3} t^2 \cdot f(x-t^3)\,dt$. Then find $f(x)$.

Sol. $f(x) = x^2 + 3\int_0^{x^{1/3}} e^{-t^3} t^2 f(x-t^3)\,dt$ ...(1)

Put $x - t^3 = z$ in second term

$\therefore \quad f(x) = x^2 + \int_0^x e^{z-x} f(z)\,dz$

$= x^2 + e^{-x}\int_0^x e^{z} f(z)\,dz$

Differentiating w.r.t. x we get

$f'(x) = 2x + e^{-x}e^{x} f(x) - e^{-x}\int_0^x e^{z} f(z)\,dz$

$\therefore \quad f'(x) = 2x + x^2$

$\therefore \quad f(x) = \dfrac{x^3}{3} + x^2 + c$

From (1), $f(0) = 0$

$\therefore \quad f(x) = \dfrac{x^3}{3} + x^2$

ILLUSTRATION 8.98

If $y = \int_0^x f(t)\sin\{k(x-t)\}\,dt$, then prove that $\dfrac{d^2y}{dx^2} + k^2 y = k f(x)$.

Sol. $y = \int_0^x f(t)\sin\{k(x-t)\}\,dt$

$= \int_0^x f(t)[\sin kx\cos kt - \sin kt\cos kx]\,dt$

$= \sin kx\int_0^x f(t)\cos kt\,dt - \cos kx\int_0^x f(t)\sin kt\,dt$ (1)

$\therefore \quad \dfrac{dy}{dx} = k\cos kx\int_0^x f(t)\cos kt\,dt + \sin kx\,[f(x)\cos kx]$

$+ k\sin kx\int_0^x f(t)\sin kt\,dt - \cos kx\,[f(x)\sin kx]$

$= k\cos kx\int_0^x f(t)\cos kt\,dt + k\sin kx\int_0^x f(t)\sin kt\,dt$ (2)

Again differentiating equation (2) w.r.t. x, we get

$\dfrac{d^2y}{dx^2} = -k^2\sin kx\int_0^x f(t)\cos kt\,dt + k\cos kx\,[f(x)\cos kx]$

$+ k^2\cos kx\int_0^x f(t)\sin tdt + k\sin kx\,[f(x)\sin kx]$

$= -k^2 y + kf(x)$

$\therefore \quad \dfrac{d^2y}{dx^2} + k^2 y = k f(x)$

CONCEPT APPLICATION EXERCISE 8.9

1. If $\int_{\pi/3}^{x}\sqrt{(3-\sin^2 t)}\,dt + \int_0^y \cos t\,dt = 0$, then evaluate $\dfrac{dy}{dx}$.
2. If $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \dfrac{t\,dt}{1+t^4}$, then find the value of $f'(2)$.
3. Evaluate $\lim\limits_{x\to 4}\int_4^x \dfrac{(4t - f(t))}{(x-4)}\,dt$.
4. Evaluate $\lim\limits_{x\to 0}\dfrac{\int_0^x \cos t^2\,dt}{x}$.
5. Find the points of minima for $f(x) = \int_0^x t(t-1)(t-2)\,dt$.
6. Find the equation of tangent to $y = \int_{x^2}^{x^3}\dfrac{dt}{\sqrt{1+t^2}}$ at $x = 1$.
7. If $f(x) = \int_{\pi^2/16}^{x^2}\dfrac{\sin x\sin\sqrt{\theta}}{1+\cos^2\sqrt{\theta}}\,d\theta$, then find the value of $f'\left(\dfrac{\pi}{2}\right)$.
8. Let $f(x)$ be a continuous and differentiable function such that $f(x) = \int_0^x \sin(t^2 - t + x)\,dt$. Then prove that $f''(x) + f(x) = \cos x^2 + 2x\sin x^2$.
9. Let $f(x)$ be a differentiable function satisfying $f(x) = \int_0^x e^{(2tx - t^2)}\cos(x-t)\,dt$, then find the value of $f''(0)$.

ANSWERS

1. $\dfrac{-\sqrt{3-\sin^2 x}}{\cos y}$	2. 2/17	3. $16 - f(4)$
4. 1	5. $x = 0, 2$	6. $\sqrt{2}y = x - 1$
7. π	9. 0	

DEFINITE INTEGRATION BY TYPICAL SUBSTITUTION

We know that in evaluating the integrals main idea is substitution. It is the substitution which transform the integrand to simple function and in definite integration with the help of the properties, transforms the integrand to integrable function. Also in many cases using substitution we can find the value of the integral in the form of the given integral. Let study different techniques of substitution for different aims with the following illustrations.

ILLUSTRATION 8.99

Prove that $\int_0^x e^{xt}e^{-t^2}\,dt = e^{x^2/4}\int_0^x e^{-t^2/4}\,dt$.

Sol. Let $I = \int_0^x e^{xt}e^{-t^2}\,dt$

$= e^{x^2/4}\int_0^x e^{-x^2/4}e^{xt}e^{-t^2}\,dt$

$$= e^{x^2/4}\int_0^x e^{-(x^2/4 - tx + t^2)}dt$$

$$= e^{x^2/4}\int_0^x e^{-(x/2 - t)^2}dt$$

The result clearly suggests that we have to substitute $y/2$ for $x/2 - t$. Then $dt = -dy/2$. Also, when $t = 0$, $y = x$, and when $t = x$, $y = -x$. Thus,

$$I = e^{x^2/4}\int_x^{-x} e^{-y^2/4}(-dy/2)$$

$$= \frac{e^{x^2/4}}{2}\int_{-x}^{x} e^{-y^2/4}dy$$

$$= \frac{e^{x^2/4}}{2}\, 2\int_0^x e^{-y^2/4}dy \qquad [e^{-y^2/4} \text{ is an even function}]$$

$$= e^{x^2/4}\int_0^x e^{-t^2/4}dt$$

ILLUSTRATION 8.100

Evaluate $\int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}dx$.

Sol. $I = \int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}dx$

$$= \int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{1/3}^{2/3} e^{(3x-2)^2}dx$$

$$= I_1 + I_2$$

Note that in both I_1 and I_2, function has same form, i.e., e^{t^2}.

Also, e^{t^2} is non-integrable.

Now, in I_1, let $x + 5 = y$ and in I_2, let $3x - 2 = -t$. Then

$$I = \int_1^0 e^{y^2}dy + \int_1^0 e^{t^2}(-dt) = 0.$$

ILLUSTRATION 8.101

Compute the following integrals.

(i) $\int_0^\infty f\left(x^n + x^{-n}\right)\log x\,\frac{dx}{x}$

(ii) $\int_0^\infty f\left(x^n + x^{-n}\right)\log x\,\frac{dx}{1+x^2}$

(iii) $\int_{1/e}^{e} \frac{1}{x}\sin\left(x - \frac{1}{x}\right)dx$

Sol. Here, limits (reciprocal) and type of functions (reciprocal terms are present, i.e., x and $1/x$) suggest that we must substitute $1/t$ for x.

(i) Let $t = 1/x$ or $x = 1/t$. So, $dx = -\frac{1}{t^2}dt$.

Also, when $x \to 0$, $t \to \infty$; when $x \to \infty$, $t \to 0$. Thus,

$$I = \int_0^\infty f\left(x^n + x^{-n}\right)\ln x\frac{dx}{x}$$

$$= \int_\infty^0 f\left(t^{-n} + t^n\right)\ln\left(\frac{1}{t}\right)\frac{-\frac{dt}{t^2}}{\frac{1}{t}}$$

$$= -\int_0^\infty f\left(t^n + t^{-n}\right)\ln(t)\frac{dt}{t}$$

$$= -I$$

or $\quad 2I = 0$ or $I = 0$

(ii) Let $I = \int_0^\infty f\left(x^n + x^{-n}\right)\ln x\,\frac{dx}{1+x^2}$

Let $t = 1/x$ or $x = 1/t$. So, $dx = -\frac{1}{t^2}dt$.

Also, when $x \to 0$, $t \to \infty$; when $x \to \infty$, $t \to 0$. Thus,

$$I = \int_0^\infty f\left(x^n + x^{-n}\right)\ln x\,\frac{dx}{1+x^2}$$

$$= \int_\infty^0 f\left(t^{-n} + t^n\right)\ln\left(\frac{1}{t}\right)\frac{-\frac{dt}{t^2}}{1+\frac{1}{t^2}}$$

$$= -\int_0^\infty f\left(t^n + t^{-n}\right)\ln(t)\,\frac{dt}{1+t^2}$$

$$= -I$$

or $2I = 0$ or $I = 0$

(iii) $I = \int_{1/e}^{e} \frac{1}{x}\sin\left(x - \frac{1}{x}\right)dx$

Put $x = \frac{1}{t}$; $dx = -\frac{1}{t^2}dt$. Thus,

$$I = \int_e^{1/e} t\sin\left(\frac{1}{t} - t\right)\left(-\frac{1}{t^2}\right)dt$$

$$= \int_e^{1/e}\frac{1}{t}\sin\left(t - \frac{1}{t}\right)dt$$

$$= -\int_{1/e}^{e}\frac{1}{t}\sin\left(t - \frac{1}{t}\right)dt$$

or $\quad I = -I$ or $2I = 0$ or $I = 0$

ILLUSTRATION 8.102

Let $A = \int_0^\infty \frac{\log x}{1+x^3}dx$. Then find the value of $\int_0^\infty \frac{x\log x}{1+x^3}dx$ in terms of A.

Sol. $B = \int_0^\infty \frac{x\log x}{1+x^3}dx$

$$= \int_0^\infty \frac{(x+1)\log x - \log x}{1+x^3}dx$$

$$= \int_0^\infty \frac{\log x}{x^2 - x + 1}dx - A$$

Let $\quad I = \int_0^\infty \frac{\log x}{x^2 - x + 1}dx$

Put $\quad x = \frac{1}{t}$

$\therefore \quad I = \int_{\infty}^{0} \dfrac{\log \frac{1}{t}}{\frac{1}{t^2} - \frac{1}{t} + 1}\left(\dfrac{-dt}{t^2}\right)$

$= -\int_{0}^{\infty} \dfrac{\log t}{t^2 - t + 1} dt$

$= -I$

or $\quad 2I = 0$

or $\quad I = 0$

$\therefore \quad B = -A$

ILLUSTRATION 8.103

If $\int_0^1 \dfrac{\sin t}{1+t} dt = \alpha$, then find the value of $\int_{4\pi-2}^{4\pi} \dfrac{\sin \frac{t}{2}}{4\pi + 2 - t} dt$

Sol. $I = \dfrac{1}{2}\int_{4\pi-2}^{4\pi} \dfrac{\sin \frac{t}{2}}{1 + \left(2\pi - \frac{t}{2}\right)} dt$

Put $\quad 2\pi - \dfrac{t}{2} = z$

$\therefore \quad -\dfrac{1}{2} dt = dz$ or $dt = -2dz$

When $t = 4\pi - 2$, $z = 2\pi - 2\pi + 1 = 1$

when $t = 4\pi$, $z = 2\pi - 2\pi = 0$

$\therefore \quad I = \dfrac{1}{2}\int_1^0 \dfrac{\sin(2\pi - z)(-2dz)}{1+z}$

$= \int_0^1 \dfrac{-\sin z\, dz}{z+1}$

$= -\int_0^1 \dfrac{\sin t}{1+t} dt$

$= -\alpha$

ILLUSTRATION 8.104

Prove that $\int_0^1 \dfrac{\tan^{-1} x}{x} dx = \dfrac{1}{2}\int_0^{\frac{\pi}{2}} \dfrac{x}{\sin x} dx$.

Sol. $I = \int_0^1 \dfrac{\tan^{-1} x}{x} dx$

Putting $x = \tan \theta$,

$\therefore \quad dx = \sec^2 \theta \, d\theta$

$\therefore \quad I = \int_0^{\pi/4} \dfrac{\theta}{\tan \theta} . \sec^2 \theta d\theta$

$= \int_0^{\pi/4} \dfrac{2\theta}{\sin 2\theta} d\theta$

Putting $2\theta = t$, $\therefore 2d\theta = dt$,

$\therefore \quad I = \int_0^{\pi/2} \dfrac{t}{\sin t} \cdot \dfrac{dt}{2}$

$= \dfrac{1}{2}\int_0^{\pi/2} \dfrac{x}{\sin x} dx$

ILLUSTRATION 8.105

For $x > 0$, let $f(x) = \int_1^x \dfrac{\log t}{1+t} dt$. Find the function $f(x) + f\left(\dfrac{1}{x}\right)$ and find the value of $f(e) + f\left(\dfrac{1}{e}\right)$.

Sol. Let $F(x) = f(x) + f\left(\dfrac{1}{x}\right)$

$= \int_1^x \dfrac{\log t}{1+t} dt + \int_1^{1/x} \dfrac{\log t}{1+t} dt$

In second integral, let $t = 1/y$. So, $dt = -\dfrac{1}{y^2} dy$

$\therefore \quad F(x) = \int_1^x \dfrac{\log t}{1+t} dt + \int_1^x \dfrac{-\log y}{1 + \frac{1}{y}}\left(-\dfrac{dy}{y^2}\right)$

$= \int_1^x \dfrac{\log t}{1+t} dt + \int_1^x \dfrac{\log y}{y(1+y)} dy$

$= \int_1^x \dfrac{\log t}{1+t} dt + \int_1^x \dfrac{\log t}{t(1+t)} dt$

$= \int_1^x \dfrac{\log t}{t} dt = \dfrac{1}{2}(\log x)^2$

$\therefore \quad F(e) = \dfrac{1}{2}$

CONCEPT APPLICATION EXERCISE 8.10

1. If $\int_0^1 \dfrac{e^t dt}{t+1} = a$, then evaluate $\int_{b-1}^{b} \dfrac{e^{-t} dt}{t-b-1}$.

2. If $f(x) = \int_1^x \dfrac{\log t}{1+t+t^2} dt \;\forall\, x \geq 1$, then prove that $f(x) = f\left(\dfrac{1}{x}\right)$.

3. If $f(x) = \int_1^x \dfrac{\tan^{-1}(t)}{t} dt \;\forall\, x \in R^+$, then find the value of $f(e^2) - f\left(\dfrac{1}{e^2}\right)$.

4. Evaluate $\int_{\sqrt{2}-1}^{\sqrt{2}+1} \dfrac{(x^2-1)}{(x^2+1)^2} dx$.

5. Evaluate $\int_0^{e-1} \dfrac{e^{\frac{x^2+2x-1}{2}}}{x+1} dx + \int_1^e x \log x e^{\frac{x^2-2}{2}} dx$.

6. Find the value of $\int_{\frac{1}{2}}^{2} e^{\left|x - \frac{1}{x}\right|} dx$

7. If $I_1 = \int_0^1 \frac{dx}{e^x(1+x)}$ and $I_2 = \int_0^{\pi/4} \frac{e^{\tan^2\theta}\sin\theta}{(2-\tan^2\theta)\cos^3\theta}d\theta$, then find the value of $\frac{I_1}{I_2}$

ANSWERS

1. $-ae^{-b}$ 3. π 4. 0 5. $e^{\frac{e^2-2}{2}}$

6. $e\sqrt{e}-1$ 7. $2/e$

DEFINITE INTEGRATION BY REDUCTION FORMULA

Integration by reduction formula is a technique of integration, in the form of a recurrence relation. It is used when an expression containing an integer parameter, usually in the form of powers of elementary functions, or products of transcendental functions and polynomials of arbitrary degree, can't be integrated directly. But using different methods of integration a reduction formula can be used to obtain the integral of the same or similar expression with a lower integer parameter, progressively simplifying the integral until it can be evaluated.

ILLUSTRATION 8.106

Determine a positive integer n such that $\int_0^{\pi/2} x^n \sin x dx = \frac{3}{4}(\pi^2 - 8)$.

Sol. Let $I_n = \int_0^{\pi/2} x^n \sin x\, dx$

Integrate by parts and choose $\sin x$ as the second function.

Therefore, $I_n = \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1}(-\cos x)dx$

$$= 0 + n\int_0^{\pi/2} x^{n-1} \cos x\, dx$$

Again integrating by parts, we get

$$I_n = n\left\{x^{n-1}\sin x\right\}_0^{\pi/2} - n(n-1)\int_0^{\pi/2} x^{n-2}\sin x\, dx$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, I_{n-2}$$

R.H.S. contains π^2. Therefore, put $n = 3$.

$$\therefore \quad I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3\times 2I_1$$

$$= \frac{3\pi^2}{4} - 6\int_0^{\pi/2} x \sin x\, dx$$

$$= \frac{3\pi^2}{4} - 6\left\{x(-\cos x) + \sin x\right\}_0^{\pi/2}$$

$$= \frac{3\pi^2}{4} - 6\{1\}$$

$$= \frac{3}{4}(\pi^2 - 8) \text{ which is true}$$

Hence, $n = 3$.

ILLUSTRATION 8.107

Determine a positive integer $n \le 5$ such that $\int_0^1 e^x (x-1)^n = 16 - 6e$.

Sol. Given $\int_0^1 e^x (x-1)^n\, dx = 16 - 6e$

where $n \in N$ and $n \le 5$.

To find the value of n, let

$$I_n = \int_0^1 e^x (x-1)^n\, dx$$

$$= [(x-1)^n e^x]_0^1 - \int_0^1 n(x-1)^{n-1} e^x dx$$

$$= -(-1)^n - \int_0^1 n(x-1)^{n-1} e^x dx$$

or $\quad I_n = (-1)^{n+1} - nI_{n-1}$. (1)

Also, $I_1 = \int_0^1 e^x (x-1)\, dx$

$$= \left[e^x(x-1)\right]_0^1 - \int_0^1 e^x\, dx$$

$$= -(-1) - (e^x)_0^1$$

$$= 1 - (e-1) = 2 - e$$

Using equation (1), we get

$$I_2 = (-1)^3 - 2I_1 = -1 - 2(2-e) = 2e - 5$$

Similarly, $I_3 = (-1)^4 - 3I_2 = 1 - 3(2e-5)$

$$= 16 - 6e$$

Thus, $n = 3$

ILLUSTRATION 8.108

Prove that $I_n = \int_0^\infty x^{2n+1} e^{-x^2} dx = \frac{n!}{2}, n \in N$.

Sol. $I_n = \int_0^\infty (x^2)^n x e^{-x^2} dx$

Put $x^2 = t$ or $x\, dx = dt/2$

$$\therefore \quad I_n = \frac{1}{2}\int_0^\infty t^n e^{-t} dt$$

$$= \frac{1}{2}\left[\left[-t^n e^{-t}\right]_0^\infty + n\int_0^\infty t^{n-1} e^{-t} dt\right]$$

$$= \frac{1}{2}\left[0 + n\int_0^\infty t^{n-1} e^{-t} dt\right]$$

$$= \frac{n}{2}\int_0^\infty t^{n-1} e^{-t} dt = n\, I_{n-1}$$

or $\quad I_{n-1} = (n-1)\, I_{n-2}$

or $\quad I_n = n(n-1)(n-2)\ldots 1 I_0$

$$= n!\, I_0 = n!\,\frac{1}{2}\int_0^\infty e^{-t} dt$$

$$= n!\,\frac{1}{2}\left[-e^{-t}\right]_0^\infty = \frac{n!}{2}$$

ILLUSTRATION 8.109

If $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$; $n \in N$, then prove that $2n\, I_{n+1} = 2^{-n} + (2n-1)\, I_n$.

Sol. $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$

$$= \left|\frac{1}{(1+x^2)^n}\cdot x\right|_0^1 - \int_0^1 -n(1+x^2)^{-n-1}2x\cdot x\,dx$$

$$= \frac{1}{2^n} + n\int_0^1 \frac{2x^2}{(1+x^2)^{n+1}}dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}}dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{dx}{(1+x^2)^n} - 2n\int_0^1 \frac{dx}{(1+x^2)^{n+1}}$$

$$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$$

or $\quad (2n-1)I_n + \frac{1}{2^n} = 2n\, I_{n+1}$

ILLUSTRATION 8.110

If $I_n = \int_0^{\pi/2} \sin^n x\, dx$, then show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$.

Hence, prove that

$$I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{1}{2}\right)\frac{\pi}{2} & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{2}{3}\right)1 & \text{if } n \text{ is odd} \end{cases}$$

Sol. $I_n = \int_0^{\pi/2} \sin^n x\, dx$

$$= \int_0^{\pi/2} \sin^{n-1} x \sin x\, dx$$

$$= \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\pi/2} (n-1)\sin^{n-2} x \cos^2 x\, dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,(1-\sin^2 x)\, dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\, dx - (n-1)\int_0^{\pi/2} \sin^n x\, dx$$

or $\quad I_n + (n-1)\, I_n = (n-1)\, I_{n-2}$

or $\quad I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

$$= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots I_0 \text{ or } I_1$$

Accordingly, if n is even or odd,

$$I_0 = \frac{\pi}{2},\ I_1 = 1$$

Hence,

$$I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{1}{2}\right)\frac{\pi}{2} & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{2}{3}\right)1 & \text{if } n \text{ is odd.} \end{cases}$$

CONCEPT APPLICATION EXERCISE 8.11

1. If $I_k = \int_1^e (\ln x)^k\, dx\,(k \in I^+)$, then find the value of I_4.

2. Given $I_m = \int_1^e (\log x)^m dx$, then prove that $\frac{I_m}{1-m} + mI_{m-2} = e$

3. If $I_n = \int_0^\pi x^n \sin x dx$, then find the value of $I_5 + 20I_3$.

4. If $L(m, n) = \int_0^1 t^m\,(1+t)^n\, dt$, then prove that
$$L(m, n) = \frac{2^n}{m+1} - \frac{n}{m+1}\, L(m+1, n-1)$$

5. If $I_n = \int_0^1 x^n(\tan^{-1} x)dx$, then prove that $(n+1)I_n + (n-1)$
$$I_{n-2} = -\frac{1}{n} + \frac{\pi}{2}.$$

6. If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\, dx$, then show that
$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,\,n} \qquad (m, n \in N)$$
Hence, prove that
$$I_{m,n} = \begin{cases} \frac{(m-1)(m-3)(m-5)\cdots(n-1)(n-3)(n-5)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots 2}\,\frac{\pi}{2} & \text{when both } m \text{ and } n \text{ are even} \\ \frac{(m-1)(m-3)(m-5)\cdots(n-1)(n-3)(n-5)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots}. & \text{otherwise} \end{cases}$$

ANSWERS

1. $9e - 24$ 3. π^5

Solved Examples

EXAMPLE 8.1

f, g, h are continuous in $[0, a]$, $f(a-x) = f(x)$, $g(a-x) = -g(x)$, $3h(x) - 4h(a-x) = 5$. Then prove that $\int_0^a f(x)\,g(x)\,h(x)\,dx = 0$.

Sol. $I = \int_0^a f(x)g(x)h(x)dx$

$$= \int_0^a f(a-x)g(a-x)h(a-x)dx$$

$$= \int_0^a f(x)(-g(x))\left(\frac{3h(x)-5}{4}\right)dx$$

$$= -\frac{3}{4}\int_0^a f(x)g(x)h(x)dx + \frac{5}{4}\int_0^a f(x)g(x)dx$$

$$= -\frac{3}{4}I + \frac{5}{4}\int_0^a f(x)g(x)dx$$

$$\therefore \quad I = \frac{5}{7}\int_0^a f(x)g(x)dx$$

$$= \frac{5}{7}\int_0^a f(a-x)g(a-x)dx$$

$$= \frac{5}{7}\int_0^a f(x)(-g(x))dx$$

$$= -I$$

or $\quad 2I = 0$ or $I = 0$

EXAMPLE 8.2

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}dx$.

Sol. $I = \underbrace{\int_0^{\frac{\pi}{2}} \frac{\sin 3x - \sin x}{\sin x + \cos x}dx}_{I_1} + \underbrace{\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}dx}_{I_2}$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}dx$$

$$\therefore \quad I_2 = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}dx$$

Adding, we get

$$2I_2 = \frac{\pi}{2}$$

$$\Rightarrow \quad I_2 = \frac{\pi}{4}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{2(\cos 2x)\sin x}{\sin x + \cos x}dx$$

$$= 2\int_0^{\frac{\pi}{2}} \sin x(\cos x - \sin x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin 2x\,dx - \int_0^{\frac{\pi}{2}} 2\sin^2 x\,dx$$

$$= \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= 1 - \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 1 - \frac{\pi}{4}$$

Therefore, $I = 1$

EXAMPLE 8.3

Given a function $f: [0, 4] \to R$ is differentiable, then prove that for some $\alpha, \beta \in (0, 2)$, $\int_0^4 f(t)dt = 2\alpha f(\alpha^2) + 2\beta f(\beta^2)$.

Sol. $I = \int_0^4 f(t)dt$, put $t = x^2$

$$\therefore \quad I = 2\int_0^2 xf(x^2)dx$$

Using LMVT, we have

$$\frac{\int_0^2 2xf(x^2)dx - \int_0^0 2xf(x^2)dx}{2-0} = 2yf(y^2) \text{ for some } y \in (0, 2)$$

$$\Rightarrow \quad \int_0^2 2xf(x^2)dx = 2.2yf(y^2) = 2\left\{\frac{2\alpha f(\alpha^2) + 2\beta f(\beta^2)}{2}\right\}$$

[where $0 < \beta < y < \alpha < 2$, and using intermediate Value Theorem]

$$\Rightarrow \quad \int_0^2 2xf(x^2)dx = 2\alpha f(\alpha^2) + 2\beta f(\beta^2)$$

EXAMPLE 8.4

Prove that $\int_0^\infty \frac{\sin^2 x}{x^2}dx = \int_0^\infty \frac{\sin x}{x}dx$.

Sol. $I = \int_0^\infty \sin^2 x \cdot \frac{1}{x^2}dx$

$$= \left|\sin^2 x\left(-\frac{1}{x}\right)\right|_0^\infty - \int_0^\infty 2\sin x\cos x\left(-\frac{1}{x}\right)dx$$

$$\lim_{x\to\infty} \frac{\sin^2 x}{x} = 0 \text{ and } \lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0}(\sin x)\lim_{x\to 0}\frac{\sin x}{x} = 0$$

$\therefore \quad I = 0 + \int_0^{\infty} \frac{\sin 2x}{x} dx$

$= \int_0^{\infty} \frac{\sin t}{t/2} \cdot \frac{dt}{2}$ (Putting $2x = t$)

$= \int_0^{\infty} \frac{\sin x}{x} dx$

Example 8.5

If $\int_0^{\pi/2} \log \sin \theta d\theta = k$, then find the value of $\int_0^{\pi/2} (\theta/\sin\theta)^2 d\theta$ in terms of k.

Sol. $I = \int_0^{\pi/2} \left(\frac{\theta}{\sin\theta}\right)^2 d\theta$

$= \int_0^{\pi/2} \theta^2 \operatorname{cosec}^2 \theta d\theta$

$= [\theta^2(-\cot\theta)]_0^{\pi/2} - \int_0^{\pi/2} 2\theta \cdot (-\cot\theta) d\theta$

(Integrating by parts)

$= [\lim_{\theta \to 0} \theta^2 \cdot \cot\theta] + 2\int_0^{\pi/2} \theta \cot\theta d\theta$

$= 0 + 2\left[[\theta \log \sin\theta]_0^{\pi/2} - \int_0^{\pi/2} \log \sin\theta d\theta\right]$

(Integrating by parts)

$= 2\left[-\lim_{\theta \to 0} \theta \ln \sin\theta - k\right]$

$= -2k$

Example 8.6

Evaluate $\int_0^{\pi} \frac{x \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} dx.$

Sol. Let $I = \int_0^{\pi} \frac{x \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} dx$ (1)

Then $I = \int_0^{\pi} \frac{(\pi - x) \sin(2\pi - 2x) \sin\left(\frac{\pi}{2} \cos(\pi - x)\right)}{2(\pi - x) - \pi} dx$

$= \int_0^{\pi} \frac{(\pi - x)(-\sin 2x) \sin\left(-\frac{\pi}{2} \cos x\right)}{\pi - 2x} dx,$

or $I = \int_0^{\pi} \frac{(x - \pi) \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} dx$ (2)

Adding (1) and (2), we get

$2I = \int_0^{\pi} \frac{(2x - \pi) \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} dx$

$= \int_0^{\pi} \sin 2x \sin\left(\frac{\pi}{2} \cos x\right) dx$

$= \int_0^{\pi} 2 \sin x \cos x \sin\left(\frac{\pi}{2} \cos x\right) dx$

or $I = \int_0^{\pi} \sin x \cos x \sin\left(\frac{\pi}{2} \cos x\right) dx$

Put $z = \frac{\pi}{2} \cos x$. Then $dz = -\frac{\pi}{2} \sin x\, dx$.

When $x = 0$, $z = \frac{\pi}{2}$, and when $x = \pi$, $z = -\frac{\pi}{2}$.

$\therefore \quad I = -\frac{2}{\pi} \int_{\pi/2}^{-\pi/2} \frac{2z}{\pi} \sin z\, dz = \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} z \sin z\, dz = \frac{8}{\pi^2}$

Example 8.7

Find the value of $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx.$

Sol. Let $I = \int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$

$= \int_{-\pi/3}^{\pi/3} \frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx + \int_{-\pi/3}^{\pi/3} \frac{4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$

The second integral becomes zero as integrand being an odd function of x.

$\therefore \quad I = 2\pi \int_0^{\pi/3} \frac{dx}{2 - \cos\left(x + \frac{\pi}{3}\right)}$

Let $x + \pi/3 = y$ or $dx = dy$.

Also, as $x \to 0$, $y \to \pi/3$, and as $x \to \pi/3$, $y \to 2\pi/3$.

$\therefore \quad I = 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2 - \cos y}$

$= 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2 - \frac{1 - \tan^2 y/2}{1 + \tan^2 y/2}}$

$= 2\pi \int_{\pi/3}^{2\pi/3} \frac{\sec^2 y/2}{3\tan^2 y/2 + 1} dy$

$= \frac{4\pi}{3} \int_{\pi/3}^{2\pi/3} \frac{\frac{1}{2}\sec^2 y/2}{\tan^2 y/2 + (1/\sqrt{3})^2} dy$

$= \frac{4\pi\sqrt{3}}{3} \left[\tan^{-1}(\sqrt{3} \tan y/2)\right]_{\pi/3}^{2\pi/3}$

$= \frac{4\pi}{\sqrt{3}} [\tan^{-1} 3 - \tan^{-1} 1]$

$= \frac{4\pi}{\sqrt{3}} [\tan^{-1} 3 - \pi/4]$

Example 8.8

It is known that $f(x)$ is an odd function and has a period p. Prove that $\int_a^x f(t)dt$ is also periodic function with the same period.

Sol. Let $F(x) = \int_a^x f(t)\,dt$

$\therefore$ $F(x+p) = \int_a^{x+p} f(t)\,dt = \int_a^x f(t) + \int_x^{x+p} f(t)dt$

$= F(x) + \int_x^{x+p} f(t)\,dt$ (1)

Obviously, now we have to prove that $\int_x^{x+p} f(t)\,dt$ is zero.

Given that $f(x)$ has period p. Then $\int_x^{x+p} f(t)\,dt$ is independent of x.

Let $x = -p/2$. Then $\int_x^{x+p} f(t)\,dt = \int_{-p/2}^{p/2} f(t)\,dt = 0$

[As given $f(x)$ is an odd function]

$\therefore$ $F(x+p) = F(x)$

Thus, $F(x)$ is periodic with period P.

Example 8.9

Evaluate $\int_0^{\pi/4}\left(\tan^{-1}\left(\frac{2\cos^2\theta}{2-\sin 2\theta}\right)\right)\sec^2\theta d\theta$.

Sol. Given integral is $\int_0^{\pi/4} \tan^{-1}\left(\frac{2\cos^2\theta}{2-\sin 2\theta}\right)\sec^2\theta d\theta$

$$= \int_0^{\pi/4} \tan^{-1}\left(\frac{1}{\sec^2\theta - \tan\theta}\right)\sec^2\theta d\theta$$

$$= \int_0^{\pi/4} \tan^{-1}\left(\frac{1}{1+\tan^2\theta - \tan\theta}\right)\sec^2\theta d\theta$$

Put $\tan\theta = t$. Then $\sec^2\theta\, d\theta = dt$

The given integral reduces to

$$\int_0^1 \tan^{-1}\left(\frac{1}{1+t^2-t}\right)dt = \int_0^1 \tan^{-1}\left(\frac{t-(t-1)}{1+t(t-1)}\right)dt$$

$$= \int_0^1 \tan^{-1} t\,dt - \int_0^1 \tan^{-1}(t-1)dt$$

$$= \int_0^1 \tan^{-1} t\,dt - \int_0^1 \tan^{-1}((1-t)-1)dt$$

$$= 2\int_0^1 1\tan^{-1} t\,dt$$

$$= 2\left[t\tan^{-1} t\right]_0^1 - 2\int_0^1 \frac{t}{1+t^2}dt$$

(Integrating by parts)

$$= \frac{\pi}{2} - \left[\ln(1+t^2)\right]_0^1 = \frac{\pi}{2} - \ln 2.$$

Example 8.10

If $f(x) = \frac{\sin x}{x}\ \forall x \in (0, \pi]$, prove that

$$\frac{\pi}{2}\int_0^{\pi/2} f(x)\, f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi} f(x)dx.$$

Sol. $\int_0^{\pi} f(x)dx = \int_0^{\pi}\frac{\sin x}{x}dx$

Let $I = \int_0^{\pi/2} f(x)\, f\left(\frac{\pi}{2}-x\right)dx$

$$= \int_0^{\pi/2} \frac{\sin x}{x}\,\frac{\sin\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)}dx$$

$$= \int_0^{\pi/2} \frac{2\sin x\cos x}{x(\pi-2x)}dx$$

$$= \int_0^{\pi/2} \frac{\sin 2x}{x(\pi-2x)}dx$$

Let $2x = t$. So, $dt = 2dx$

$$\therefore \quad I = \int_0^{\pi} \frac{\sin t}{\frac{t}{2}(\pi-t)}\frac{dt}{2} = \int_0^{\pi}\frac{\sin t}{t(\pi-t)}dt$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin t\left(\frac{1}{t}+\frac{1}{\pi-t}\right)dt$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt + \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{(\pi-t)}dt$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt + \frac{1}{\pi}\int_0^{\pi}\frac{\sin(\pi-t)}{\pi-(\pi-t)}dt$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt + \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt = \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt$$

or $$\frac{\pi}{2}\int_0^{\pi/2} f(x)f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi}\frac{\sin x}{x}dx$$

Example 8.11

Let $f(x)$ be a continuous function $\forall\ x \in R$, except at $x = 0$, such that $\int_0^a f(x)dx$, $a \in R^+$ exists. If $g(x) = \int_x^a \frac{f(t)}{t}dt$, prove that $\int_0^a f(x)dx = \int_0^a g(x)dx$.

Sol. We have $g(x) = \int_x^a \frac{f(t)}{t}dt$ (1)

Differentiating both sides w.r.t. x, we get

$$g'(x) = -\frac{f(x)}{x} \text{ or } f(x) = -x\,g'(x)$$

or $$\int_0^a f(x)dx = -\int_0^a xg'(x)dx$$

$$= -x\,g(x)\Big|_0^a + \int_0^a g(x)dx$$

$$= -a\,g(a) + \int_0^a g(x)\,dx$$

$$= \int_0^a g(x)\,dx \qquad \text{[as from (1) } g(a) = 0\text{]}$$

EXAMPLE 8.12

If $x\int_0^x \sin(f(t))dt = (x+2)\int_0^x t\sin(f(t))dt$, where $x > 0$, then show that $f'(x)\cot f(x) + \frac{3}{1+x} = 0$.

Sol. Given equation is

$$x\int_0^x (1-t)\sin(f(t))\,dt = 2\int_0^x t\sin(f(t))\,dt$$

Differentiating w.r.t. x, we get

$$\int_0^x (1-t)\sin[f(t)]dt + x(1-x)\sin[f(x)] = 2x\sin[f(x)]$$

or $$\int_0^x (1-t)\sin[f(t)]dt = x^2\sin[f(x)] + x\sin[f(x)]$$

Again, differentiating w.r.t. x, we get

$$(1-x)\sin[f(x)] = 2x\sin[f(x)] + x^2\cos[f(x)]f'(x) + \sin[f(x)] + x\cos[f(x)]f'(x)$$

or $$-3x\sin[f(x)] = (x + x^2)\cos[f(x)]f'(x)$$

or $$\frac{-3x}{x(1+x)} = \cot[f(x)]f'(x)$$

or $$f'(x)\cot f(x) + \frac{3}{1+x} = 0$$

EXAMPLE 8.13

Show that $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$.

Sol. Let $$I = \int_0^{\pi/2} f(\sin 2x)\sin x\,dx \qquad (1)$$

$$= \int_0^{\pi/2} f\left\{\sin 2\left(\frac{1}{2}\pi - x\right)\right\}\sin\left(\frac{1}{2}\pi - x\right)dx$$

$\therefore$ $$I = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx \qquad (2)$$

Then adding (1) and (2), we have

$$2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x)\,dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\sin 2x)\sin\left(x + \frac{\pi}{4}\right)dx$$

To get $f(\cos 2x)$ on RHS, we have to substitute $\frac{\pi}{2} - 2\theta = 2x$.

$\therefore$ $dx = -d\theta$.

Also, when $x = 0$, $\theta = \pi/4$, and when $x = \pi/2$, $\theta = -\pi/4$.

$\therefore$ $$2I = \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2\theta)\cos\theta\,d\theta$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2\theta)\cos\theta\,d\theta$$

[as $g(\theta) = f(\cos 2\theta)\cos\theta$ is an even function]

$\therefore$ $$I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx \qquad (3)$$

EXAMPLE 8.14

Let $a + b = 4$, where $a < 2$, and let $g(x)$ be a differentiable function. If $\frac{dg}{dx} > 0$ for all x, prove that $\int_0^a g(x)\,dx + \int_0^b g(x)dx$ increases as $(b - a)$ increases.

Sol. $a + b = 4$ or $b = 4 - a$

Let $$f(a) = \int_0^a g(x)dx + \int_0^b g(x)\,dx$$

$$= \int_0^a g(x)dx + \int_0^{4-a} g(x)\,dx$$

$\therefore$ $$\frac{df(a)}{da} = g(a) - g(4-a)$$

Now, $$\frac{df(a)}{d(b-a)} = \frac{df(a)}{d(4-2a)} = \frac{df(a)}{-2da} = (g(4-a) - g(a))/2$$

given $a < 2$

or $2a < 4$

or $4 - a > a$

or $g(4 - a) > g(a)$ $\quad$ [$\because \frac{dg(x)}{dx} > 0$. So, $g(x)$ is an increasing function]

Thus, $$\frac{df(a)}{d(b-a)} > 0$$

Therefore, $f(a) = \int_0^a g(x)dx + \int_0^b g(x)\,dx$ increases as $(b - a)$ increases.

EXAMPLE 8.15

If f is a continous function with $\int_0^x f(t)dt \to \infty$ as $|x| \to \infty$, then show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)dt = 2$.

Sol. We are given that f is a continuous function and

$$\int_0^x f(t)\,dt \to \infty \text{ as } |x| \to \infty$$

We have to show that every line $y = mx$ intersects the curve

$$y^2 + \int_0^x f(t)\,dt = 2$$

If possible, let $y = mx$ intersects the given curve. Then substituting $y = mx$ in the curves, we get

$$m^2x^2 + \int_0^x f(t)\,dt = 2 \quad (1)$$

Consider $F(x) = m^2x^2 + \int_0^x f(t)\,dt - 2$

Then $F(x)$ is a continuous function as $f(x)$ is given to be continuous.

Also, $F(x) \to \infty$ as $|x| \to \infty$

But $F(0) = -2$

So, the graph of $y = F(x)$ must cross x-axis to reach infinity as $|x|$ approaches infinity.

Hence, $y = mx$ intersects the given curves.

EXAMPLE 8.16

If $f(x + f(y)) = f(x) + y\ \forall\ x, y \in R$ and $f(0) = 1$, then prove that $\int_0^2 f(2-x)dx = 2\int_0^1 f(x)dx$.

Sol. It is given that $f(x + f(y)) = f(x) + y$.

Putting $y = 0$, we get $f(x + f(0)) = f(x) + 0$

or $f(x + 1) = f(x)$

Now, using the property

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx, \text{ we get}$$

$$\int_0^2 f(2-x)\,dx = \int_0^1 f(2-x)\,dx + \int_0^1 f(2-(2-x))\,dx$$

$$= \int_0^1 f(2-(1-x))\,dx + \int_0^1 f(x)\,dx$$

$$= \int_0^1 f(1+x)\,dx + \int_0^1 f(x)\,dx$$

$$= 2\int_0^1 f(x)\,dx$$

Alternative method

It is given that $f(x + f(y)) = f(x) + y$.

Putting $y = 0$, we get $f(x + f(0)) = f(x) + 0$

or $f(x + 1) = f(x)$

Thus, $f(x)$ is periodic with period 1.

Now, $I = \int_0^2 f(2-x)dx$

Putting $2 - x = t$, we get

$$I = \int_0^2 f(t)\,dt = 2\int_0^1 f(t)\,dt$$

$$= 2\int_0^1 f(x)\,dx$$

EXAMPLE 8.17

Suppose f is a real-valued differentiable function defined on $[1, \infty)$ with $f(1) = 1$. Moreover, suppose that f satisfies $f'(x) = \dfrac{1}{x^2 + f^2(x)}$. Show that $f(x) < 1 + \dfrac{\pi}{4}\ \forall\ x \geq 1$.

Sol. Here, $f'(x) = \dfrac{1}{x^2 + (f(x))^2} > 0\ \forall\ x \geq 1$

Thus, $f(x)$ is an increasing function $\forall\ x \geq 1$.

Given $f(1) = 1$. So, $f(x) \geq 1\ \forall\ x \geq 1$.

Hence, $f'(x) \leq \dfrac{1}{1+x^2}\ \forall\ x \geq 1$

or $\displaystyle\int_1^x f'(x)dx \leq \int_1^x \frac{1}{1+x^2}dx$

or $f(x) - f(1) \leq \tan^{-1} x - \tan^{-1} 1$

or $f(x) \leq \tan^{-1} x + 1 - \dfrac{\pi}{4}$

or $f(x) < \dfrac{\pi}{2} + 1 - \dfrac{\pi}{4}$ $\left(\text{as } \tan^{-1} x < \dfrac{\pi}{2}\ \forall\ x \geq 1\right)$

i.e., $f(x) < 1 + \dfrac{\pi}{4}\ \forall\ x \geq 1$

EXAMPLE 8.18

Let f be a continuous function on $[a, b]$. If

$$F(x) = \left(\int_a^x f(t)\,dt - \int_x^b f(t)\,dt\right)(2x - (a+b)),$$

then prove that there exist some $c \in (a, b)$ such that

$$\int_a^c f(t)\,dt - \int_c^b f(t)dt = f(c)\,(a + b - 2c).$$

Sol. Given $F(x) = \left(\int_a^x f(t)\,dt - \int_x^b f(t)\,dt\right)(2x - (a+b)) \quad (1)$

As f is continuous, $F(x)$ is also continuous.

Now, $F(a) = \left(-\int_a^b f(t)\,dt\right)(a-b) = (b-a)\int_a^b f(t)\,dt$

and $F(b) = \left(\int_a^b f(t)\,dt\right)(b-a)$

Thus, $F(a) = F(b)$

Hence, Rolle's Theorem is applicable to $F(x)$.

Therefore, there exists at least one $c \in (a, b)$ such that $F'(c) = 0$.

$\therefore\ 2\left(\int_a^c f(t)\,dt - \int_c^b f(t)\,dt\right) + (f(c) - (-f(c))).\,(2c - (a+b)) = 0.$

Hence, proved.

EXAMPLE 8.19

$f(x)$ is a continuous and bijective function on R. If $\forall\ t \in R$, area bounded by $y = f(x)$, $x = a - t$, $x = a$, and x-axis is equal to area bounded by $y = f(x)$, $x = a + t$, $x = a$, and x-axis, then prove that $\int_{-\lambda}^{\lambda} f^{-1}(x)dx = 2a\lambda$ [given that $f(a) = 0$].

Sol. Given $\left|\int_{a-t}^{a} f(x)dx\right| = \left|\int_{a}^{a+t} f(x)dx\right| \ \forall\, t \in R$

or $\int_{a-t}^{a} f(x)dx = -\int_{a}^{a+t} f(x)dx$

[since $f(a) = 0$ and $f(x)$ is monotonic]

or $f(a - t) = -f(a + t)$ (differentiating both sides w.r.t. t)

or $f(a + t) = -f(a - t) = x$ (say)

or $t = f^{-1}(x) - a$ (2)

and $t = a - f^{-1}(-x)$ (3)

From equations (3) and (2), $(a - f^{-1}(x)) + (a - f^{-1}(-x)) = 0$

or $\int_{-\lambda}^{\lambda} f^{-1}(x)dx = \frac{1}{2}\int_{-\lambda}^{\lambda} (f^{-1}(x) + f^{-1}(-x))\, dx = 2a\lambda$

EXAMPLE 8.20

If $f(x) = x + \int_0^1 t(x+t)\, f(t)\, dt$, then find the value of the definite integral $\int_0^1 f(x)\, dx$.

Sol. $f(x) = x + x\int_0^1 tf(t)\,dt + \int_0^1 t^2 f(t)\,dt$

$\therefore \quad f(x) = x(1 + A) + B$

where $A = \int_0^1 tf(t)\,dt$ and $B = \int_0^1 t^2 f(t)\,dt$

$$A = \int_0^1 t[t(1+A) + B]dt = \left[(1+A)\frac{t^3}{3} + B\frac{t^2}{2}\right]_0^1$$

$\therefore \quad A = \frac{1+A}{3} + \frac{B}{2}$

or $\quad 4A - 3B = 2$

$$B = \int_0^1 t^2[t(1+A) + B]dt$$

$$= \left[\frac{t^4(1+A)}{4} + \frac{Bt^3}{3}\right]_0^1$$

$$= \frac{1+A}{4} + \frac{B}{3}$$

or $\quad 8B - 3A = 3$

Solving (1) and (2), we get $A = \frac{25}{23}$ and $B = \frac{18}{23}$

$\therefore \quad f(x) = \frac{48x + 18}{23}$

Exercises

Single Correct Answer Type

1. Let $f(x) = \lim_{n\to\infty} \frac{1}{n}\left(\left(x+\frac{1}{n}\right)^2 + \left(x+\frac{2}{n}\right)^2 + \ldots + \left(x+\frac{n-1}{n}\right)^2\right)$.

Then the minimum value of $f(x)$ is

(1) 1/4 (2) 1/6 (3) 1/9 (4) 1/12

2. If $S_n = \left[\frac{1}{1+\sqrt{n}} + \frac{1}{2+\sqrt{2n}} + \cdots + \frac{1}{n+\sqrt{n^2}}\right]$, then $\lim_{n\to\infty} S_n$ is equal to

(1) log 2 (2) log 4

(3) log 8 (4) none of these

3. The value of $\lim_{n\to\infty} \sum_{r=1}^{4n} \frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$ is equal to

(1) $\frac{1}{35}$ (2) $\frac{1}{14}$ (3) $\frac{1}{10}$ (4) $\frac{1}{5}$

4. The value of

$$\lim_{n\to\infty} \frac{(1^2+2^2+\ldots+n^2)(1^3+2^3+\ldots+n^3)(1^4+2^4+\ldots+n^4)}{(1^5+2^5+\ldots+n^5)^2}$$

is equal to

(1) $\frac{3}{5}$ (2) $\frac{4}{5}$ (3) $\frac{2}{5}$ (4) $\frac{1}{5}$

5. The value of $\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\cdots\tan\frac{n\pi}{2n}\right]^{1/n}$ is

(1) e (2) e^2 (3) 1 (4) e^3

6. $\int_{2-a}^{2+a} f(x)dx$ is equal to [where $f(2-\alpha) = f(2+\alpha)\ \forall\ \alpha \in R$]

(1) $2\int_2^{2+a} f(x)dx$ (2) $2\int_0^a f(x)dx$

(3) $2\int_2^2 f(x)dx$ (4) none of these

7. Let $f(x) = \min(\{x\}, \{-x\})\ \forall\ x \in R$, where $\{\cdot\}$ denotes the fractional part of x. Then $\int_{-100}^{100} f(x)dx$ is equal to

(1) 50 (2) 100

(3) 200 (4) none of these

8. Which of the following is incorrect?

(1) $\int_{a+c}^{b+c} f(x)dx = \int_a^b f(x+c)dx$

(2) $\int_{ac}^{bc} f(x)dx = c\int_a^b f(cx)dx$

(3) $\int_{-a}^{a} f(x)dx = \frac{1}{2}\int_{-a}^{a} (f(x)+f(-x))\,dx$

(4) None of these

9. $\int_{-1}^{1/2} \frac{e^x\left(2-x^2\right)dx}{(1-x)\sqrt{1-x^2}}$ is equal to

(1) $\frac{\sqrt{e}}{2}(\sqrt{3}+1)$ (2) $\frac{\sqrt{3e}}{2}$

(3) $\sqrt{3e}$ (4) $\sqrt{\frac{e}{3}}$

10. If $\int_{\log 2}^{x} \frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$, then x is equal to

(1) 4 (2) ln 8

(3) ln 4 (4) none of these

11. $\int_{5/2}^{5} \frac{\sqrt{(25-x^2)^3}}{x^4} dx$ is equal to

(1) $\frac{\pi}{6}$ (2) $\frac{2\pi}{3}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{3}$

12. If $f(x)$ satisfies the condition of Rolle's theorem in [1, 2], then $\int_1^2 f'(x)\,dx$ is equal to

(1) 1 (2) 3

(3) 0 (4) none of these

13. The value of the integral $\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}dx$ is

(1) $3+2\pi$ (2) $4-\pi$

(3) $2+\pi$ (4) none of these

14. The value of the integral $\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1}$, $0<\alpha<\pi$ is equal to

(1) $\sin\alpha$ (2) $\alpha\sin\alpha$

(3) $\frac{\alpha}{2\sin\alpha}$ (4) $\frac{\alpha}{2}\sin\alpha$

15. $\int_0^\infty \frac{dx}{[x+\sqrt{x^2+1}]^3}$ is equal to

(1) $\frac{3}{8}$ (2) $\frac{1}{8}$

(3) $-\frac{3}{8}$ (4) none of these

16. If $f(y) = e^y$, $g(y) = y$, $y > 0$, and $F(t) = \int_0^t f(t-y)\,g(y)\,dt$, then

(1) $F(t) = e^t - (1+t)$ (2) $F(t) = te^t$

(3) $F(t) = te^{-t}$ (4) $F(t) = 1 - e^t(1+t)$

17. If $P(x)$ is a polynomial of the least degree that has a maximum equal to 6 at $x = 1$, and a minimum equal to 2 at $x = 3$, then $\int_0^1 P(x)dx$ equals

(1) $\frac{17}{4}$ (2) $\frac{13}{4}$ (3) $\frac{19}{4}$ (4) $\frac{5}{4}$

18. The number of positive continuous $f(x)$ defined in $[0, 1]$ for which $I_1 = \int_0^1 f(x)\,dx = 1$, $I_2 = \int_0^1 x f(x)\,dx = a$, $I_3 = \int_0^1 x^2 f(x)\,dx = a^2$ is/are

(1) 1 (2) ∞ (3) 2 (4) 0

19. Suppose that $F(x)$ is an anti-derivative of $f(x) = \frac{\sin x}{x}$, where $x > 0$. Then $\int_1^3 \frac{\sin 2x}{x} dx$ can be expressed as

(1) $F(6) - F(2)$ (2) $\frac{1}{2}(F(6) - F(2))$

(3) $\frac{1}{2}(F(3) - F(1))$ (4) $2(F(6) - F(2))$

20. $\int_{-\pi/3}^{0} \left[\cot^{-1}\left(\frac{2}{2\cos x - 1}\right) + \cot^{-1}\left(\cos x - \frac{1}{2}\right)\right] dx$ is equal to

(1) $\frac{\pi^2}{6}$ (2) $\frac{\pi^2}{3}$ (3) $\frac{\pi^2}{8}$ (4) $\frac{3\pi^2}{8}$

21. The value of $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$ is

(1) $\frac{1}{20}\log 3$ (2) $\frac{1}{40}\log 3$

(3) $\frac{1}{20}\log 6$ (4) $10 \log 3$

22. $\int_{-1}^{1} \frac{e^{-1/x}}{x^2(1 + e^{-2/x})} dx$ is equal to

(1) $\frac{\pi}{2} - 2\tan^{-1} e$ (2) $\frac{\pi}{2} - 2\cot^{-1} e$

(3) $2\tan^{-1} e$ (4) $\pi - 2\tan^{-1} e$

23. If $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$, then $\int_0^{\infty} \frac{\sin^3 x}{x} dx$ is equal to

(1) $\pi/2$ (2) $\pi/4$ (3) $\pi/6$ (4) $3\pi/2$

24. The range of the function $f(x) = \int_{-1}^{1} \frac{\sin x\, dt}{(1 - 2t\cos x + t^2)}$ is

(1) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (2) $[0, \pi]$

(3) $\{0, \pi\}$ (4) $\left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$

25. If the function $f : [0, 8] \to R$ is differentiable, then for $0 < \beta < 1$ and $0 < \alpha < 2$, $\int_0^8 f(t)\,dt$ is equal to

(1) $3[\alpha^3 f(\alpha^2) + \beta^2 f(\beta^2)]$

(2) $3[\alpha^3 f(\alpha) + \beta^3 f(\beta)]$

(3) $3[\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3)]$

(4) $3[\alpha^2 f(\alpha^2) + \beta^2 f(\beta^2)]$

26. If $f(x) = x^5 + 5x - 1$ then $\int_5^{41} \frac{dx}{(f^{-1}(x))^5 + 5f^{-1}(x)}$ equals

(1) 0 (2) $\log_e 3$ (3) $\log_e 4$ (4) $\log_e 7$

27. Let $f(0) = 0$ and $\int_0^2 f'(2t) e^{f(2t)} dt = 5$. Then the value of $f(4)$ is

(1) $\log 2$ (2) $\log 7$ (3) $\log 11$ (4) $\log 13$

28. If $f(x) = \cos(\tan^{-1} x)$, then the value of the integral $\int_0^1 x f''(x)\,dx$ is

(1) $\frac{3 - \sqrt{2}}{2}$ (2) $\frac{3 + \sqrt{2}}{2}$

(3) 1 (4) $1 - \frac{3}{2\sqrt{2}}$

29. The equation of the curve is $y = f(x)$. The tangents at $[1, f(1)]$, $[2, f(2)]$, and $[3, f(3)]$ make angles $\frac{\pi}{6}, \frac{\pi}{3}$, and $\frac{\pi}{4}$ respectively, with the positive direction of x-axis. Then the value of $\int_2^3 f'(x) f''(x)\,dx + \int_1^3 f''(x)\,dx$ is equal to

(1) $-1/\sqrt{3}$ (2) $1/\sqrt{3}$

(3) 0 (4) none of these

30. The value of $\int_1^e \left(\frac{\tan^{-1} x}{x} + \frac{\log x}{1 + x^2}\right) dx$ is

(1) $\tan e$ (2) $\tan^{-1} e$

(3) $\tan^{-1}(1/e)$ (4) none of these

31. If $f(\pi) = 2$ and $\int_0^{\pi} (f(x) + f''(x)) \sin x\, dx = 5$, then $f(0)$ is equal to (it is given that $f(x)$ is continuous in $[0, \pi]$)

(1) 7 (2) 3 (3) 5 (4) 1

32. If $\int_1^2 e^{x^2} dx = a$, then $\int_e^{e^4} \sqrt{\ln x}\, dx$ is equal to

(1) $2e^4 - 2e - a$ (2) $2e^4 - e - a$

(3) $2e^4 - e - 2a$ (4) $e^4 - e - a$

33. If $f(x)$ is continuous for all real values of x, then $\sum_{r=1}^{n} \int_0^1 f(r - 1 + x)\,dx$ is equal to

(1) $\int_0^n f(x)\,dx$ (2) $\int_0^1 f(x)\,dx$

(3) $n\int_0^1 f(x)\,dx$ (4) $(n - 1)\int_0^1 f(x)\,dx$

34. The value of $\int_0^{\pi/2} \sin|2x - \alpha|\,dx$, where $\alpha \in [0, \pi]$, is

(1) $1 - \cos\alpha$ (2) $1 + \cos\alpha$

(3) 1 (4) $\cos\alpha$

35. $f(x)$ is a continuous function for all real values of x and satisfies $\int_n^{n+1} f(x)\,dx = \frac{n^2}{2}\ \forall\ n \in I$. Then $\int_{-3}^{5} f(|x|)\,dx$ is equal to

(1) 19/2 (2) 35/2

(3) 17/2 (4) none of these

36. If $f(x)=\int_{-1}^{x}|t|dt$,, then for any $x \geq 0, f(x)$ equals

(1) $\frac{1}{2}(1-x^2)$ (2) $\frac{1}{2}x^2$

(3) $\frac{1}{2}(1+x^2)$ (4) none of these

37. If $a>0$ and $A=\int_0^a \cos^{-1} x dx$, and

$\int_{-a}^{a}(\cos^{-1}x-\sin^{-1}\sqrt{1-x^2})dx=\pi a-\lambda A$. Then λ is

(1) 0 (2) 2

(3) 3 (4) none of these

38. The value of $\int_1^a [x]f'(x)dx$, where $a>1$, and $[x]$ denotes the greatest integer not exceeding x, is

(1) $af(a)-(f(1)+f(2)+\cdots+f([a]))$

(2) $[a]f(a)-(f(1)+f(2)+\cdots+f([a]))$

(3) $[a]f([a])-(f(1)+f(2)+\cdots+f(a))$

(4) $af([a])-(f(1)+f(2)+\cdots+f(a))$

39. $\int_3^{10}[\log[x]]dx$ is equal to (where [.] represents the greatest integer function)

(1) 9 (2) $16-e$ (3) 10 (4) $10+e$

40. $\int_{-1}^{2}\left[\frac{[x]}{1+x^2}\right]dx$, where [·] denotes the greatest integer function, is equal to

(1) –2 (2) –1

(3) zero (4) none of these

41. The value of $\int_{-2}^{1}\left[x\left[1+\cos\left(\frac{\pi x}{2}\right)\right]+1\right]dx$, where [.] denotes the greatest integer function, is

(1) 1 (2) 1/2

(3) 2 (4) none of these

42. The value of $\int_0^{2\pi}[2\sin x]dx$, where [.] represents the greatest integral function, is

(1) $\frac{-5\pi}{3}$ (2) $-\pi$

(3) $\frac{5\pi}{3}$ (4) -2π

43. $I_1=\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$, $I_2=\int_0^{2\pi}\cos^6 x\,dx$,

$I_3=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^3 x\,dx$, $I_4=\int_0^1 \ln\left(\frac{1}{x}-1\right)dx$. Then

(1) $I_2=I_3=I_4=0, I_1\neq 0$

(2) $I_1=I_2=I_3=0, I_4\neq 0$

(3) $I_1=I_3=I_4=0, I_2\neq 0$

(4) $I_1=I_2=I_3=0, I_4\neq 0$

44. Given $\int_0^{\pi/2}\frac{dx}{1+\sin x+\cos x}=A$. Then the value of the definite integral $\int_0^{\pi/2}\frac{\sin x}{1+\sin x+\cos x}dx$ is equal to

(1) $\frac{1}{2}A$ (2) $\frac{\pi}{2}-A$

(3) $\frac{\pi}{4}-\frac{1}{2}A$ (4) $\frac{\pi}{2}+A$

45. If $I_1=\int_{-100}^{101}\frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$ and

$I_2=\int_{-100}^{101}\frac{dx}{5+2x-2x^2}$, then $\frac{I_1}{I_2}$ is

(1) 2 (2) $\frac{1}{2}$ (3) 1 (4) $-\frac{1}{2}$

46. $\int_0^{\infty}\frac{xdx}{(1+x)(1+x^2)}$ is equal to

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{2}$

(3) π (4) none of these

47. For any integer n, the integral $\int_0^{\pi}e^{\cos^2 x}\cos^3(2n+1)x\,dx$ has the value

(1) π (2) 1

(3) 0 (4) none of these

48. Let f be a positive function. Let $I_1=\int_{1-k}^{k}x f[x(1-x)]dx$, $I_2=\int_{1-k}^{k}f[x(1-x)]dx$, where $2k-1>0$. Then $\frac{I_1}{I_2}$ is

(1) 2 (2) k (3) $\frac{1}{2}$ (4) 1

49. If $f(x)=\frac{e^x}{1+e^x}$, $I_1=\int_{f(-a)}^{f(a)}xg(x(1-x))dx$, and

$I_2=\int_{f(-a)}^{f(a)}g(x(1-x))dx$, then the value of $\frac{I_2}{I_1}$ is

(1) –1 (2) –2 (3) 2 (4) 1

50. The value of $\int_1^{\frac{1+\sqrt{5}}{2}}\frac{x^2+1}{x^4-x^2+1}\log\left(1+x-\frac{1}{x}\right)dx$ is

(1) $\frac{\pi}{8}\log_e 2$ (2) $\frac{\pi}{2}\log_e 2$

(3) $-\frac{\pi}{2}\log_e 2$ (4) none of these

51. The value of the definite integral $\int_0^{\pi/2}\sqrt{\tan x}\,dx$ is

(1) $\sqrt{2}\pi$ (2) $\frac{\pi}{\sqrt{2}}$ (3) $2\sqrt{2}\pi$ (4) $\frac{\pi}{2\sqrt{2}}$

52. $f(x) > 0\ \forall\ x \in R$ and is bounded. If

$$\lim_{n\to\infty}\left[\int_0^a \frac{f(x)dx}{f(x)+f(a-x)} + a\int_a^{2a}\frac{f(x)dx}{f(x)+f(3a-x)} + a^2\int_{2a}^{3a}\frac{f(x)dx}{f(x)+f(5a-x)} + \cdots + a^{n-1}\int_{(n-1)a}^{na}\frac{f(x)dx}{f(x)+f[(2n-1)a-x]}\right] = 7/5$$

(where $0 < a < 1$), then a is equal to

(1) $\frac{2}{7}$ (2) $\frac{1}{7}$ (3) $\frac{14}{19}$ (4) $\frac{9}{14}$

53. If $\int_0^1 \cot^{-1}(1-x+x^2)dx = \lambda\int_0^1 \tan^{-1} x dx$, then λ is equal to

(1) 1 (2) 2 (3) 3 (4) 4

54. The value of the definite integral $\int_{-1}^{1}(1+x)^{1/2}(1-x)^{3/2}\,dx$ equals

(1) π (2) $\frac{3\pi}{4}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{2}$

55. The value of the integral $\int_{-3\pi/4}^{5\pi/4}\frac{(\sin x+\cos x)}{e^{x-\pi/4}+1}dx$ is

(1) 0 (2) 1

(3) 2 (4) none of these

56. $I_1 = \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx$, $I_2 = \int_{-\pi/4}^{\pi/4}\ln(\sin x+\cos x)\,dx$. Then

(1) $I_1 = 2I_2$ (2) $I_2 = 2I_1$

(3) $I_1 = 4I_2$ (4) $I_2 = 4I_1$

57. If $I_1 = \int_0^{\pi/2}\frac{\cos^2 x}{1+\cos^2 x}dx$, $I_2 = \int_0^{\pi/2}\frac{\sin^2 x}{1+\sin^2 x}dx$,

$I_3 = \int_0^{\pi/2}\frac{1+2\cos^2 x\sin^2 x}{4+2\cos^2 x\sin^2 x}dx$, then

(1) $I_1 = I_2 > I_3$ (2) $I_3 > I_1 = I_2$

(3) $I_1 = I_2 = I_3$ (4) none of these

58. The value of $\int_0^{\pi/2}\frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ is

(1) $\frac{\pi^2}{2}$ (2) $\frac{\pi^2}{4}$ (3) $\frac{\pi^2}{8}$ (4) $\frac{\pi^2}{16}$

59. For $x \in R$ and a continuous function f, let

$I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x f\{x(2-x)\}dx$ and $I_2 = \int_{\sin^2 t}^{1+\cos^2 t} f\{x(2-x)\}dx$.

Then $\frac{I_1}{I_2}$ is

(1) -1 (2) 1 (3) 2 (4) 3

60. If $\int_{-\pi/4}^{3\pi/4}\frac{e^{\pi/4}dx}{(e^x+e^{\pi/4})(\sin x+\cos x)} = k\int_{-\pi/2}^{\pi/2}\sec x dx$, then the value of k is

(1) $\frac{1}{2}$ (2) $\frac{1}{\sqrt{2}}$ (3) $\frac{1}{2\sqrt{2}}$ (4) $-\frac{1}{\sqrt{2}}$

61. The value of the definite integral $\int_2^4 (x(3-x)(4+x)(6-x)(10-x)+\sin x)\,dx$ equals

(1) $\cos 2+\cos 4$ (2) $\cos 2-\cos 4$

(3) $\sin 2+\sin 4$ (4) $\sin 2-\sin 4$

62. If $I = \int_{-20\pi}^{20\pi}|\sin x|[\sin x]dx$ (where [.] denotes the greatest integer function), then the value of I is

(1) -40 (2) 40 (3) 20 (4) -20

63. The functions f and g are positive and continuous. If f is increasing and g is decreasing, then

$\int_0^1 f(x)[g(x)-g(1-x)]\,dx$

(1) is always non-positive

(2) is always non-negative

(3) can take positive and negative values

(4) none of these

64. $\int_0^{\pi}\frac{x\tan x}{\sec x+\cos x}dx$ is

(1) $\frac{\pi^2}{4}$ (2) $\frac{\pi^2}{2}$ (3) $\frac{3\pi^2}{2}$ (4) $\frac{\pi^2}{3}$

65. If $f(x) = \int_0^{\pi}\frac{t\sin t\,dt}{\sqrt{1+\tan^2 x\sin^2 t}}$ for $0 < x < \frac{\pi}{2}$, then

(1) $f(0^+) = -\pi$

(2) $f\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8}$

(3) f is continuous and differentiable in $\left(0, \frac{\pi}{2}\right)$

(4) f is continuous but not differentiable in $\left(0, \frac{\pi}{2}\right)$

66. $\int_{-3}^{3} x^8\{x^{11}\}dx$ is equal to (where {.} is the fractional part of x)

(1) 3^8 (2) 3^7

(3) 3^9 (4) none of these

67. The value of $\int_0^{4\pi}\log_e|3\sin x+3\sqrt{3}\cos x|\,dx$ then the value of I is equal to

(1) $\pi\log_e 3$ (2) $2\pi\log_e 3$

(3) $4\pi\log_e 3$ (4) $8\pi\log_e 3$

68. The value of $\int_0^{\pi}\frac{|x|\sin^2 x}{1+2|\cos x|\sin x}dx$ is equal to

(1) $\pi/4$ (2) $\pi/2$ (3) π (4) 2π

69. The value of the integral $\int_{-\pi}^{\pi}\sin mx\sin nx\,dx$, for $m \neq n$ $(m, n \in I)$, is

(1) 0 (2) π (3) $\pi/2$ (4) 2π

70. If $f(x)$ and $g(x)$ are continuous functions, then $\int_{\ln \lambda}^{\ln(1/\lambda)} \frac{f(x^2/4)[f(x)-f(-x)]}{g(x^2/4)[g(x)+g(-x)]}dx$ is

(1) dependent on λ (2) a non-zero constant

(3) zero (4) none of these

71. The value of $\int_0^1 \frac{\tan^{-1}\left(\frac{x}{x+1}\right)}{\tan^{-1}\left(\frac{1+2x-2x^2}{2}\right)}dx$ is

(1) 1/4 (2) 1/2 (3) 1 (4) 2

72. $\int_{-\pi/2}^{\pi/2} \frac{e^{|\sin x|}\cos x}{1+e^{\tan x}}dx$ is equal to

(1) $e+1$ (2) $2e$ (3) $e-1$ (4) $e-2$

73. The value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$ is

(1) π (2) π^2 (3) $2\pi^2$ (4) $\pi^2/2$

74. The value of $\int_{-\pi}^{\pi} \sum_{r=0}^{999} \cos rx \left(1+\sum_{r=1}^{999} \sin rx\right)dx$, is

(1) 2π (2) $999\,\pi$ (3) 0 (4) π

75. Let $T > 0$ be a fixed real number. Suppose f is continuous function such that for all $x \in R$, $f(x+T) = f(x)$. If $I = \int_0^T f(x)\,dx$, then the value of $\int_3^{3+3T} f(2x)dx$ is

(1) $3/2I$ (2) $2I$ (3) $3I$ (4) $6I$

76. $\int_1^4 \{x-0.4\}dx$ equals (where $\{x\}$ is a fractional part of x)

(1) 13 (2) 6.3 (3) 1.5 (4) 7.5

77. The value of $\int_0^x [\cos t]dt$, $x \in \left[(4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2}\right]$ and $n \in N$, is equal to (where $[\cdot]$ represents greatest integer function

(1) $\frac{\pi}{2}(2n-1)-2x$ (2) $\frac{\pi}{2}(2n-1)+x$

(3) $\frac{\pi}{2}(2n+1)-x$ (4) $\frac{\pi}{2}(2n+1)+x$

78. $\int_0^x |\sin t|\,dt$, where $x \in (2n\pi, (2n+1)\pi)$, $n \in N$, is equal to

(1) $4n - \cos x$ (2) $4n - \sin x$

(3) $4n + 1 - \cos x$ (4) $4n - 1 - \cos x$

79. $\int_0^x \frac{2^t}{2^{[t]}}dt$, where [.] denotes the greatest integer function, and $x \in R^+$, is equal to

(1) $\frac{1}{\ln 2}([x]+2^{\{x\}}-1)$ (2) $\frac{1}{\ln 2}([x]+2^{\{x\}})$

(3) $\frac{1}{\ln 2}([x]-2^{\{x\}})$ (4) $\frac{1}{\ln 2}([x]+2^{\{x\}}+1)$

80. f is an odd function. It is also known that $f(x)$ is continuous for all values of x and is periodic with period 2. If $g(x) = \int_0^x f(t)\,dt$, then

(1) $g(x)$ is odd (2) $g(n) = 0$, $n \in N$

(3) $g(2n) = 0$, $n \in N$ (4) $g(x)$ is non-periodic

81. If $g(x) = \int_0^x (|\sin t|+|\cos t|)dt$, then $g\left(x+\frac{\pi n}{2}\right)$ is equal to, where $n \in N$,

(1) $g(x) + g(\pi)$ (2) $g(x) + ng\left(\frac{\pi}{2}\right)$

(3) $g(x) + g\left(\frac{\pi}{2}\right)$ (4) none of these

82. If $x = \int_c^{\sin t} \sin^{-1} z\,dz$, $y = \int_k^{\sqrt{t}} \frac{\sin z^2}{z}dz$, then $\frac{dy}{dx}$ is equal to

(1) $\frac{\tan t}{2t}$ (2) $\frac{\tan t}{t^2}$ (3) $\frac{\tan t}{2t^2}$ (4) $\frac{\tan t^2}{2t^2}$

83. Let $f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}$ and g be the inverse of f. Then the value of $g'(0)$ is

(1) 1 (2) 17

(3) $\sqrt{17}$ (4) none of these

84. If $f(x)$ is differentiable and $\int_0^{t^2} x f(x)\,dx = \frac{2}{5}t^5$, then $f\left(\frac{4}{25}\right)$ equals

(1) 2/5 (2) –5/2 (3) 1 (4) 5/2

85. If $f(x) = \cos x - \int_0^x (x-t) f(t)\,dt$, then $f''(x) + f(x)$ is equal to

(1) $-\cos x$ (2) $-\sin x$

(3) $\int_0^x (x-t)\, f(t)\,dt$ (4) 0

86. A function f is continuous for all x (and not everywhere zero) such that $f^2(x) = \int_0^x f(t)\frac{\cos t}{2+\sin t}dt$. Then $f(x)$ is

(1) $\frac{1}{2}\ln\left(\frac{x+\cos x}{2}\right)$

(2) $\frac{1}{2}\ln\left(\frac{3}{2+\cos x}\right)$

(3) $\frac{1}{2}\ln\left(\frac{2+\sin x}{2}\right)$

(4) $\frac{\cos x+\sin x}{2+\sin x}$

87. $\lim_{x\to 0}\frac{1}{x}\left[\int_y^a e^{\sin^2 t}dt - \int_{x+y}^a e^{\sin^2 t}dt\right]$ is equal to

(1) $e^{\sin^2 y}$ (2) $\sin 2y\, e^{\sin^2 y}$

(3) 0 (4) none of these

88. $f(x) = \int_1^x \frac{e^t}{t} dt$, where $x \in R^+$. Then the complete set of values of x for which $f(x) \le \ln x$ is

(1) $(0, 1]$ (2) $[1, \infty)$
(3) $(0, \infty)$ (4) none of these

89. If $\int_0^x f(t)dt = x + \int_x^1 tf(t)dt$, then the value of $f(1)$ is

(1) 1/2 (2) 0 (3) 1 (4) −1/2

90. If $f(x) = 1 + \frac{1}{x}\int_1^x f(t)dt$, then the value of $f(e^{-1})$ is

(1) 1 (2) 0
(3) −1 (4) none of these

91. If $\int_{\cos x}^{1} t^2 f(t)dt = 1 - \cos x \ \forall x \in \left(0, \frac{\pi}{2}\right)$, then the value of $\left[f\left(\frac{\sqrt{3}}{4}\right)\right]$ is ([.] denotes the greatest integer function)

(1) 4 (2) 5 (3) 6 (4) − 7

92. $f(x)$ is a continuous function for all real values of x and satisfies $\int_0^x f(t)dt = \int_x^1 t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^6}{3} + a$ then value of 'a' is equal to

(1) $-\frac{1}{24}$ (2) $\frac{17}{168}$ (3) $\frac{1}{7}$ (4) $-\frac{167}{840}$

93. The value of $\int_{1/e}^{\tan x} \frac{t\,dt}{1+t^2} + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$, where $x \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$, is equal to

(1) 0 (2) 2
(3) 1 (4) none of these

94. $\lim_{x\to\infty} \frac{\int_0^x \tan^{-1} t\,dt}{\sqrt{x^2+1}}$ is equal to

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{4}$ (3) 1 (4) π

95. A function f is defined by $f(x) = \int_0^{\pi} \cos t \cos(x-t)dt$, $0 \le x \le 2\pi$ then the minimum value of $f(x)$ is

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{2}$ (3) $\frac{-\pi}{2}$ (4) $\frac{-\pi}{4}$

96. If 'f' is a differentiable function satisfying $f(x) = \int_0^x \sqrt{1-f^2(t)}dt + \frac{1}{2}$ then the value of $f(\pi)$) is equal to

(1) $-\frac{\sqrt{3}}{2}$ (2) $-\frac{1}{2}$ (3) $\frac{\sqrt{3}}{2}$ (4) $\frac{1}{2}$

97. If $\int_0^1 e^{x^2}(x-\alpha)dx = 0$, then

(1) $1 < \alpha < 2$ (2) $\alpha < 0$
(3) $0 < \alpha < 1$ (4) $\alpha = 0$

98. The value of the integral $\int_0^1 e^{x^2} dx$ lies in the interval

(1) (0, 1) (2) (−1, 0)
(3) $(1, e)$ (4) none of these

99. Given that f satisfies $|f(u) - f(v)| \le |u - v|$ for u and v in $[a, b]$. Then $\left|\int_a^b f(x)\,dx - (b-a)f(a)\right| \le$

(1) $\frac{(b-a)}{2}$ (2) $\frac{(b-a)^2}{2}$
(3) $(b-a)^2$ (4) none of these

100. The value of the integral $\int_0^{\infty} \frac{x \log x}{(1+x^2)^2}dx$ is

(1) 0 (2) log 7
(3) 5 log 13 (4) none of these

101. $\int_0^{\infty}\left(\frac{\pi}{1+\pi^2x^2} - \frac{1}{1+x^2}\right)\log x\,dx$ is equal to

(1) $-\frac{\pi}{2}\ln \pi$ (2) 0
(3) $\frac{\pi}{2}\ln 2$ (4) none of these

102. If $A = \int_0^{\pi} \frac{\cos x}{(x+2)^2}dx$, then $\int_0^{\pi/2} \frac{\sin 2x}{x+1}dx$ is equal to

(1) $\frac{1}{2} + \frac{1}{\pi+2} - A$ (2) $\frac{1}{\pi+2} - A$
(3) $1 + \frac{1}{\pi+2} - A$ (4) $A - \frac{1}{2} - \frac{1}{\pi+2}$

103. $\int_0^4 \frac{(y^2-4y+5)\sin(y-2)dy}{[2y^2-8y+1]}$ is equal to

(1) 0 (2) 2
(3) − 2 (4) none of these

104. $\int_{\sin\theta}^{\cos\theta} f(x\tan\theta)dx \left(\text{where } \theta \ne \frac{n\pi}{2}, n \in I\right)$ is equal to

(1) $-\cos\theta\int_1^{\tan\theta} f(x\sin\theta)dx$

(2) $-\tan\theta\int_{\cos\theta}^{\sin\theta} f(x)dx$

(3) $\sin\theta\int_1^{\tan\theta} f(x\cos\theta)dx$

(4) $\frac{1}{\tan\theta}\int_{\sin\theta}^{\sin\theta\tan\theta} f(x)dx$

105. Let $I_1 = \int_0^1 \frac{e^x dx}{1+x}$ and $I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)}$. Then $\frac{I_1}{I_2}$ is equal to

(1) $3/e$ (2) $e/3$ (3) $3e$ (4) $1/3e$

106. Let $I_1 = \int_{-2}^{2} \frac{x^6+3x^5+7x^4}{x^4+2} dx$ and

$I_2 = \int_{-3}^{1} \frac{2(x+1)^2+11(x+1)+14}{(x+1)^4+2} dx$.

Then the value of $I_1 + I_2$ is

(1) 8
(2) 200/3
(3) 100/3
(4) none of these

107. Let f be integrable over $[0, a]$ for any real value of a.

If $I_1 = \int_0^{\pi/2} \cos\theta\, f\left(\sin\theta + \cos^2\theta\right) d\theta$ and

$I_2 = \int_0^{\pi/2} \sin 2\theta\, f(\sin\theta + \cos^2\theta)\, d\theta$, then

(1) $I_1 = -2I_2$
(2) $I_1 = I_2$
(3) $2I_1 = I_2$
(4) $I_1 = -I_2$

108. The value of $\int_a^b (x-a)^3(b-x)^4 dx$ is

(1) $\frac{(b-a)^4}{6^4}$
(2) $\frac{(b-a)^8}{280}$
(3) $\frac{(b-a)^7}{7^3}$
(4) none of these

109. If $I(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$, $(m, n \in I, m, n \geq 0)$, then

(1) $I(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m-n}} dx$

(2) $I(m, n) = \int_0^{\infty} \frac{x^{m}}{(1+x)^{m+n}} dx$

(3) $I(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$

(4) $I(m, n) = \int_0^{\infty} \frac{x^{n}}{(1+x)^{m+n}} dx$

110. The value of the definite integral $\int_0^{\pi/2} \frac{\sin 5x}{\sin x} dx$ is

(1) 0
(2) $\frac{\pi}{2}$
(3) π
(4) 2π

111. If $I_n = \int_0^{\pi} e^x (\sin x)^n dx$, then $\frac{I_3}{I_1}$ is equal to

(1) 3/5
(2) 1/5
(3) 1
(4) 2/5

112. If $f'(x) = f(x) + \int_0^1 f(x) dx$, given $f(0) = 1$, then the value of $f(\log_e 2)$ is

(1) $\frac{1}{3+e}$
(2) $\frac{5-e}{3-e}$
(3) $\frac{2+e}{e-2}$
(4) none of these

113. Let $f(x)$ be positive, continuous, and differentiable on the interval (a, b) and $\lim_{x\to a^+} f(x) = 1$, $\lim_{x \to b^-} f(x) = 3^{1/4}$. If $f'(x) \geq f^3(x) + \frac{1}{f(x)}$, then the greatest value of $b - a$ is

(1) $\frac{\pi}{48}$
(2) $\frac{\pi}{36}$
(3) $\frac{\pi}{24}$
(4) $\frac{\pi}{12}$

Multiple Correct Answers Type

1. If $f(x)$ is integrable over $[1, 2]$, then $\int_1^2 f(x)\,dx$ is equal to

(1) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$
(2) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=n+1}^{2n} f\left(\frac{r}{n}\right)$
(3) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r+n}{n}\right)$
(4) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} f\left(\frac{r}{n}\right)$

2. If $L = \lim_{n\to\infty} \frac{n^3(e^{1/n} + e^{2/n} + \ldots + e)}{(n+1)^m (1^m + 4^m + \ldots + n^{2m})}$ is non-zero finite real, then

(1) $L = 3(e-1)$
(2) $L = 2(e-1)$
(3) $m = 1/3$
(4) $m = 1/3$

3. Let $p = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{120}}$ and

$q = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{121}}$, then

(1) $p > 20$
(2) $q < 20$
(3) $p + q < 40$
(4) $p + q > 40$

4. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$ for $n = 1, 2, 3, \ldots$. Then

(1) $S_n < \frac{\pi}{3\sqrt{3}}$
(2) $S_n > \frac{\pi}{3\sqrt{3}}$
(3) $T_n < \frac{\pi}{3\sqrt{3}}$
(4) $T_n > \frac{\pi}{3\sqrt{3}}$

5. The value of $\int_0^1 \frac{2x^2+3x+3}{(x+1)(x^2+2x+2)} dx$ is

(1) $\frac{\pi}{4} + 2\log 2 - \tan^{-1} 2$
(2) $\frac{\pi}{4} + 2\log 2 - \tan^{-1}\frac{1}{3}$
(3) $2\log 2 - \cot^{-1} 3$
(4) $-\frac{\pi}{4} + \log 4 + \cot^{-1} 2$

6. Let $f(x)=\int\limits_{1}^{x}\frac{3^t}{1+t^2}dt$, where $x>0$. Then

(1) for $0<\alpha<\beta$, $f(\alpha)<f(\beta)$
(2) for $0<\alpha<\beta$, $f(\alpha)>f(\beta)$
(3) $f(x)+\pi/4<\tan^{-1}x\ \forall\ x\geq 1$
(4) $f(x)+\pi/4>\tan^{-1}x\ \forall\ x\geq 1$

7. If $\int_a^b|\sin x|\,dx=8$ and $\int_0^{a+b}|\cos x|\,dx=9$, then

(1) $a+b=\frac{9\pi}{2}$ (2) $|a-b|=4\pi$
(3) $\frac{a}{b}=15$ (4) $\int_a^b \sec^2 x dx=0$

8. If $g(x)=\int\limits_0^x 2|t|\,dt$, then

(1) $g(x)=x|x|$
(2) $g(x)$ is monotonic
(3) $g(x)$ is differentiable at $x=0$
(4) $g'(x)$ is differentiable at $x=0$

9. If $A_n=\int\limits_0^{\pi/2}\frac{\sin(2n-1)x}{\sin x}dx$, $B_n=\int\limits_0^{\pi/2}\left(\frac{\sin nx}{\sin x}\right)^2dx$, for $n\in N$, then

(1) $A_{n+1}=A_n$ (2) $B_{n+1}=B_n$
(3) $A_{n+1}-A_n=B_{n+1}$ (4) $B_{n+1}-B_n=A_{n+1}$

10. The value of $\int\limits_0^{\infty}\frac{dx}{1+x^4}$ is

(1) same as that of $\int\limits_0^{\infty}\frac{x^2+1dx}{1+x^4}$

(2) $\frac{\pi}{2\sqrt{2}}$

(3) same as that of $\int\limits_0^{\infty}\frac{x^2dx}{1+x^4}$

(4) $\frac{\pi}{\sqrt{2}}$

11. The value of $\int\limits_0^1 e^{x^2-x}dx$ is

(1) <1 (2) >1 (3) $>e^{-\frac{1}{4}}$ (4) $<e^{-\frac{1}{4}}$

12. If $\int\limits_a^b\frac{f(x)}{f(x)+f(a+b-x)}dx=10$, then

(1) $b=22, a=2$ (2) $b=15, a=-5$
(3) $b=10, a=-10$ (4) $b=10, a=-2$

13. The values of a for which the integral $\int\limits_0^2|x-a|dx\geq 1$ is satisfied are

(1) $[2,\infty)$ (2) $(-\infty,0]$
(3) $(0,2)$ (4) none of these

14. If $f(x)=\int\limits_0^x|t-1|\,dt$, where $0\leq x\leq 2$, then

(1) range of $f(x)$ is $[0, 1]$
(2) $f(x)$ is differentiable at $x=1$
(3) $f(x)=\cos^{-1}x$ has two real roots
(4) $f'(1/2)=1/2$

15. If $f(2-x)=f(2+x)$ and $f(4-x)=f(4+x)$ for all x and $f(x)$ is a function for which $\int\limits_0^2 f(x)\,dx=5$, then $\int\limits_0^{50} f(x)\,dx$ is equal to

(1) 125 (2) $\int\limits_{-4}^{46} f(x)dx$

(3) $\int\limits_1^{51} f(x)\,dx$ (4) $\int\limits_2^{52} f(x)dx$

16. If $f(x)=\int\limits_0^x(\cos(\sin t)+\cos(\cos t))dt$, then $f(x+\pi)$ is

(1) $f(x)+f(\pi)$ (2) $f(x)+2f(\pi)$
(3) $f(x)+f\left(\frac{\pi}{2}\right)$ (4) $f(x)+2f\left(\frac{\pi}{2}\right)$

17. If $I_n=\int\limits_0^{\pi/4}\tan^n x\,dx$, ($n>1$ and is an integer), then

(1) $I_n+I_{n-2}=\frac{1}{n+1}$

(2) $I_n+I_{n-2}=\frac{1}{n-1}$

(3) $I_2+I_4, I_4+I_6, \ldots$, are in H.P.

(4) $\frac{1}{2(n+1)}<I_n<\frac{1}{2(n-1)}$

18. If $I_n=\int\limits_0^1\frac{dx}{(1+x^2)^n}$, where $n\in N$, which of the following statements hold good?

(1) $2n\,I_{n+1}=2^{-n}+(2n-1)I_n$
(2) $I_2=\frac{\pi}{8}+\frac{1}{4}$
(3) $I_2=\frac{\pi}{8}-\frac{1}{4}$
(4) $I_3=\frac{3\pi}{32}+\frac{1}{4}$

19. Let $f:[1,\infty)\to R$ and $f(x)=x\int\limits_1^x\frac{e^t}{t}dt-e^x$. Then

(1) $f(x)$ is an increasing function
(2) $\lim\limits_{x\to\infty}f(x)\to\infty$
(3) $f'(x)$ has a maxima at $x=e$
(4) $f(x)$ is a decreasing function

20. If $f(x)=\int_a^x [f(x)]^{-1}dx$ and $\int_a^1 [f(x)]^{-1}dx=\sqrt{2}$, then

(1) $f(2)=2$ (2) $f'(2)=1/2$

(3) $f^{-1}(2)=2$ (4) $\int_0^1 f(x)dx=\sqrt{2}$

21. A function $f(x)$ satisfies the relation

$f(x)=e^x+\int\limits_0^1 e^x f(t)\,dt$. Then

(1) $f(0)<0$

(2) $f(x)$ is a decreasing function

(3) $f(x)$ is an increasing function

(4) $\int\limits_0^1 f(x)\,dx>0$

22. $\int_0^x\left\{\int_0^u f(t)\,dt\right\}du$ is equal to

(1) $\int_0^x (x-u)f(u)du$ (2) $\int_0^x uf(x-u)du$

(3) $x\int_0^x f(u)du$ (4) $x\int_0^x uf(u-x)du$

23. Which of the following statement(s) is/are true?

(1) If function $y=f(x)$ is continuous at $x=c$ such that $f(c)\neq 0$, then $f(x)f(c)>0\ \forall\ x\in(c-h, c+h)$, where h is sufficiently small positive quantity.

(2) $\lim\limits_{n\to\infty}\frac{1}{n}\ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right)=1+2\ln 2.$

(3) Let f be a continuous and non-negative function defined on $[a, b]$. If $\int\limits_a^b f(x)dx=0$, then $f(x)=0\ \forall\ x\in[a,b]$.

(4) Let f be a continuous function defined on $[a, b]$ such that $\int\limits_a^b f(x)dx=0$. Then there exists at least one $c\in(a, b)$ for which $f(c)=0$.

24. If $\int_0^x [x]dx=\int_0^{[x]} x\,dx$, (where [.] and {.} denotes the greatest integer and fractional part respectively), then

(1) $x\in[0, 1)$ (2) $\{x\}=1/2$

(3) $\{x\}=1/3$ (4) $x>0$

25. Consider the function $f(\theta)=\int\limits_0^1 \frac{\left|\sqrt{1-x^2}-\sin\theta\right|}{\sqrt{1-x^2}}dx$, where $0\le\theta\le\frac{\pi}{2}$, then

(1) $f_{min.}=\sqrt{2}-1$ (2) $f_{min.}=\sqrt{2}+1$

(3) $f_{max.}=1$ (4) $f_{max.}=\frac{\pi}{2}-1$

26. $f:[0,1]\to R$ be a non increasing function, then for $\alpha\in(0, 1)$

(1) $\alpha\int\limits_0^1 f(x)\,dx\le\int\limits_0^\alpha f(x)\,dx$

(2) $\alpha\int\limits_0^1 f(x)\,dx\ge\int\limits_0^\alpha f(x)\,dx$

(3) $\alpha^2\int\limits_0^1 f(x)\,dx\le\int\limits_0^\alpha f(x)\,dx$

(4) $\sqrt{\alpha}\int\limits_0^1 f(x)\,dx\ge\int\limits_0^\alpha f(x)\,dx$

27. Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x)=f(1-x)$ and $f'\left(\frac{1}{4}\right)=0$. Then

(1) $f'(x)$ vanishes at least twice on [0, 1]

(2) $f'\left(\frac{1}{2}\right)=0$

(3) $\int\limits_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x\,dx=0$

(4) $\int\limits_0^{1/2} f(t)e^{\sin\pi t}\,dt=\int\limits_{1/2}^1 f(1-t)\,e^{\sin\pi t}\,dt$

Linked Comprehension Type

For Problems 1–3

$y=f(x)$ satisfies the relation $\int\limits_2^x f(t)dt=\frac{x^2}{2}+\int\limits_x^2 t^2 f(t)dt.$

1. The range of $y=f(x)$ is

(1) $[0,\infty)$ (2) R

(3) $(-\infty, 0]$ (4) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

2. The value of $\int\limits_{-2}^2 f(x)dx$ is

(1) 0 (2) -2

(3) $2\log_e 2$ (4) none of these

3. The value of x for which $f(x)$ is increasing is

(1) $(-\infty, 1]$ (2) $[-1, \infty)$

(3) $[-1, 1]$ (4) none of these

For Problems 4–6

Let $f: R\to R$ be a differentiable function such that

$f(x)=x^2+\int\limits_0^x e^{-t}f(x-t)dt.$

4. $f(x)$ increases for

(1) $x>1$ (2) $x<-2$

(3) $x>2$ (4) none of these

5. $y=f(x)$ is

(1) injective but not surjective

(2) surjective but not injective

(3) bijective

(4) neither injective nor surjective

6. The value of $\int_0^1 f(x)dx$ is

(1) $\frac{1}{4}$ (2) $-\frac{1}{12}$ (3) $\frac{5}{12}$ (4) $\frac{12}{7}$

For Problems 7–9

$f(x)$ satisfies the relation $f(x) - \lambda \int_0^{\pi/2} \sin x \cos t\, f(t)dt = \sin x$.

7. If $\lambda > 2$, then $f(x)$ decreases in which of the following interval?

(1) $(0, \pi)$ (2) $(\pi/2, 3\pi/2)$
(3) $(-\pi/2, \pi/2)$ (4) None of these

8. If $f(x) = 2$ has at least one real root, then

(1) $\lambda \in [1, 4]$ (2) $\lambda \in [-1, 2]$
(3) $\lambda \in [0, 1]$ (4) $\lambda \in [1, 3]$

9. If $\int_0^{\pi/2} f(x)dx = 3$, then the value of λ is

(1) 1 (2) 3/2
(3) 4/3 (4) none of these

For Problems 10–13

Let $f(x)$ and $\phi(x)$ are two continuous functions on R satisfying $\phi(x) = \int_a^x f(t)dt$, $a \neq 0$ and another continuous function $g(x)$ satisfying $g(x + \alpha) + g(x) = 0\ \forall\ x \in R$, $\alpha > 0$, and $\int_b^{2k} g(t)dt$ is independent of b.

10. If $f(x)$ is an odd function, then

(1) $\phi(x)$ is also an odd function
(2) $\phi(x)$ is an even function
(3) $\phi(x)$ is neither an even nor an odd function
(4) for $\phi(x)$ to be an even function, it must satisfy $\int_0^a f(x)dx = 0$

11. If $f(x)$ is an even function, then

(1) $\phi(x)$ is also an even function
(2) $\phi(x)$ is an odd function
(3) if $f(a - x) = -f(x)$, then $\phi(x)$ is an even function
(4) if $f(a - x) = -f(x)$, then $\phi(x)$ is an odd function

12. Least positive value of c if c, k, b are in A.P. is

(1) 0 (2) 1 (3) α (4) 2α

13. If m, n are even integers and $p, q \in R$, then $\int_{p+m\alpha}^{q+n\alpha} g(t)dt$ is equal to

(1) $\int_p^q g(x)dx$

(2) $(n - m)\int_0^\alpha g(x)dx$

(3) $\int_p^q g(x)dx + (n - m)\int_0^\alpha g(2x)dx$

(4) $\int_p^q g(x)dx + \frac{(n-m)}{2}\int_0^{2\alpha} g(x)dx$

For Problems 14–17

Evaluating integrals dependent on a parameter:

Differentiate I with respect to the parameter within the sign of integrals taking variable of the integrand as constant. Now, evaluate the integral so obtained as a function of the parameter and then integrate the result to get I. Constant of integration can be computed by giving some arbitrary values to the parameter and the corresponding value of I.

14. The value of $\int_0^1 \frac{x^a - 1}{\log x}dx$ is

(1) $\log(a - 1)$ (2) $\log(a + 1)$
(3) $a\log(a + 1)$ (4) none of these

15. The value of $\int_0^{\pi/2} \log(\sin^2\theta + k^2\cos^2\theta)d\theta$, where $k \geq 0$, is

(1) $\pi\log(1 + k) + \pi\log 2$ (2) $\pi\log(1 + k)$
(3) $\pi\log(1 + k) - \pi\log 2$ (4) $\log(1 + k) - \log 2$

16. The value of $\frac{dI}{da}$ when $I = \int_0^{\pi/2} \log\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{dx}{\sin x}$ (where $|a| < 1$) is

(1) $\frac{\pi}{\sqrt{1-a^2}}$ (2) $-\pi\sqrt{1-a^2}$
(3) $\sqrt{1-a^2}$ (4) $\frac{\sqrt{1-a^2}}{\pi}$

17. If $\int_0^\pi \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{a^2-1}}$, then the value of $\int_0^\pi \frac{dx}{(\sqrt{10}-\cos x)^3}$ is

(1) $\frac{\pi}{81}$ (2) $\frac{7\pi}{162}$
(3) $\frac{7\pi}{81}$ (4) none of these

For Problems 18–20

$f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t\cos x)f(t)dt$

18. The range of $f(x)$ is

(1) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ (2) $\left[-\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{3}\right]$
(3) $\left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$ (4) none of these

19. $f(x)$ is not invertible for

(1) $x \in \left[-\frac{\pi}{2} - \tan^{-1}2, \frac{\pi}{2} - \tan^{-1}2\right]$

(2) $x \in \left[\tan^{-1}\frac{1}{2}, \pi + \tan^{-1}\frac{1}{2}\right]$

(3) $x \in \left[\pi + \cot^{-1}2, 2\pi + \cot^{-1}2\right]$

(4) none of these

20. The value of $\int\limits_0^{\pi/2} f(x)dx$ is

(1) 1 (2) -2 (3) -1 (4) 2

For Problems 21 and 22

Let $u = \int\limits_0^{\infty} \frac{dx}{x^4 + 7x^2 + 1}$ and $v = \int\limits_0^{\infty} \frac{x^2 dx}{x^4 + 7x^2 + 1}$.

21. The value of u is

(1) $\pi/3$ (2) $\pi/6$ (3) $\pi/12$ (4) $\pi/9$

22. The value of v is

(1) $\pi/3$ (2) $\pi/6$ (3) $\pi/12$ (4) $\pi/9$

For Problems 23 and 24

If $f(x) = \int_0^1 \frac{dt}{1 + |x - t|}$, $x \in R$.

23. The value of $f'(1/2)$ is equal to

(1) 1/2 (2) 0 (3) 1 (4) 2

24. Which of the following is not true about $f(x)$?

(1) $f(x)$ is decreasing for $x > 1$

(2) $f(x)$ is increasing for $x < 0$

(3) $f(1) = \log_e 2$

(4) $f(1/2) = \log_e(3/2)$

For Problems 25 and 26

Let f be a differentiable function satisfying

$$\int\limits_0^{f(x)} f^{-1}(t)dt - \int\limits_0^{x}(\cos t - f(t))dt = 0 \text{ and } f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

25. The value of $\int\limits_0^{\pi/2} f(x)dx$ lies in the interval

(1) $\left(\frac{2}{\pi}, 1\right)$ (2) $\left(1, \frac{\pi}{2}\right)$

(3) $\left(\frac{3}{2}, \frac{\pi}{2}\right)$ (4) $\left(0, \frac{2}{\pi}\right)$

26. The value of $\lim\limits_{x\to 0} \frac{\cos x}{f(x)}$ is equal to, where [.] denotes greatest integer function.

(1) 0 (2) 1 (3) 1/2 (4) 2

For Problems 27 and 28

If $U_n = \int_0^{\pi} \frac{1 - \cos nx}{1 - \cos x} dx$, where n is positive integer or zero, then

27. The value of U_n is

(1) $\pi/2$ (2) π (3) $n\pi/2$ (4) $n\pi$

28. The value of $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta} d\theta$ is

(1) $\pi/2$ (2) π (3) $n\pi/2$ (4) $n\pi$

For Problems 29–31

Let the definite integral be defined by the formula $\int_a^b f(x)dx = \frac{b-a}{2}(f(a) + f(b))$. For more accurate result, for $c \in (a, b)$, we can use $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = F(c)$ so that for $c = \frac{a+b}{2}$, we get $\int_a^b f(x)dx = \frac{b-a}{4}(f(a) + f(b) + 2f(c))$.

29. $\int_0^{\pi/2} \sin x dx$ is equal to

(1) $\frac{\pi}{8}(1+\sqrt{2})$ (2) $\frac{\pi}{4}(1+\sqrt{2})$

(3) $\frac{\pi}{8\sqrt{2}}$ (4) $\frac{\pi}{4\sqrt{2}}$

30. If $\lim\limits_{x\to a} \frac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x) + f(a))}{(x-a)^3} = 0$, then $f(x)$ is of maximum degree

(1) 4 (2) 3 (3) 2 (4) 1

31. If $f''(x) < 0\ \forall\ x \in (a, b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to

(1) $\frac{f(b) - f(a)}{b - a}$ (2) $\frac{2(f(b) - f(a))}{b - a}$

(3) $\frac{2f(b) - f(a)}{2b - a}$ (4) 0

Matrix Match Type

1. If [·] denotes the greatest integer function, then match the following lists:

List I	List II
a. $\int_{-1}^{1} [x + [x + [x]]]\, dx$	**p.** 3
b. $\int_2^5 ([x] + [-x])\, dx$	**q.** 5
c. $\int_{-1}^{3} \text{sgn}(x - [x])dx$	**r.** 4
d. $25\int_0^{\pi/4}\left(\tan^6(x - [x]) + \tan^4(x - [x])\right)dx$	**s.** -3

2. Match the following lists:

List I	List II
a. $\lim_{n\to\infty}\left[\dfrac{\int_0^2\left(1+\frac{t}{n+1}\right)^n dt}{n+1}\right]$ is equal to	**p.** $e-\frac{1}{2}e^2-\frac{3}{2}$
b. Let $f(x)$ be a function satisfying $f'(x)=f(x)$ with $f(0)=1$ and g be the function satisfying $f(x)+g(x)=x^2$. Then the value of the integral $\int_0^1 f(x)g(x)dx$ is	**q.** e^2
c. $\int_0^1 e^{e^x}(1+xe^x)dx$ is equal to	**r.** e^2-1
d. $\lim_{k\to 0}\frac{1}{k}\int_0^k (1+\sin 2x)^{\frac{1}{x}}dx$ is equal to	**s.** e^e

3. Match the following lists:

List I	List II
a. If $f(x)$ is an integrable function for $x\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]$ and $I_1=\int_{\pi/6}^{\pi/3}\sec^2\theta f(2\sin 2\theta)\,d\theta$, and $I_2=\int_{\pi/6}^{\pi/3}\text{cosec}^2\theta f(2\sin 2\theta)d\theta$, then $I_1/I_2=$	**p.** 3
b. If $f(x+1)=f(3+x)\ \forall\ x$, and the value of $\int_a^{a+b} f(x)dx$ is independent of a, then the value of b can be	**q.** 1
c. The value of $2\int_1^4\dfrac{\tan^{-1}[x^2]}{\tan^{-1}[x^2]+\tan^{-1}[25+x^2-10x]}dx$ (where [.]denotes the greatest integer function) is	**r.** 2
d. If $I=\int_0^2\sqrt{x+\sqrt{x+\sqrt{x+\cdots\infty}}}\,dx$ (where $x>0$), then $[I]$ is equal to (where [.] denotes the greatest integer function)	**s.** 4

4. Match the following lists:

List I	List II
a. If $I=\int_{-2}^{2}(\alpha x^3+\beta x+\gamma)dx$, then I is	**p.** independent of α
b. Let α,β be the distinct positive roots of the equation $\tan x=2x$. Then $\gamma\int_0^1(\sin\alpha x\cdot\sin\beta x)dx$ (where $\gamma\neq 0$) is	**q.** independent of β
c. If $f(x+\alpha)+f(x)=0$, where $\alpha>0$, then $\int_\beta^{\beta+2\gamma\alpha}f(x)dx$, where $\gamma\in N$, is	**r.** independent of γ
d. $\gamma\int_0^\alpha[\sin x]dx$ is, where $\gamma\neq 0$, $\alpha\in[(2\beta+1)\pi,(2\beta+2)\pi]$, $n\in N$, and where [.] denotes the greatest integer function,	**s.** depends on α

5. Let $\int_0^\infty\frac{\sin x}{x}dx=\alpha$. Then match the following lists and choose the correct code.

List I	List II
a. $\int_0^\infty\frac{\sin(5x)}{x}dx=$	**p.** 0
b. $\int_0^\infty\frac{\sin^2 x}{x^2}dx+\alpha=$	**q.** α
c. $\int_0^\infty\frac{\sin^3 x}{x}dx=$	**r.** $\alpha/2$
d. $\int_0^\infty\frac{\sin(k_1x).\cos k_2x}{x}dx-\alpha=$ (where $k_1>k_2>0$)	**s.** 2α

Codes:

	a	b	c	d
(1)	r	q	p	s
(2)	p	q	s	q
(3)	s	p	q	r
(4)	q	s	r	p

6. If $I(m)=\int_0^\pi\log_e\left(1-2m\cos x+m^2\right)dx$, then match the following lists and choose the correct code.

List I	List II
a. $I(1)$	**p.** I(−1)
b. $\frac{I(9)}{I(3)}$	**q.** 2
c. $\frac{I(81)}{I(3)}$	**r.** 0
d. $\frac{I(25)}{I(5)}$	**s.** 4

Codes:

	a	b	c	d
(1)	r	q	p	s
(2)	p	q	s	q
(3)	s	p	q	r
(4)	q	p	s	r

Numerical Value Type

1. If the value of $\lim_{n\to\infty}(n^{-3/2})\cdot\sum_{j=1}^{6n}\sqrt{j}$ is equal to $\sqrt{N}$, then the value of N is____.

2. $\lim_{n\to\infty}\frac{n}{2^n}\int_0^2 x^n\,dx$ equals____.

3. A continuous real function f satisfies $f(2x) = 3f(x)$ $\forall x \in R$. If $\int_0^1 f(x)\,dx = 1$, then the value of definite integral $\int_1^2 f(x)\,dx$ is____

4. Consider the polynomial $f(x) = ax^2 + bx + c$. If $f(0) = 0$, $f(2) = 2$, then the minimum value of $\int_0^2 |f'(x)|\,dx$ is____.

5. If $I = \int_0^{3\pi/4}((1+x)\sin x + (1-x)\cos x)\,dx$, then the value of $(\sqrt{2}-1)\,I$ is____.

6. If $\int_0^{100} f(x)dx = 7$, then $\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)dx\right) =$ ____.

7. The value of $\int_0^{\frac{3\pi}{2}}\frac{|\tan^{-1}\tan x| - |\sin^{-1}\sin x|}{|\tan^{-1}\tan x| + |\sin^{-1}\sin x|}\,dx$ is equal to____.

8. Let $f(x) = x^3 - \frac{3x^2}{2} + x + \frac{1}{4}$. Then the value of $\left(\int_{1/4}^{3/4} f(f(x))\,dx\right)^{-1}$ is____.

9. The value of $\int_0^1 \frac{\tan^{-1}x}{\cot^{-1}(1-x+x^2)}\,dx$ is ____.

10. Let $f(x)$ be differentiate function symmetric about $x = 2$, then the value of $\int_0^4 \cos(\pi x)\,f'(x)\,dx$ is equal to____.

11. Let f: $[0, \infty) \to R$ be a continuous strictly increasing function, such that $f^3(x) = \int_0^x t\cdot f^2(t)\,dt$ for every $x \geq 0$. Then value of $f(6)$ is____.

12. If f is continuous function and
$$F(x) = \int_0^x\left((2t+3)\cdot\int_t^2 f(u)\,du\right)dt,$$
then $|F''(2)/f(2)|$ is equal to____.

13. If the value of the definite integral $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2 - x + 1}\,dx$ is $\frac{\pi^2}{\sqrt{n}}$ (where $n \in N$), then the value of n is____.

14. Let $f(x) = \int_0^x \frac{dt}{\sqrt{1+t^3}}$ and $g(x)$ be the inverse of $f(x)$. Then the value of $4\frac{g''(x)}{(g(x))^2}$ is____.

15. Let $g(x)$ be differentiable on R and $\int_{\sin t}^{1} x^2 g(x)\,dx = (1 - \sin t)$, where $t \in \left(0, \frac{\pi}{2}\right)$. Then the value of $g\left(\frac{1}{\sqrt{2}}\right)$ is____.

16. If $\int_0^\infty x^{2n+1}\cdot e^{-x^2}\,dx = 360$, then the value of n is____

17. Let $f(x)$ be a derivable function satisfying $f(x) = \int_0^x e^t \sin(x-t)dt$ and $g(x) = f''(x) - f(x)$. Then the possible integers in the range of $g(x)$ is____.

18. If $F(x) = \frac{1}{x^2}\int_4^x [4t^2 - 2F'(t)]dt$, then $(9F'(4))/4$ is____

19. If the value of the definite integral $\int_0^1 {}^{207}C_7\, x^{200}\cdot(1-x)^7\,dx$ is equal to $\frac{1}{k}$, where $k \in N$, then the value of k is____

20. If $I_n = \int_0^1 (1-x^5)^n\,dx$, then $\frac{55}{7}\frac{I_{10}}{I_{11}}$ is equal to____

21. The value of $5050\frac{\int_0^1 (1-x^{50})^{100}\,dx}{\int_0^1 (1-x^{50})^{101}\,dx}$ is ____.

22. Let $J = \int_{-5}^{-4} (3-x^2)\tan(3-x^2)\,dx$ and $K = \int_{-2}^{-1} (6-6x+x^2)\tan(6x - x^2 - 6)dx$. Then $(J + K)$ equals____.

23. The value of the definite integral $\int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{x^4+x^2+2}{(x^2+1)^2}\,dx$ equals____

24. Consider a real-valued continuous function f such that $f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t f(t))\,dt$. If M and m are maximum and minimum values of the function f, then the value of M/m is____

25. If $f(x) = x + \int_0^1 t(x+t)\,f(t)\,dt$, then the value of $\frac{23}{2}f(0)$ is equal to____.

26. Let $y = f(x) = 4x^3 + 2x - 6$, then the value of $\int_0^2 f(x)dx + \int_0^{30} f^{-1}(y)dy$ is equal to____.

27. The value of $\int_1^3 \left(\sqrt{1+(x-1)^3} + (x^2-1)^{\frac{1}{3}} + 1 \right) dx$ is ______.

28. The value of $\int_0^1 \cos^{-1}\left((x-x^2) - \sqrt{(1-x^2)(2x-x^2)}\right) dx$ is equal to ______.

Archives

JEE Main

Single Correct Answer Type

1. $\int_0^{\pi} [\cot x]\, dx$, where [.] denotes the greatest integer function, is equal to

(1) $\frac{\pi}{2}$ (2) 1

(3) -1 (4) $-\frac{\pi}{2}$ **(AIEEE 2009)**

2. Let $p(x)$ be a function defined on R such that $p'(x) = p'(1-x)$, for all $x \in [0, 1]$, $p(0) = 1$, and $p(1) = 41$. Then $\int_0^1 p(x)\, dx$ is equal to

(1) 42 (2) $\sqrt{41}$ (3) 21 (4) 41 **(AIEEE 2010)**

3. The value of $\int_0^1 \frac{8 \log(1+x)}{1+x^2} dx$ is

(1) log 2 (2) $\pi \log 2$

(3) $\frac{\pi}{8}\log 2$ (4) $\frac{\pi}{2}\log 2$ **(AIEEE 2011)**

4. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\, \sin t\, dt$. Then f has

(1) local maximum at π and local minima at 2π
(2) local maximum at π and 2π
(3) local minimum at π and 2π
(4) local minimum at π and local maximum at 2π

(AIEEE 2011)

5. If $g(x) = \int_0^x \cos^4 t\, dt$, then $g(x+\pi)$ equals

(1) $\frac{g(x)}{g(\pi)}$ (2) $g(x) + g(\pi)$

(3) $g(x) - g(\pi)$ (4) $g(x) \cdot g(\pi)$

(AIEEE 2012)

6. **Statement I:**

The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ is equal to $\frac{\pi}{6}$.

Statement II:

$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(1) Statement I is true; statement II is true; statement II is a correct explanation for statement I
(2) Statement I is true; statement II is true; statement II is a not a correct explanation for statement I.
(3) Statement I is true; statement II is false.
(4) Statement I is false; statement II is true.

(JEE Main 2013)

7. The intercepts on x-axis made by tangents to the curve, $y = \int_0^x |t|\, dt$, $x \in R$, which are parallel to the line $y = 2x$, are equal to

(1) ± 1 (2) ± 2 (3) ± 3 (4) ± 4

(JEE Main 2013)

8. The integral $\int_0^{\pi} \sqrt{1 + 4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\, dx$ equals

(1) $\pi - 4$ (2) $\frac{2\pi}{3} - 4 - 4\sqrt{3}$

(3) $4\sqrt{3} - 4$ (4) $4\sqrt{3} - 4 - \frac{\pi}{3}$

(JEE Main 2014)

9. The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$ is equal to

(1) 2 (2) 4 (3) 1 (4) 6

(JEE Main 2015)

10. $\lim_{n\to\infty} \left(\frac{(n+1)(n+2)\ldots 3n}{n^{2n}}\right)^{1/n}$ is equal to

(1) $\frac{27}{e^2}$ (2) $\frac{9}{e^2}$ (3) $3\log 3 - 2$ (4) $\frac{18}{e^4}$

(JEE Main 2016)

11. The integral $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to

(1) -1 (2) -2 (3) 2 (4) 4

(JEE Main 2017)

12. The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$ is

(1) $\pi/4$ (2) $\pi/8$ (3) $\pi/2$ (4) 4π

(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\,dt, 0 \le x \le 1$, and $f(0)=0$, then

(1) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(2) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(3) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

(4) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$ **(IIT-JEE, 2009)**

2. The value of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are)

(1) $\frac{22}{7}-\pi$ (2) $\frac{2}{105}$

(3) 0 (4) $\frac{71}{15}-\frac{3\pi}{2}$

(IIT-JEE, 2010)

3. Let f be a real-valued function defined on the interval $(-1, 1)$ such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$, for all $x \in (-1, 1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to

(1) 1 (2) 1/3 (3) 1/2 (4) $1/e$

(IIT-JEE, 2010)

4. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)}dx$ is

(1) $\frac{1}{4}\ln\frac{3}{2}$ (2) $\frac{1}{2}\ln\frac{3}{2}$

(3) $\ln\frac{3}{2}$ (4) $\frac{1}{6}\ln\frac{3}{2}$

(IIT-JEE, 2011)

5. Let $f: [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} x f(x)\,dx$, and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$, and the x-axis. Then

(1) $R_1 = 2R_2$ (2) $R_1 = 3R_2$

(3) $2R_1 = R_2$ (4) $3R_1 = R_2$

(IIT-JEE, 2011)

6. Let $f:\left[\frac{1}{2}, 1\right] \to R$ (the set of all real numbers) be a positive, non-constant, and differentiable function such that $f'(x) < 2f(x)$ and $f(1/2) = 1$. Then the value of $\int_{1/2}^{1} f(x)\,dx$ lies in the interval

(1) $(2e-1, 2e)$ (2) $(e-1, 2e-1)$

(3) $\left(\frac{e-1}{2}, e-1\right)$ (4) $\left(0, \frac{e-1}{2}\right)$

(JEE Advanced, 2013)

7. Let $f: [0, 2] \to R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$. Let $F(x) = \int_0^{x^2} f(\sqrt{t})\,dt$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$, then $F(2)$ equals

(1) $e^2 - 1$ (2) $e^4 - 1$

(3) $e - 1$ (4) e^4

(JEE Advanced 2014)

8. The following integral $\int_{\pi/4}^{\pi/2} (2 \operatorname{cosec} x)^{17} dx\ dx$ is equal to

(1) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16}\,du$

(2) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{17}\,du$

(3) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{17}\,du$

(4) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16}\,du$ **(JEE Advanced 2014)**

9. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)\,dx \le M$, then the possible values of m and M are

(1) $m = 13, M = 24$ (2) $m = \frac{1}{4}, M = \frac{1}{2}$

(3) $m = -11, M = 0$ (4) $m = 1, M = 12$

(JEE Advanced 2015)

10. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x}dx$ is equal to

(1) $\frac{\pi^2}{4} - 2$ (2) $\frac{\pi^2}{4} + 2$

(3) $\pi^2 - e^{\frac{\pi}{2}}$ (4) $\pi^2 + e^{\frac{\pi}{2}}$

(JEE Advanced 2016)

Multiple Correct Answers Type

1. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}\,dx$, $n = 0, 1, 2, \ldots$, then

(1) $I_n = I_{n+2}$ (2) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(3) $\sum_{m=1}^{10} I_{2m} = 0$ (4) $I_n = I_{n+1}$

(IIT-JEE, 2009)

2. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is (are) true?

(1) $f''(x)$ exists for all $x \in (0, \infty)$.

(2) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$ but not differentiable on $(0, \infty)$.

(3) There exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(4) There exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$. **(IIT-JEE, 2010)**

3. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$. Then

(1) $S \ge \frac{1}{e}$ (2) $S \ge 1 - \frac{1}{e}$

(3) $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (4) $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

(IIT-JEE, 2012)

4. For $a \in R$ (the set of all real numbers), $a \ne -1$,

$$\lim_{n\to\infty} \frac{(1^a + 2^a + \ldots + n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+\cdots+(na+n)]} = \frac{1}{60}.$$

Then $a =$

(1) 5 (2) 7 (3) $\frac{-15}{2}$ (4) $\frac{-17}{2}$

(JEE Advanced 2013)

5. Let $f : [a, b] \to [1, \infty)$ be a continuous function and let $g : R \to R$ be defined as

$$g(x) = \begin{cases} 0 & \text{if } x < a \\ \int_a^x f(t)\,dt & \text{if } a \le x \le b \\ \int_a^b f(t)\,dt & \text{if } x > b \end{cases}$$ Then

(1) $g(x)$ is continuous but not differentiable at a

(2) $g(x)$ is differentiable on R

(3) $g(x)$ is continuous but not differentiable at b

(4) $g(x)$ is continuous and differentiable at either a or b but not both **(JEE Advanced 2014)**

6. Let $f : (0, \infty) \to R$ be given by $f(x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$, then

(1) $f(x)$ is monotonically increasing on $[1, \infty)$

(2) $f(x)$ is monotonically decreasing on $(0, 1)$

(3) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$

(4) $f(2^x)$ is an odd function of x on R

(JEE Advanced 2014)

7. The option(s) with the values of a and L that satisfy the following equation is (are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)\,dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)\,dt}$$

(1) $a = 2, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ (2) $a = 2, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

(3) $a = 4, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ (4) $a = 4, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

(JEE Advanced 2015)

8. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is (are)

(1) $\int_0^{\pi/4} xf(x)dx = \frac{1}{12}$ (2) $\int_0^{\pi/4} f(x)dx = 0$

(3) $\int_0^{\pi/4} xf(x) = \frac{1}{6}$ (4) $\int_0^{\pi/4} f(x)dx = 1$

(JEE Advanced 2015)

9. Let $f(x) = \lim_{n\to\infty} \left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right)\cdots\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right)\cdots\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}}$, for all $x > 0$. Then

(1) $f\left(\frac{1}{2}\right) \ge f(1)$ (2) $f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$

(3) $f'(2) \le 0$ (4) $\frac{f'(3)}{f(3)} \ge \frac{f'(2)}{f(2)}$

(JEE Advanced 2016)

10. Let $f : R \to (0, 1)$ be a continuous function. Then, which of the following function(s) has (have) the value zero at some point in the interval $(0, 1)$?

(1) $e^x - \int_0^x f(t)\sin t\,dt$ (2) $x^9 - f(x)$

(3) $f(x)+\int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$ (4) $x-\int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

(JEE Advanced 2017)

11. If $I=\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx$, then

(1) $I>\log_e 99$ (2) $I<\log_e 99$

(3) $I<\frac{49}{50}$ (4) $I>\frac{49}{50}$

(JEE Advanced 2017)

12. If $g(x)=\int_{\sin x}^{\sin(2x)}\sin^{-1}(t)dt$, then

(1) $g'\left(\frac{\pi}{2}\right)=-2\pi$ (2) $g'\left(-\frac{\pi}{2}\right)=2\pi$

(3) $g'\left(\frac{\pi}{2}\right)=2\pi$ (4) $g'\left(-\frac{\pi}{2}\right)=-2\pi$

(JEE Advanced 2017)

Linked Comprehension Type

For Problems 1 and 2

Given that for each $a\in(0,1)$, $\lim_{h\to 0^-}\int_h^{1-h} t^{-a}(1-t)^{a-1}dt$ exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$. **(JEE Advanced 2014)**

1. The value of $g\left(\frac{1}{2}\right)$ is

(1) π (2) 2π (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$

2. The value of $g'\left(\frac{1}{2}\right)$ is

(1) $\frac{\pi}{2}$ (2) π (3) $-\frac{\pi}{2}$ (4) 0

For Problems 3 and 4

Let $F: R\to R$ be a thrice differentiable function. Suppose that $F(1)=0$, $F(3)=-4$ and $F'(x)<0$ for all $x\in(1/2, 3)$. Let $f(x)=xF(x)$ for all $x\in R$. **(JEE Advanced 2015)**

3. The correct statement(s) is (are)

(1) $f'(1)<0$

(2) $f(2)<0$

(3) $f'(x)\neq 0$ for any $x\in(1,3)$

(4) $f'(x)=0$ for some $x\in(1,3)$

4. If $\int_1^3 x^2F'(x)dx=-12$ and $\int_1^3 x^3F''(x)dx=40$, then the correct expression(s) is (are)

(1) $9f'(3)+f'(1)-32=0$ (2) $\int_1^3 f(x)dx=12$

(3) $9f'(3)-f'(1)+32=0$ (4) $\int_1^3 f(x)dx=-12$

Matrix Match Type

1. Match the statements/expressions given in List I with the values given in List II.

List I	List II
a. The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0)=0$ and $\int_0^1 f(x)dx=1$, is	**p.** 8
b. The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x)=\sin(x^2)+\cos(x^2)$ attains its maximum value, is	**q.** 2
c. $\int_{-2}^{2}\frac{3x^2}{1+e^x}dx$ equals	**r.** 4
d. $\dfrac{\int_{-1/2}^{1/2}\cos 2x\cdot\log\left(\frac{1+x}{1-x}\right)dx}{\int_0^{1/2}\cos 2x\cdot\log\left(\frac{1+x}{1-x}\right)dx}$ equals	**s.** 0

(JEE Advanced 2014)

Codes:

	a	b	c	d
(1)	r	q	s	p
(2)	q	r	s	p
(3)	r	q	p	s
(4)	q	r	p	s

Numerical Value Type

1. For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real-valued function defined on the interval $[-10, 10]$ by

$$f(x)=\begin{cases}x-[x], & \text{if } [x] \text{ is odd}\\ 1+[x]-x, & \text{if } [x] \text{ is even}\end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x\,dx$ is ______.

(IIT-JEE, 2010)

2. Let $y'(x)+y(x)g'(x)=g(x)g'(x)$, $y(0)=0$, $x\in R$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0)=g(2)=0$. Then the value of $y(2)$ is ______.

3. The value of $\int_0^1 4x^3 \left\{ \frac{d^2}{dx^2}(1-x^2)^5 \right\} dx$ is ________.

(JEE Advanced 2014)

4. Let $f: R \to R$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)\,dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t|f(f(t))|\,dt$ for all $x \in [-1, 2]$. If $\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is

(JEE Advanced 2015)

5. If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$ where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is ________.

(JEE Advanced 2015)

6. Let $F(x) = \int_{x}^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x \in R$ and $f: \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0, y = 0, y = f(x)$ and $x = a$, then $f(0)$ is ________.

(JEE Advanced 2015)

7. Let $f: R \to R$ be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$ where $[x]$ is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)}dx$, then the value of $(4I - 1)$ is

(JEE Advanced 2015)

8. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x - 1$ is ________.

(JEE Advanced 2016)

9. Let $f: R \to R$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$.

If $g(x) = \int_x^{\frac{\pi}{2}} [f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)]dt$ for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x \to 0} g(x) =$ ________.

(JEE Advanced 2017)

10. For each positive integer n, let

$$y_n = \frac{1}{n}((n+1)(n+2)\ldots(n+n))^{1/n}.$$

For $x \in R$, let $[x]$ be the greatest integer less than or equal to x. If $\lim_{n \to \infty} y_n = L$, then the value of $[L]$ is ________.

(JEE Advanced 2018)

11. The value of the integral $\int_0^{1/2} \frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{1/4}}dx$ is ________.

(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (4)	2. (2)	3. (3)	4. (1)	5. (3)
6. (1)	7. (1)	8. (4)	9. (3)	10. (3)
11. (4)	12. (3)	13. (2)	14. (3)	15. (1)
16. (1)	17. (3)	18. (4)	19. (1)	20. (1)
21. (1)	22. (4)	23. (2)	24. (4)	25. (3)
26. (4)	27. (3)	28. (4)	29. (1)	30. (2)
31. (2)	32. (2)	33. (1)	34. (3)	35. (2)
36. (3)	37. (2)	38. (2)	39. (1)	40. (2)
41. (3)	42. (2)	43. (3)	44. (3)	45. (2)
46. (1)	47. (3)	48. (3)	49. (3)	50. (1)
51. (2)	52. (3)	53. (2)	54. (4)	55. (1)
56. (1)	57. (3)	58. (4)	59. (2)	60. (3)
61. (2)	62. (1)	63. (1)	64. (1)	65. (3)
66. (2)	67. (3)	68. (2)	69. (1)	70. (3)
71. (2)	72. (3)	73. (2)	74. (1)	75. (3)
76. (3)	77. (3)	78. (3)	79. (1)	80. (3)
81. (2)	82. (3)	83. (3)	84. (1)	85. (1)
86. (3)	87. (1)	88. (1)	89. (1)	90. (2)
91. (2)	92. (4)	93. (3)	94. (1)	95. (3)
96. (2)	97. (3)	98. (3)	99. (2)	100. (1)
101. (1)	102. (1)	103. (1)	104. (1)	105. (3)
106. (3)	107. (2)	108. (2)	109. (3)	110. (2)
111. (1)	112. (2)	113. (3)		

Multiple Correct Answers Type

1. (2), (3)	2. (1), (3)
3. (1), (2), (4)	4. (1), (4)
5. (1), (3), (4)	6. (1), (4)
7. (1), (2)	8. (1), (2), (3)
9. (1), (4)	10. (2), (3)
11. (1), (3)	12. (1), (2), (3)
13. (1), (2), (3)	14. (1), (2), (4)
15. (1), (2), (4)	16. (1), (4)
17. (2), (3), (4)	18. (1), (2), (4)

19. (1), (2)
20. (1), (2), (3)
21. (1), (2)
22. (1), (2)
23. (1), (3), (4)
24. (1), (2)
25. (1), (3)
26. (1), (3)
27. (1), (2), (3), (4)

Linked Comprehension Type

1. (4)	2. (1)	3. (3)	4. (2)	5. (2)
6. (3)	7. (3)	8. (4)	9. (3)	10. (2)
11. (4)	12. (4)	13. (4)	14. (2)	15. (3)
16. (1)	17. (3)	18. (2)	19. (4)	20. (3)
21. (2)	22. (2)	23. (2)	24. (4)	25. (2)
26. (2)	27. (4)	28. (3)	29. (1)	30. (4)
31. (2)				

Matrix Match Type

1. a → s; b → s; c → r; d → q
2. a → r; b → p; c → s; d → q
3. a → q; b → r, s; c → p; d → p.
4. a → p, q; b → p, q, r; c → q, s; d → s
5. (4) 6. (2)

Numerical Value Type

1. (96)	2. (2)	3. (5)	4. (2)	5. (2)
6. (7)	7. (0)	8. (4)	9. (0.5)	10. (0)
11. (6)	12. (7)	13. (108)	14. (6)	15. (2)
16. (6)	17. (3)	18. (32)	19. (208)	20. (8)
21. (5051)	22. (0)	23. (2)	24. (3)	25. (9)
26. (56)	27. (8)	28. (2)		

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (4)	2. (3)	3. (2)	4. (1)	5. (2)
6. (4)	7. (1)	8. (4)	9. (3)	10. (1)
11. (3)	12. (1)			

JEE ADVANCED

Single Correct Answer Type

1. (3)	2. (1)	3. (2)	4. (1)	5. (3)
6. (4)	7. (2)	8. (1)	9. (4)	10. (1)

Multiple Correct Answers Type

1. (1), (2), (3)	2. (2), (3)
3. (1), (2), (4)	4. (2), (4)
5. (1), (3)	6. (1), (3), (4)
7. (1), (3)	8. (1), (2)
9. (2), (3)	10. (2), (4)
11. (2), (4)	12. None

Linked Comprehension Type

1. (1) 2. (4) 3. (1), (2), (3) 4. (3), (4)

Matrix Match Type

1. (4)

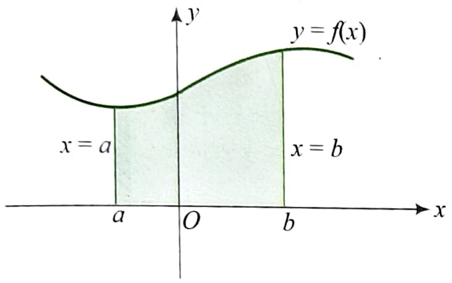
Numerical Value Type

1. (4)	2. (0)	3. (2)	4. (7)
5. (9)	6. (3)	7. (0)	8. (1)
9. (2)	10. (1)	11. (2)	

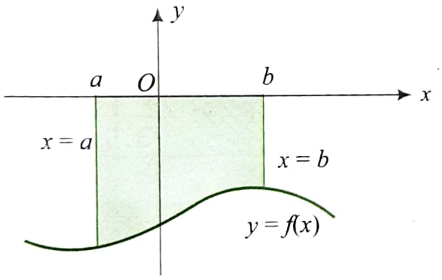
9 Area

AREA BOUNDED BY CURVE AND AXIS

In definite integration we learned that if $f(x) \geq 0\ \forall\ x \in [a, b]$ then $\int_a^b f(x)dx$ represents the area bounded by $y = f(x)$, x-axis and lines $x = a$ and $x = b$.



If $f(x) \leq 0\ \forall\ x \in [a, b]$ then area bounded by $y = f(x)$, x-axis and lines $x = a$ and $x = b$ is $\left|\int_a^b f(x)dx\right|$



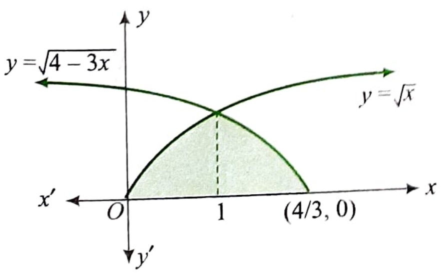
Note:

- Area bounded by $y = \sin x$ and x-axis between $x = 0$ and $x = \pi$ is 2 sq. units.
- Area bounded by $x = 0$, $y = 0$ and $y = \log_e x$ is 1 sq. units.

ILLUSTRATION 9.1

Find the area of the closed figure bounded by the curves $y = \sqrt{x}$, $y = \sqrt{4x - 3x}$, and $y = 0$.

Sol.



$$A = \int_0^1 \left(\sqrt{x}dx\right) + \int_1^{4/3} \sqrt{4-3x}dx$$

$$= \left(\frac{x^{3/2}}{3/2}\right)_0^1 + \left(\frac{(4-3x)^{3/2}}{-3(3/2)}\right)_1^{4/3}$$

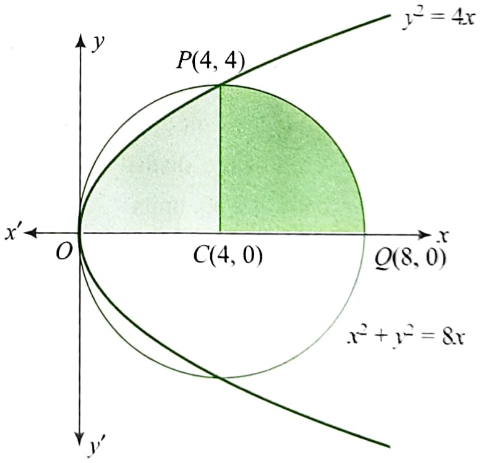
$$= \frac{2}{3} + \frac{2}{3}\left[\frac{1}{3}\right]$$

$$= \frac{2}{3} + \frac{2}{9} = \frac{8}{9} \text{ sq. units}$$

ILLUSTRATION 9.2

Find the area, lying above the x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

Sol. Solving the curves, we get $x^2 + 4x = 8x \Rightarrow x = 0, 4$



$$\text{Required area} = \int_0^4 y_{\text{parabola}}\, dx + \int_4^8 y_{\text{circle}}\, dx$$

Circle is $(x - 4)^2 + y^2 = 4^2$,

Area of quarter of circle $= \frac{1}{4}\pi 4^2 = 4\pi$

$$A = \int_0^4 2\sqrt{x}\, dx + 4\pi$$

$$= \frac{4}{3}\left[x^{3/2}\right]_0^4 + 4\pi$$

$$= \frac{4}{3} \times 4\sqrt{4} + 4\pi \text{ sq. units}$$

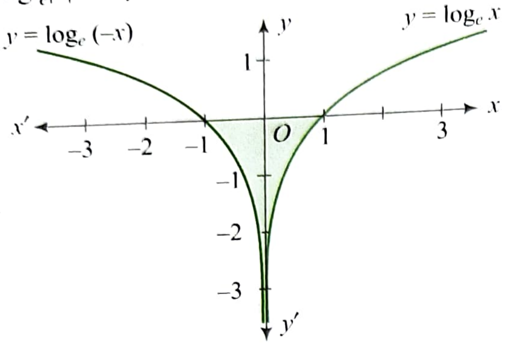
$$= \frac{32}{3} + 4\pi \text{ sq. units}$$

ILLUSTRATION 9.3

Find the area bounded by
(i) $y = \log_e|x|$ and $y = 0$ **(ii)** $y = |\log_e|x||$ and $y = 0$

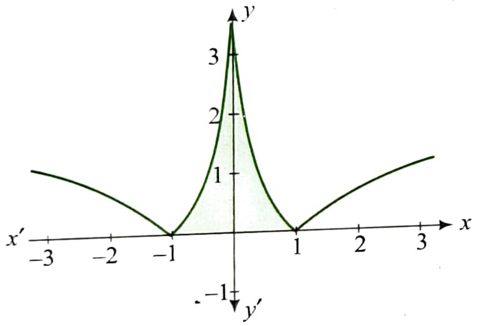
Sol.

(i) $y = \log_e|x|$ and $y = 0$



From the figure, required area = area of the shaded region $= 1 + 1 = 2$ sq. units (as we know that area bounded by $y = \log_e x$, $x = 0$, and $y = 0$ is 1 sq. units)

(ii) $y = |\log_e|x||$ and $y = 0$



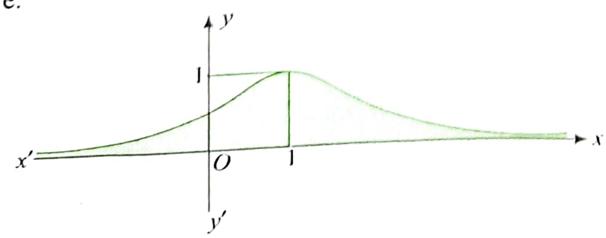
From the figure,
Required area = Area of the shaded region
$= 1 + 1 = 2$ sq. units.

ILLUSTRATION 9.4

Find the area bounded by $y = \dfrac{1}{x^2 - 2x + 2}$ and x-axis.

Sol. $y = \dfrac{1}{(x-1)^2 + 1}$

When $x = 1$, $y_{max.} = 1$
When $x \to \pm\infty$, $y \to 0$
Therefore, x-axis is the asymptote.
Also $f(1 + x) = f(1 - x)$
Hence, the graph is symmetrical about line $x = 1$
From these information the graph of function is as shown in the figure.

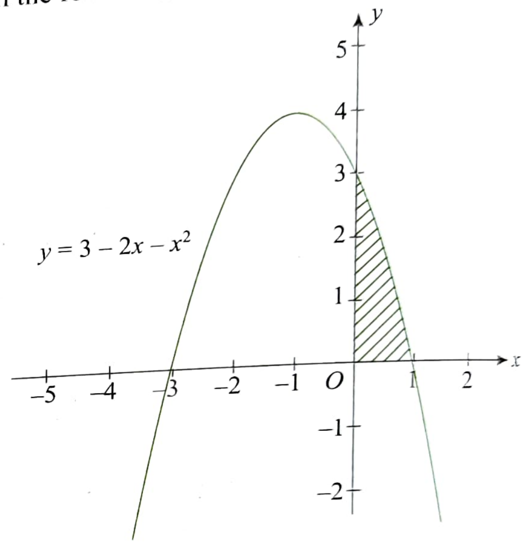


$$\text{Area} = 2\int_1^{\infty} \frac{1}{(x-1)^2 + 1}\,dx = [2\tan^{-1}(x-1)]_1^{\infty} = \pi \text{ sq. units}$$

ILLUSTRATION 9.5

The area of the region for which $0 < y < 3 - 2x - x^2$ and $x > 0$ i...

Sol. $0 < y < 3 - 2x - x^2$, $x > 0$
Consider parabola $y = 3 - 2x - x^2$
or $(x + 1)^2 = -(y - 2)$
The parabola having vertex $(-1, 2)$ and concave downward.
For $x, y > 0$, the points lie inside parabola in the first quadrant as shown in the following figure.



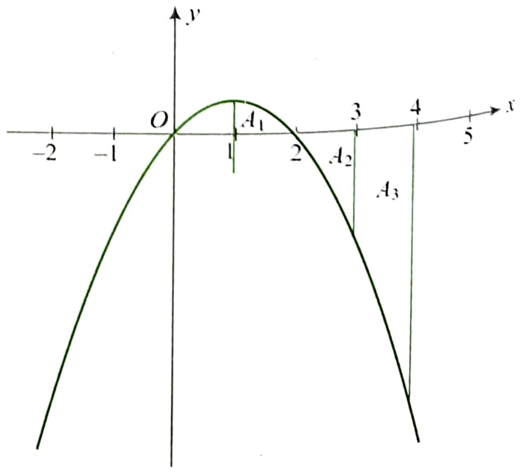
$$\therefore \quad \text{Required area} = \int_0^1 (3 - 2x - x^2)\,dx$$

$$= \left[3x - x^2 - \frac{x^3}{3}\right]_0^1 = \frac{5}{3} \text{ sq. units}$$

ILLUSTRATION 9.6

If the area bounded between x-axis and the curve of $y = 6x - 3x^2$ between the ordinates $x = 1$ and $x = a$ is 19 square units then prove that $a \in (3, 4)$ or $a \in (-2, -1)$.

Sol. We have $y = 3x(2 - x)$



Graph of the function is downward parabola intersecting x-axis at $x = 0$ and $x = 2$.

Now $I = \int (6x - 3x^2)dx = \frac{6x^2}{2} - \frac{3x^3}{3}$

$= 3x^2 - x^3$

$= x^2(3 - x) = F(x)$

Now $A_1 = F(2) - F(1) = 4 - 2 = 2$ sq. units

$A_2 = F(2) - F(3) = 4 - 0 = 4$ sq. units

$A_3 = F(3) - F(4) = 0 - (-16) = 16$ sq. units

Since $A_1 + A_2 + A_3 = 22$ sq. units.

So, $a \in (3, 4)$

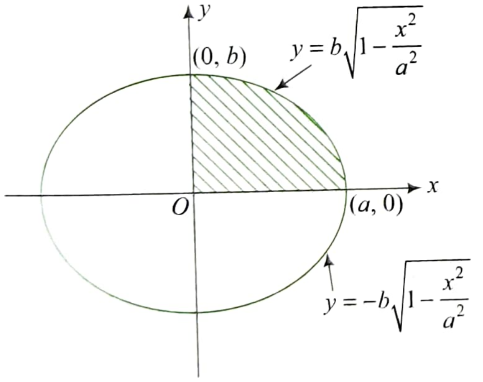
From the symmetric of the parabola, other value of a lies in $(-2, -1)$

ILLUSTRATION 9.7

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also prove that the area common to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its auxiliary circle $x^2 + y^2 = a^2$ is equal to the area of another ellipse whose semi axes are a and $b - a$, where $b > a$.

Sol. We have ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\therefore \quad y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$



Since graph of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetrical about both the axis,

Area enclosed by ellipse = 4 × Area in the first quadrant

$= 4 \times \int_0^a b\sqrt{1 - \frac{x^2}{a^2}}dx$

$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$

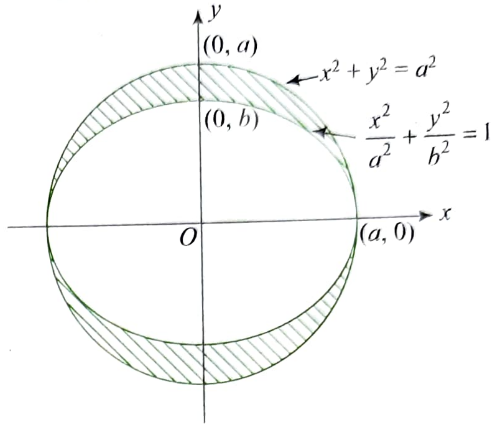
$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}1 - 0 - 0\right]$

$= \frac{4b}{a}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right]$

$= \pi ab$

In the following figure ellipse and its auxiliary are drawn.



Area of the ellipse = πab

Area of the circle = πa^2

Common area between curves is that shown as shaded region

= Area of circle – area of ellipse

$= \pi ab - \pi a^2$

$= \pi a(b - a)$

This is area of the ellipse whose semi axis are a and $(a - b)$.

One of the possible equations of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{(a-b)^2} = 1$

ILLUSTRATION 9.8

Let $f(x) = \text{Maximum } \{x^2, (1 - x)^2, 2x(1 - x)\}$, where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$, and $x = 1$.

Sol. $f(x) = \text{Maximum } \{x^2, (1 - x)^2, 2x(1 - x)\}$

We draw the graphs of

$y = x^2$ (1)

$y = (1 - x)^2$ (2)

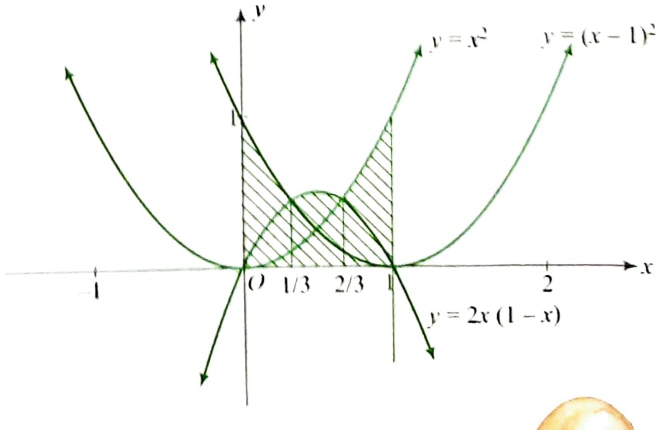
$y = 2x(1 - x)$ (3)

Solving (1) and (3), we get $x^2 = 2x(1 - x)$

or $3x^2 = 2x \Rightarrow x = 0$ or $x = 2/3$.

Solving (2) and (3) we get $(1 - x)^2 = 2x(1 - x)$

$\Rightarrow \quad x = 1/3$ and $x = 1$,



From the figure, it is clear that

$$f(x)=\begin{cases}(1-x)^2 & \text{for } 0\le x\le 1/3\\ 2x(1-x) & \text{for } 1/3\le x\le 2/3\\ x^2 & \text{for } 2/3< x\le 1\end{cases}$$

The required area A is given by

$$A=\int_0^1 f(x)\,dx$$

$$=\int_0^{1/3}(1-x)^2\,dx+\int_{1/3}^{2/3}2x(1-x)\,dx+\int_{2/3}^{1}x^2dx$$

$$=\left[-\frac{1}{3}(1-x)^3\right]_0^{1/3}+\left[\left(x^2-\frac{2x^3}{3}\right)\right]_{1/3}^{2/3}+\left[\frac{x^3}{3}\right]_{2/3}^{1}$$

$$=-\frac{1}{3}\left(\frac{2}{3}\right)^3+\frac{1}{3}+\left(\frac{2}{3}\right)^2-\frac{2}{3}\left(\frac{2}{3}\right)^3-\left(\frac{1}{3}\right)^2+\frac{2}{3}\left(\frac{1}{3}\right)^3$$
$$+\frac{1}{3}-\frac{1}{3}\left(\frac{2}{3}\right)^3$$

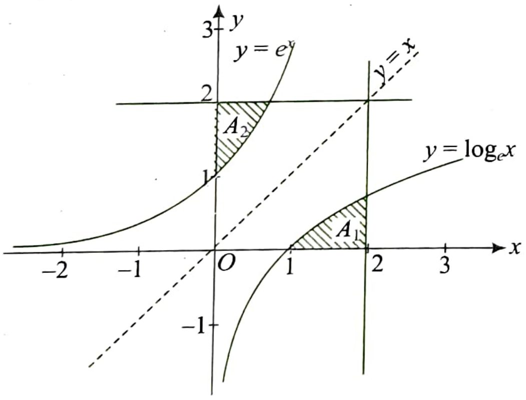
$$=\frac{17}{27}\text{ sq. units.}$$

ILLUSTRATION 9.9

Consider the region formed by the lines $x = 0$, $y = 0$, $x = 2$, $y = 2$. Area enclosed by the curves $y = e^x$ and $y = \log_e x$, within this region is being removed. Then find the area of the remaining region

Sol. $y = e^x$ and $y = \log_e x$ are inverse to each other.

So, their graphs are symmetrical about the line $y = x$.



Area of the square $OABC$ is 4 sq. units.

Areas A_1 and A_2 are same.

So, area of the shaded region $= 2A_1$

$$=2\int_1^2\log_e x\,dx$$

$$=2\left[x\log_e x-x\right]_1^2$$

$$=2[2\log_e 2-1]\text{ sq. unit}$$

Therefore, the area of the unshaded region of the square $= 4 - 2[2\log_e 2 - 1] = (6 - 4\log_e 2)$ sq. units

ILLUSTRATION 9.10

Draw a rough sketch of the curve, $y=\dfrac{x^2+3x+2}{x^2-3x+2}$ and find the area of the bounded region between the curve and x-axis.

Sol. $f(x)=\dfrac{(x+1)(x+2)}{(x-1)(x-2)}$

$f(x) = 0$. $\therefore x = -1, -2$.

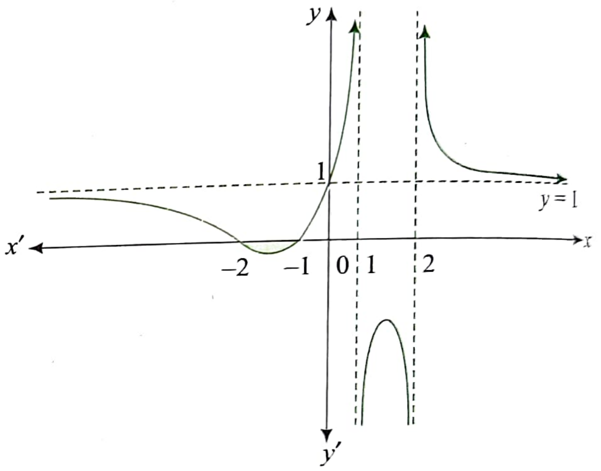
Also, $f(x)$ is discontinuous at $x = 1$ and $x = 2$.

$\lim\limits_{x\to1^-} f(x)\to+\infty$ and $\lim\limits_{x\to1^+} f(x)\to-\infty$

$\lim\limits_{x\to2^-} f(x)\to-\infty$ and $\lim\limits_{x\to2^+} f(x)\to+\infty$, $f(0) = 1$

$\lim\limits_{x\to\pm\infty} f(x)\to1$

So, graph of the function is as shown in the following figure.



The area of the shaded region is

$$\left|\int_{-2}^{-1}f(x)dx\right|=\left|\int_{-2}^{-1}\left(\frac{x^2+3x+2}{x^2-3x+2}\right)dx\right|$$

$$=\left|\int_{-2}^{-1}1+\frac{6x}{(x-1)(x-2)}dx\right|$$

$$=\left|\int_{-2}^{-1}\left[1+6\left(\frac{2}{x-2}-\frac{1}{x-1}\right)\right]dx\right|$$

$$=\left|\left[x+12\log_e|x-2|-6\log_e|x-1|\right]_{-2}^{-1}\right|$$

$$=\left|\left[1+12\log_e\frac{3}{4}-6\log_e\frac{2}{3}\right]\right|$$

$$=\left|\left[1+6\log_e\frac{9}{16}-6\log_e\frac{2}{3}\right]\right|$$

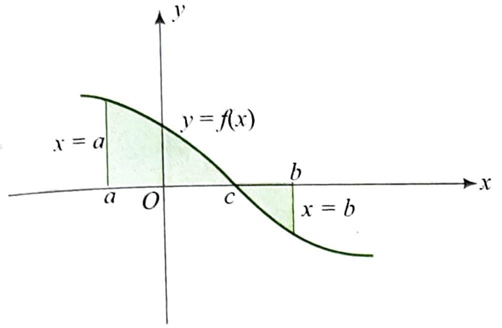
$$=\left|\left[1+6\log_e\frac{27}{32}\right]\right|$$

$$=6\log_e\frac{32}{27}-1\text{ sq. units}$$

Area Bounded by Curve and Axis when Graph of Function Intersects x-Axis

Let the graph of $y = f(x)$ for $x \in [a, b]$ intersect x-axis at $x = c$. If $f(x) \geq 0 \ \forall x \in [a, c]$ and $f(x) \leq 0 \ \forall x \in [c, b]$, then area bounded by curve and x-axis between lines $x = a$ and $x = b$ is

$$\int_a^b |f(x)|\, dx = \int_a^c f(x)dx - \int_c^b f(x)dx$$



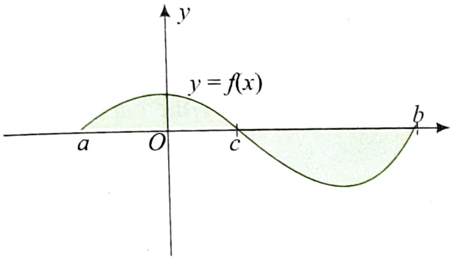


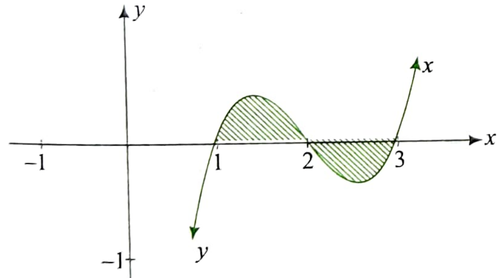
Illustration 9.11

Find the area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ lying between the ordinates $x = 1$ and $x = 3$.

Sol. $y = f(x) = (x - 1)(x - 2)(x - 3)$

$y = 0$, then $x = 1, 2, 3$.

So, graph of the function is as shown in the following figure.



Since $f(2 - x) = -f(2 + x)$

$$\int_1^2 f(x)dx = \left|\int_2^3 f(x)dx\right|$$

$\therefore$ Area of the shaded region,

$$A = 2\int_1^2 (x^3 - 6x^2 + 11x - 6)dx$$

$$= 2\left[\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right]_1^2$$

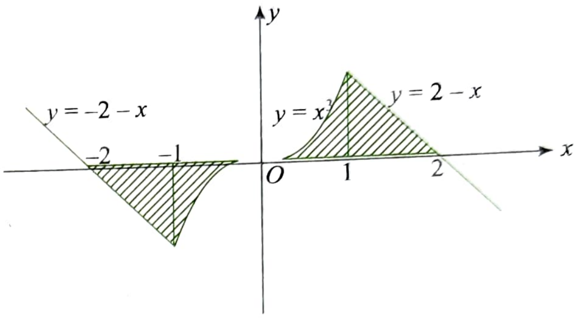
$$= 2 \times \frac{1}{4} = \frac{1}{2}$$

Illustration 9.12

Find the area bounded by the curve $x = \begin{cases} -2 - y, & y < -1 \\ y^3, & -1 \leq y \leq 1 \\ 2 - y, & y > 1 \end{cases}$ and $x = 0$ is

Sol. Required area is same as area bounded by curve

$$y = \begin{cases} -2 - x, & x < -1 \\ x^3, & -1 \leq x \leq 1 \\ 2 - x, & x > 1 \end{cases} \text{ and } y = 0.$$



$$\text{Required area } = 2\int_0^1 x^3 dx + 2\int_1^2 (2 - x)dx$$

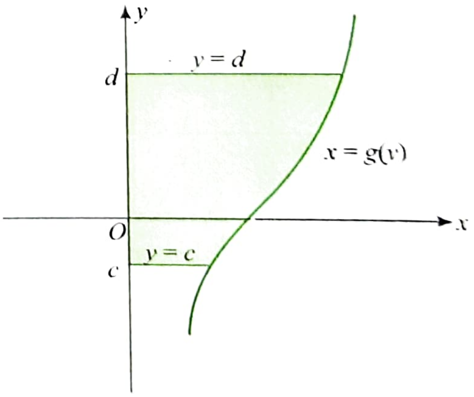
$$= 2\left[\frac{x^4}{4}\right]_0^1 + 2\left[2x - \frac{x^2}{2}\right]_1^2$$

$$= 2\left[\frac{1}{4}\right] + 2\left[4 - \frac{4}{2} - 2 + \frac{1}{2}\right]$$

$$= \frac{3}{2}$$

Area Using Integration along y-Axis

If we want to find the area bounded by curve $x = g(y)$ and y-axis between given limits of y as shown in the figure, we integrate function $x = g(y)$ along y-axis.

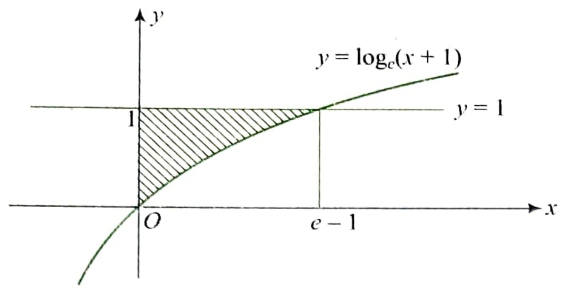


So, area (of shaded region) $= \int_c^d x\,dy = \int_c^d g(y)\,dy$.

Illustration 9.13

Find the area enclosed by the graph of $y = \log_e(x + 1)$, y-axis, and the line $y = 1$

Sol.



$$y = \log_e(x+1)$$
$$\therefore \quad x + 1 = e^y$$
$$\Rightarrow \quad x = e^y - 1$$

From the figure, required area (Intergrating along y-axis)

$$A = \int_0^1 (e^y - 1)\,dy = \left[e^y - y\right]_0^1$$
$$= e^1 - 1 - 1 + 0$$
$$= e - 2$$

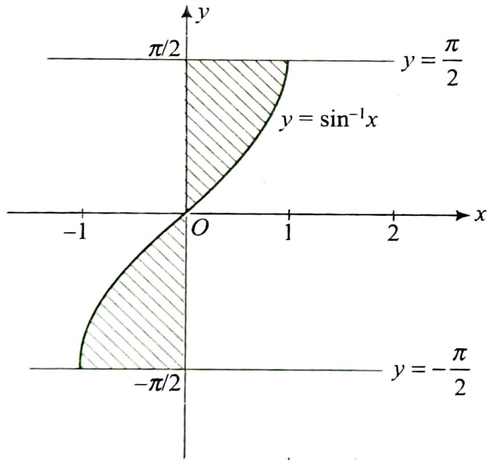
Also, $A = (e-1)\times 1 - \int_0^{e-1} \log_e(x+1)dx = (e-1) - \int_1^e \log_e x\,dx$

$$= (e-1) - 1 = e - 2$$

ILLUSTRATION 9.14

Find the area bounded by the curve $y = \sin^{-1} x$ and the lines $x = 0, |y| = \dfrac{\pi}{2}$.

Sol.



The area of the shaded region (Intergrating along y-axis)

$$= 2\int_{-\pi/2}^{\pi/2} |\sin y|\,dy = 2\int_0^{\pi/2} \sin y dy = 2 \text{ sq. units}$$

CONCEPT APPLICATION EXERCISE 9.1

1. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \dfrac{a}{\sqrt{2}}$.
2. Find the area enclosed by the curves $x^2 = y, y = x + 2$, and x-axis.
3. A curve is given by $y = \begin{cases}\sqrt{(4-x^2)}, & 0 \le x < 1 \\ \sqrt{(3x)}, & 1 \le x \le 3\end{cases}$. Find the area lying between the curve and x-axis.
4. Find the area bounded by $x = 2y - y^2$ and the y-axis.
5. Find the area bounded by the x-axis, part of the curve $y = \left(1 + \dfrac{8}{x^2}\right)$, and the ordinates at $x = 2$ and $x = 4$. If the ordinate at $x = a$ divides the area into two equal parts, then find a.
6. Find the area of the region bounded by the x-axis and the curves defined by $y = \tan x \left(\text{where } -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{3}\right)$ and $y = \cot x \left(\text{where } \dfrac{\pi}{6} \le x \le \dfrac{3\pi}{2}\right)$.
7. Find the area bounded by $y = \left|\sin x - \dfrac{1}{2}\right|$ and $y = 1$ for $x \in [0, \pi]$.
8. If the area bounded by the graph of $y = xe^{-ax}$ $(a > 0)$ and the x-axis is 1/9 then find the value of a.
9. Find the area bounded by the curve $xy^2 = 4(2 - x)$ and y-axis.

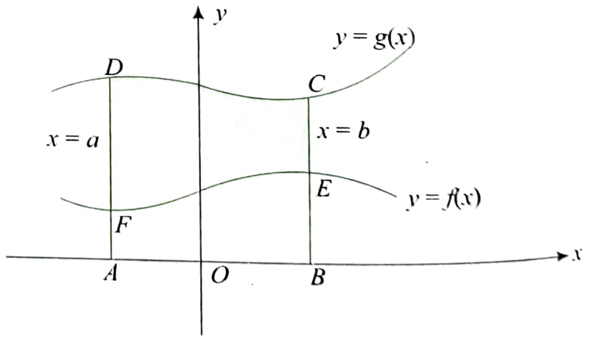
ANSWERS

1. $\dfrac{a^2}{2}\left(\dfrac{\pi}{2} - 1\right)$ sq. units	2. $\dfrac{5}{6}$ sq. units
3. $\dfrac{1}{6}\left(2\pi - \sqrt{3} + 36\right)$ sq. units	4. $\dfrac{4}{3}$ sq. units
5. 4 sq. units; $a = 2\sqrt{2}$	6. $\log\dfrac{3}{2}$ sq. units
7. $\left(\dfrac{\pi}{2} + 2\right)$ sq. units	8. 3
9. 4π sq. units	

AREA BOUNDED BY TWO CURVES

AREA BOUNDED BY CURVES IN GIVEN INTERVAL

Let us find the area bounded by the straight lines $x = a, x = b (a < b)$ and the curves $y = f(x)$ and $y = g(x)$, provided $f(x) \le g(x)$ for all $x \in [a, b]$



Clearly from the figure, area of the shaded region

= Area of the region $ABCD$ – Area of the region $ABEF$

$$= \int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b \Bigg(\underbrace{g(x)}_{\text{Upper curve}} - \underbrace{f(x)}_{\text{Lower curve}}\Bigg)dx$$

Note:

- Area bounded by ellipse whose semi-axes are a and b is πab sq. units.
- Area bounded by parabola $y^2 = 4ax$ and $x^2 = 4by$ is $\left|\frac{16ab}{3}\right|$ sq. units.

ILLUSTRATION 9.15

Find the area of the region bounded by the curves $y = \sqrt{x+2}$ and $y = \frac{1}{x+1}$ between the lines $x = 0$ and $x = 2$.

Sol. $y = \sqrt{x+2}$ ...(1)

$\therefore \quad y^2 = x + 2$

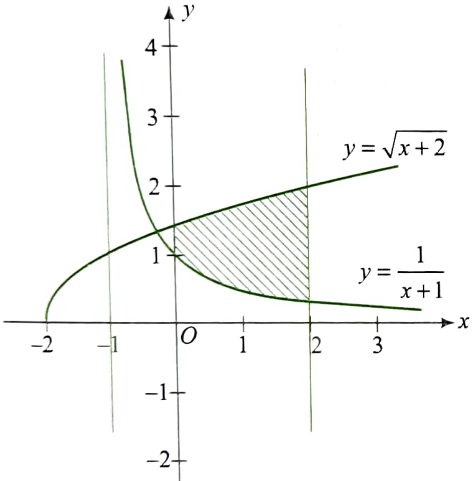
So, curve (1) is parabola for $y \geq 0$.

$$y = \frac{1}{x+1} \qquad \text{...(2)}$$

$$\lim_{x\to -1^+} \frac{1}{x+1} = \infty \text{ and } \lim_{x\to\infty} \frac{1}{x+1} = 0$$

$y(0) = 1$.

Also, y is decreasing function.



From the figure required area is

$$A = \int_0^2 \left(\sqrt{x+2} - \frac{1}{x+1}\right) dx$$

$$= \left[\frac{2}{3}(x+2)^{3/2} - \log_e(x+1)\right]_0^2$$

$$= \left(\frac{2}{3}\times 8 - \log_e 3\right) - \left(\frac{2}{3}\times 2^{3/2} - \log_e 1\right)$$

$$= \frac{16 - 4\sqrt{2}}{3} - \log_e 3$$

ILLUSTRATION 9.16

Find the area bounded by $y = \sin^{-1}x$, $y = \cos^{-1}x$, and the x-axis.

Sol. We have to find the area bounded by $y = \sin^{-1}x$, $y = \cos^{-1}x$, and the x-axis. If integration along x-axis is used then

$$A = \int_0^{1/\sqrt{2}} \sin^{-1} x\, dx + \int_{1/\sqrt{2}}^{1} \cos^{-1} x\, dx$$

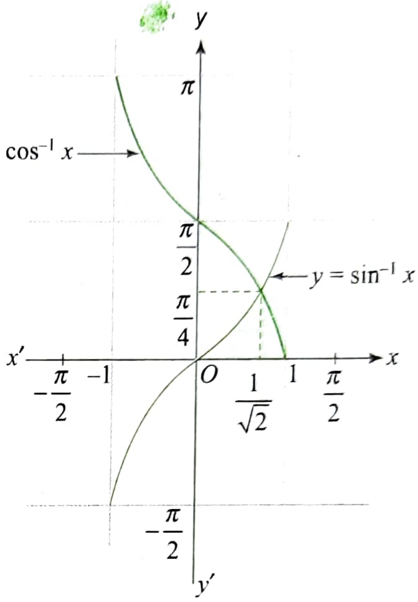
If integration along y-axis is used, then

$$A = \int_0^{\pi/4} (\cos y - \sin y)\, dy$$

$$= [\sin y + \cos y]_0^{\pi/4}$$

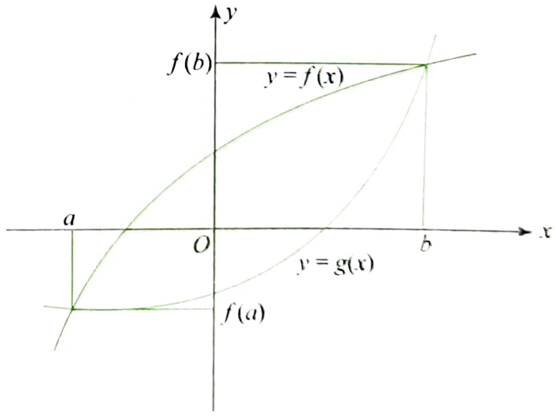
$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right]$$

$$= \sqrt{2} - 1$$



AREA BOUNDED BY TWO CURVES WHEN CURVES INTERSECT AT TWO POINTS

Consider the following graphs of the functions $y = f(x)$ and $y = g(x)$ which intersect at two points where $x = a$ and $x = b$.



Area bounded by curves $= \int_a^b [f(x) - g(x)]dx$, where a and b are roots of the equation $f(x) = g(x)$

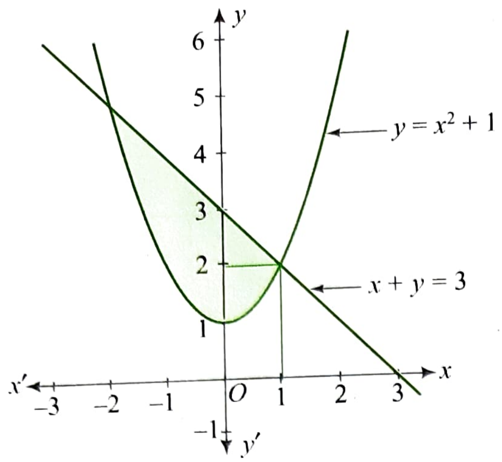
Also, area $= \int_{f(a)}^{f(b)} [g^{-1}(y) - f^{-1}(y)]dy$

if we choose to integrate along y-axis.

ILLUSTRATION 9.17

Find the area bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$.

l.



e two curves meet at points where $3 - x = x^2 + 1$,

∴ $x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$ or $x = -2, 1$

$$\text{Required area} = \int_{-2}^{1}\left[(3-x)-(x^2+1)\right]dx$$

$$= \int_{-2}^{1}(2-x-x^2)\,dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

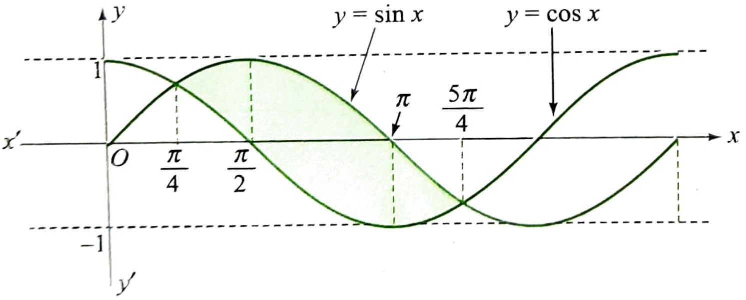
$$= \left(2-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-\frac{4}{2}+\frac{8}{3}\right)$$

$$= \frac{9}{2} \text{ sq. units}$$

ILLUSTRATION 9.18

Find the area bounded by the curves $y = \sin x$ and $y = \cos x$ between two consecutive points of the intersection.

Sol.



Two consecutive points of intersection of $y = \sin x$ and $y = \cos x$ can be taken as $x = \pi/4$ and $x = 5\pi/4$. Therefore,

$$\text{Required area} = \int_{\pi/4}^{5\pi/4}(\sin x - \cos x)\,dx$$

$$= [-\cos x - \sin x]_{\pi/4}^{5\pi/4}$$

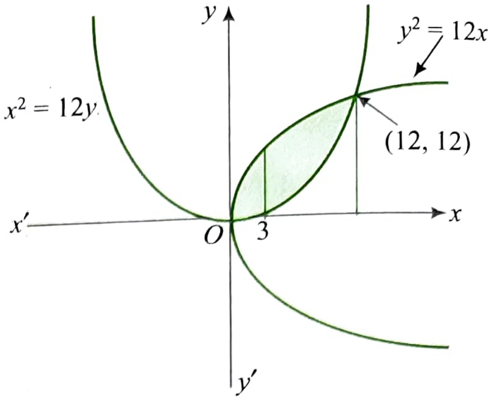
$$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$$= 2\sqrt{2} \text{ sq. units}$$

ILLUSTRATION 9.19

Find the ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$.

Sol.



A_1 = Area bounded by $y^2 = 12x$, $x^2 = 12y$, and line $x = 3$

$$= \int_0^3 \sqrt{12x}\,dx - \int_0^3 \frac{x^2}{12}\,dx$$

$$= \sqrt{12}\left|\frac{2x^{3/2}}{3}\right|_0^3 - \left|\frac{x^3}{36}\right|_0^3 = \frac{45}{4} \text{ sq. units.}$$

A_2 = Area bounded by $y^2 = 12x$ and $x^2 = 12y$

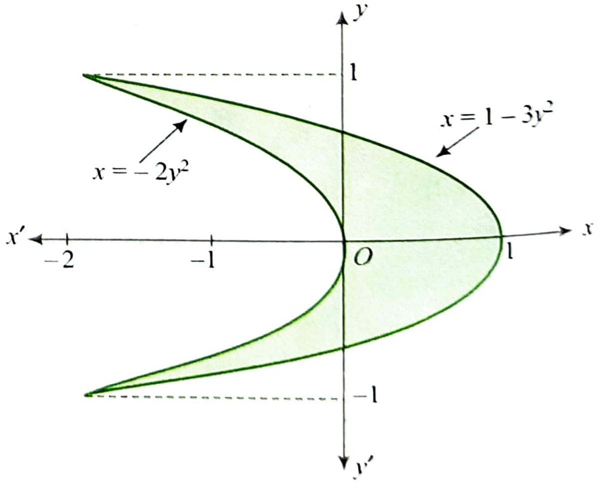
$$= \frac{16(3)(3)}{3} = 48 \text{ sq. units}$$

$$\therefore \quad \text{Required ratio} = \frac{\frac{45}{4}}{48 - \frac{45}{4}} = \frac{45}{147}$$

ILLUSTRATION 9.20

Find the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$.

Sol.



Solving the equation $x = -2y^2$, $x = 1 - 3y^2$, we find that the ordinates of the point of intersection of the two curves are $y_1 = -1$, $y_2 = 1$. The points are (−2, −1) and (−2, 1).

The required area is given by (Integrating along y-axis)

$$A = 2\int_0^1 (x_1 - x_2)\,dy$$
$$= 2\int\left[(1-3y^2) - (-2y^2)\right]dy$$
$$= 2\int_0^1 (1-y^2)\,dy$$
$$= 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3}$$

ILLUSTRATION 9.21

The area common to regions $x^2 + y^2 - 2x \le 0$ and $y \ge \sin\frac{\pi x}{2}$.

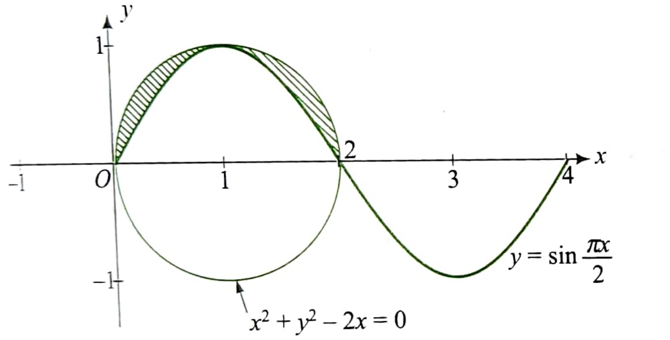
Sol. For $x^2 + y^2 - 2x \le 0$

points lie on or inside circle $(x-1)^2 + y^2 = 1$

For $y \ge \sin\frac{\pi x}{2}$, points lie on or above $y = \sin\frac{\pi x}{2}$.

$y = \sin\frac{\pi x}{2}$ has period 4.

The graphs of curves and the required region is as shown in the following figure.



From the figure, required area = area of semicircle – $\int_0^2 \sin\frac{\pi x}{2}\,dx$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\cos\frac{\pi x}{2}\right]_0^2$$
$$= \frac{\pi}{2} - \frac{4}{\pi}$$

ILLUSTRATION 9.22

Find the area of the region enclosed by the curves $y = x\log x$ and $y = 2x - 2x^2$.

Sol. Curve tracing, $y = x\log_e x$

Clearly $x > 0$,

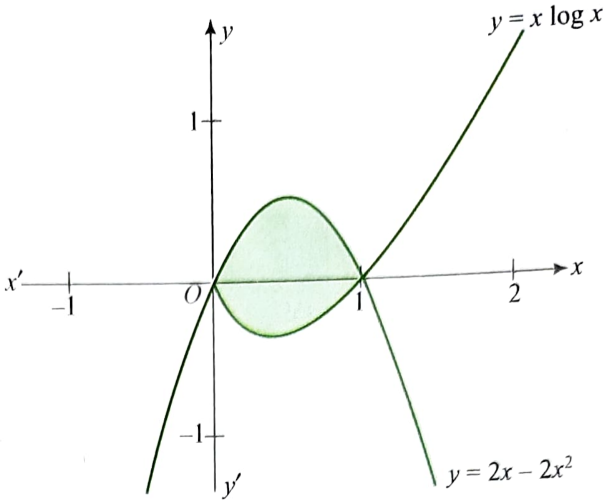
For $0 < x < 1$, $x\log_e x < 0$, and for $x > 1$, $x\log_e x > 0$

Also $x\log_e x = 0$ for $x = 1$

Further $\frac{dy}{dx} = 0 \Rightarrow 1 + \log_e x = 0$ or $x = 1/e$, which is point of minima.

$\lim_{x\to\infty} x\log_e x = \infty$. Graph of function is as shown in the figure.

$y = 2x - 2x^2$ is downward parabola intersecting x-axis at $x = 0$ and $x = 1$.



$$\text{Required area} = \int_0^1 (2x - 2x^2)\,dx - \int_0^1 x\log x\,dx$$
$$= \left[x^2 - \frac{2x^3}{3}\right]_0^1 - \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1$$
$$= \left(1 - \frac{2}{3}\right) - \left[0 - \frac{1}{4} - \frac{1}{2}\lim_{x\to 0} x^2\log x\right] = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

ILLUSTRATION 9.23

Find the area bounded by $y^2 \le 4x$, $x^2 + y^2 \ge 2x$, and $x \le y + 2$ in the first quadrant.

Sol. For $y^2 \le 4x$, points lie on or inside parabola $y^2 = 4x$

For $x^2 + y^2 \ge 2x$, points lie on or outside circle $(x-1)^2 + (y-0)^2 = 1$

For $x \le y + 2$, points lie on or to the left side (towards origin) of the line $x = y + 2$.

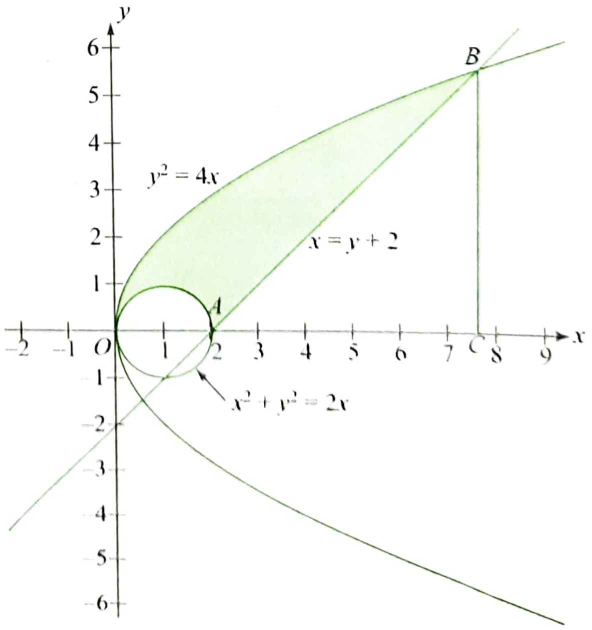
The graphs of the curves is as shown in the figure.

Solving parabola and line $y = x + 2$, we get

$(x-2)^2 = 4x$

or $x^2 - 8x + 4 = 0$

or $x = \frac{8 \pm \sqrt{48}}{2} = 2(2 \pm \sqrt{3}) = (1 \pm \sqrt{3})^2$



From the figure, required area is

$$A = \int_0^{(\sqrt{3}+1)^2} \sqrt{4x}\,dx - \text{Area of semicircle} - \text{Area of triangle } ABC$$

$$= \sqrt{4}\left[\frac{2x^{3/2}}{3}\right]_0^{(\sqrt{3}+1)^2} - \frac{\pi}{2} - \frac{1}{2}\left((\sqrt{3}+1)^2 - 2\right) 2(\sqrt{3}+1)$$

$$= \frac{4(\sqrt{3}+1)^3}{3} - \frac{\pi}{2} - 2(\sqrt{3}+1)^2$$

ILLUSTRATION 9.24

Sketch the region bounded by the curves $y = x^2$ and $y = \dfrac{2}{1+x^2}$. Find the area.

Sol. The given curves are $y = x^2$ (1)

and $y = \dfrac{2}{1+x^2}$ (2)

Solving (1) and (2), we have

$$x^2 = \frac{2}{1+x^2}$$

or $x^4 + x^2 - 2 = 0$

or $(x^2 - 1)(x^2 + 2) = 0$

or $x = \pm 1$

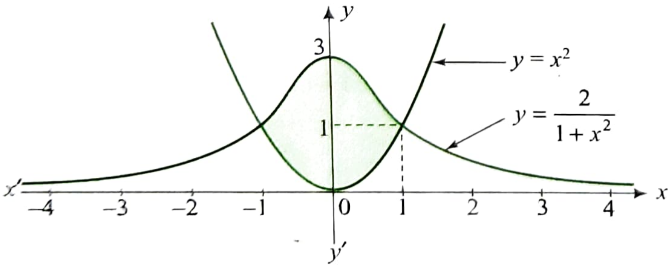
Also, $y = \dfrac{2}{1+x^2}$ is an even function.

Hence, its graph is symmetrical about y-axis.

At $x = 0$, $y = 2$, by increasing the values of x, y decreases and when $x \to \infty$, $y \to 0$.

Therefore, $y = 0$ is an asymptote of the given curve.

Thus, the graphs of the two curves are as follows:



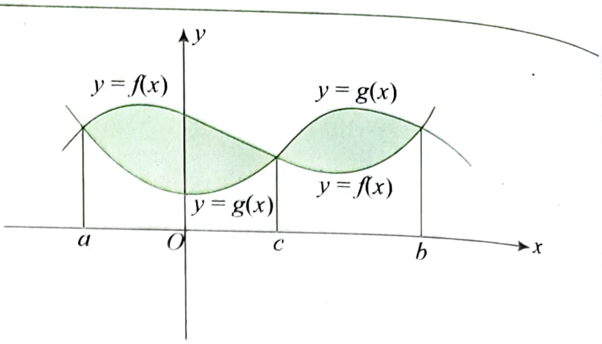
$$\text{Required area} = 2\int_0^1 \left(\frac{2}{1+x^2} - x^2\right)dx$$

$$= \left(4\tan^{-1}x - \frac{2x^3}{3}\right)_0^1$$

$$= \pi - \frac{2}{3} \text{ sq. units.}$$

AREA BOUNDED BY TWO CURVES WHEN CURVES INTERSECT AT MORE THAN TWO POINTS

Consider the following graphs of the functions $y = f(x)$ and $y = g(x)$ which intersect at three points where $x = a$, $x = c$ and $x = b$.



Here $x = a$, $x = b$ and $x = c$, roots of the equation $f(x) = g(x)$.

For $x \in (a, c)$, $f(x) > g(x)$ and for $x \in (c, b)$, $g(x) > f(x)$.

Area bounded by curves,

$$A = \int_a^b |f(x) - g(x)|\,dx$$

$$= \int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx$$

ILLUSTRATION 9.25

Find the area bounded by the curves $y = x^3 - x$ and $y = x^2 + x$.

Sol. $y = x^3 - x = x(x-1)(x+1)$ is a cubic polynomial function intersecting the x-axis at $(-1, 0)$, $(0, 0)$, $(1, 0)$.

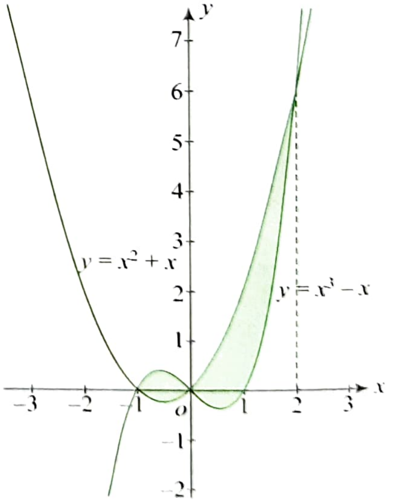
$y = x^2 + x = x(x+1)$ is a quadratic function which is concave upward and intersect x-axis at $(-1, 0)$ $(0, 0)$.

Solving curves, we get

$$x^3 - x = x^2 + x$$

$\therefore \quad x = -1, 0, 2$

The graphs of functions are as shown in the figure.



From the figure,

$$\text{Required area} = \int_{-1}^{0}\left((x^3 - x) - (x^2 + x)\right)dx + \int_0^2 (x^2 + x - (x^3 - x))dx$$

$$= \int_{-1}^{0} (x^3 - x^2 - 2x)dx + \int_0^2 (x^2 + 2x - x^3)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} + \left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\right]_0^2$$

$$= \left[0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right)\right] + \left[\frac{8}{3} + 4 - 4\right]$$

$$= \frac{37}{12} \text{ sq. units}$$

ILLUSTRATION 9.26

Find the area bounded by $y = -x^3 + x^2 + 16x$ and $y = 4x$

Sol. Solving given curve, we have

$$-x^3 + x^2 + 16x = 4x$$

$\therefore$ $x = 0$ or $x^2 - x - 12 = 0$

$\Rightarrow$ $x = 0$ or $(x - 4)(x + 3) = 0$

$\Rightarrow$ $x = 0$ or $x = -3, 4$

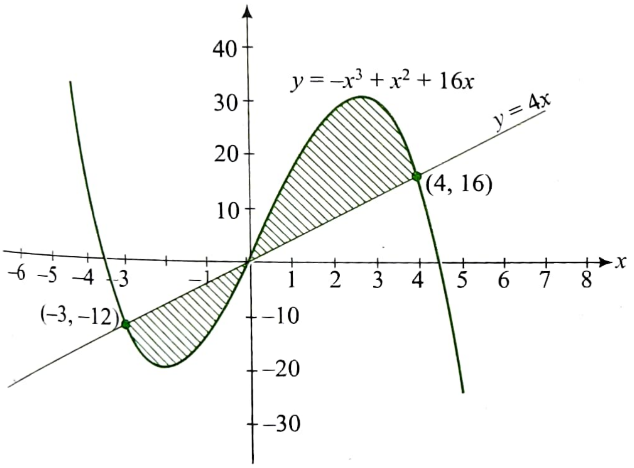
Now $y = -x^3 + x^2 + 16x$

$= -x(x^2 - x - 16)$

If $y = 0$ then $x = 0$ or $x^2 - x - 16 = 0$

$\therefore$ $x = 0$ or $x \dfrac{1 \pm \sqrt{65}}{2}$

From the above information we having following graphs.



From the figure area of the shaded region,

$$A = \int_{-3}^{0} [4x - (-x^3 + x^2 + 16x)]dx + \int_0^4 [(-x^3 + x^2 + 16x) - 4x]dx$$

$$= \int_{-3}^{0} (x^3 - x^2 - 12x)dx + \int_0^4 (-x^3 + x^2 + 12x)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - 6x^2\right]_{-3}^{0} + \left[-\frac{x^4}{4} + \frac{x^3}{3} + 6x^2\right]_0^4$$

$$= \left[-\frac{81}{4} - 9 + 54\right] + \left[-64 + \frac{64}{3} + 96\right]$$

$$= \frac{937}{12} \text{ sq. units.}$$

ILLUSTRATION 9.27

Find the area of the region enclosed by $y = 5x - x^2$ and $y = x$ on interval $[-1, 5]$

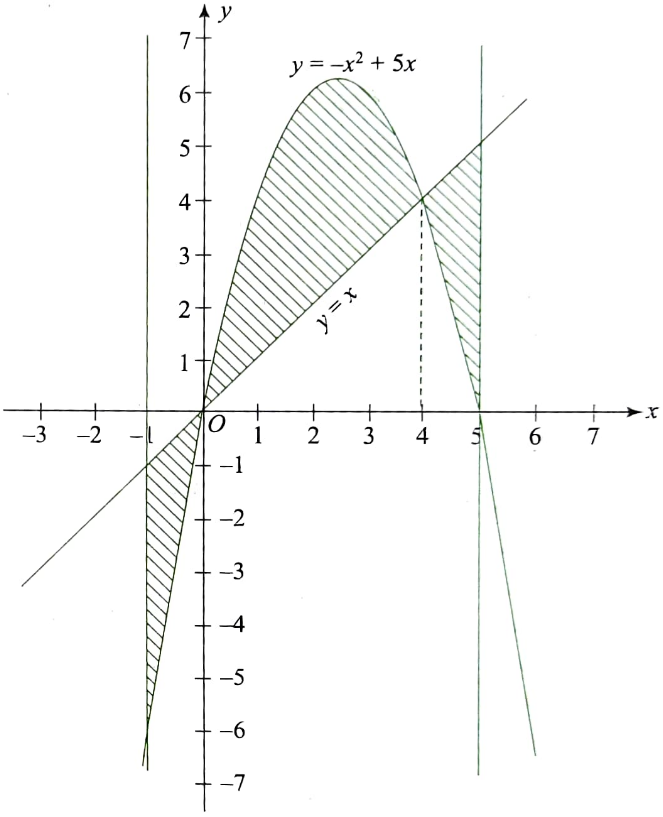
Sol. Solving the given curves we get

$$5x - x^2 = x$$

$\therefore$ $x^2 - 4x = 0$

$\Rightarrow$ $x = 0, 4$

$y = x(5 - x)$ is parabola which is concave downward and intersecting x-axis at $x = 0$ and $x = 5$.



$$\text{Required area, } = \int_{-1}^{5} |x - (5x - x^2)|\,dx$$

$$= \int_{-1}^{5} |x^2 - 4x|\,dx$$

$$= \int_{-1}^{0} (x^2 - 4x)dx + \int_0^4 (4x - x^2)dx + \int_4^5 (x^2 - 4x)dx$$

$$= \left[\frac{x^3}{3} - 2x^2\right]_{-1}^{0} + \left[2x^2 - \frac{x^3}{3}\right]_0^4 + \left[\frac{x^3}{3} - 2x^2\right]_4^5$$

$$= -\left[\frac{-1}{3} - 2\right] + \left[32 - \frac{64}{3}\right] + \left[\frac{125}{3} - 50\right] - \left[\frac{64}{3} - 32\right]$$

$$= 16 - \frac{2}{3}$$

$$= \frac{46}{3} \text{ sq. units}$$

Concept Application Exercise 9.2

1. Find the area lying in the first quadrant and bounded by the curve $y = x^3$ and the line $y = 4x$.
2. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.
3. Find the area enclosed by the figure described by the equation $x^4 + 1 = 2x^2 + y^2$.
4. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$?
5. Find the area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$.
6. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$, and $x = 3$.
7. Find the area of the region bounded by the limits $x = 0$, $x = \frac{\pi}{2}$, and $f(x) = \sin x$, $g(x) = \cos x$.
8. Find the area bounded by $y = \tan^{-1} x$, $y = \cot^{-1} x$, and y-axis in the first quadrant.
9. Find the area bounded by $y = \log_e x$, $y = -\log_e x$, $y = \log_e(-x)$, and $y = -\log_e(-x)$.
10. Find the area of the region $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$.
11. Sketch the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ and find its area.
12. Sketch the curves and identify the region bounded by $x = \frac{1}{2}$, $x = 2$, $y = \ln x$, and $y = 2^x$. Find the area of this region.
13. Find the area bounded by $y = x^2$ and $y = x^{1/3}$ for $x \in [-1, 1]$.
14. Find the smallest area bounded by the curves $y = x - \sin x$, $y = x + \cos x$.

Answers

1. 4 sq. units
2. $\frac{9}{8}$ sq. units
3. $\frac{8}{3}$ sq. units
4. 121 : 4
5. $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$ sq. units
6. $\frac{21}{2}$ sq. units
7. $2(\sqrt{2} - 1)$ sq. units
8. $\log \sqrt{2}$ sq. units
9. 4 sq. units
10. $2\left(\frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{12}\right)$ sq. units
11. $\frac{5\pi - 2}{4}$ sq. units
12. $\left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}\right)$ sq. units
13. $\frac{3}{2}$ sq. units
14. $2\sqrt{2}$ sq. units

AREA BOUNDED BY MISCELLANEOUS CURVES

Concept of Variable Area

Till now we have been finding the area bounded by curves using integration. In the reverse situation if area bounded by curves in which one of the curves is unknown is given with one of the limits variable, we can find the function using differentiation.

Illustration 9.28

If the area enclosed by curve $y = f(x)$ and $y = x^2 + 2$ between the abscissa $x = 2$ and $x = \alpha$, $\alpha > 2$, is $(\alpha^3 - 4\alpha^2 + 8)$ sq. units then find function $f(x)$. It is known that curve $y = f(x)$ lies below the parabola $y = x^2 + 2$.

Sol. According to the question,

$$\alpha^3 - 4\alpha^2 + 8 = \int_2^{\alpha} (x^2 + 2 - f(x))\,dx$$

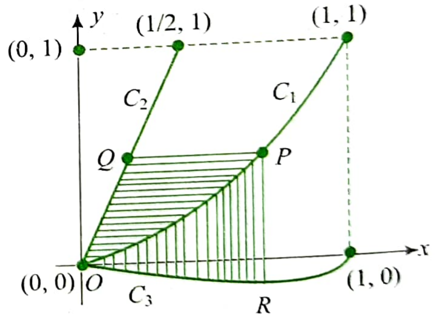
Differentiating w.r.t. α on both sides, we get

$$3\alpha^2 - 8\alpha = \alpha^2 + 2 - f(\alpha)$$

$\therefore$ $f(x) = -2x^2 + 8x + 2$

Illustration 9.29

Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, respectively, where $0 \le x \le 1$. Let C_3 be the graph of a function $y = f(x)$, where $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R, respectively (see figure). If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function $f(x)$.



Sol. Let P be on C_1, $y = x^2$ be (t, t^2)

$\therefore$ y co-ordinate of Q is also t^2

Now, Q on $y = 2x$ where $y = t^2$

$\therefore$ $x = t^2/2$

$\therefore$ $Q\left(\frac{t^2}{2}, t^2\right)$

For point R, $x = t$ and it is on $y = f(x)$

$\therefore$ $R(t, f(t))$

Given that,

Area OPQ = Area OPR

$$\Rightarrow \int_0^{t^2}\left(\sqrt{y} - \frac{y}{2}\right)dy = \int_0^t \left(x^2 - f(x)\right)dx$$

Differentiating both sides w.r.t. t, we get

$$\left(\sqrt{t^2}-\frac{t^2}{2}\right)(2t) = t^2 - f(t)$$

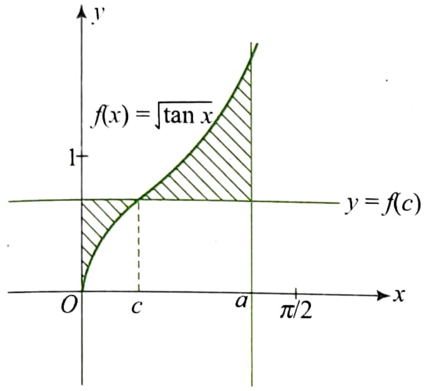
$\Rightarrow \quad f(t) = t^3 - t^2$

$\Rightarrow \quad f(x) = x^3 - x^2$

ILLUSTRATION 9.30

If the area bounded by $f(x) = \sqrt{\tan x}$, $y = f(c)$, $x = 0$ and $x = a$, $0 < c < a < \frac{\pi}{2}$ is minimum then find the value of c.

Sol. Graph of $f(x) = \sqrt{\tan x}$ is as shown in the following figure.



Required area is

$$A = \int_0^c (f(c) - f(x))dx + \int_c^a (f(x) - f(c))dx$$

$$= \int_0^c \sqrt{\tan c}\,dx - \int_0^c \sqrt{\tan x}\,dx + \int_c^a \sqrt{\tan x}\,dx - \int_c^a \sqrt{\tan c}\,dx$$

$$= (2c-a)\sqrt{\tan c} - \int_0^c \sqrt{\tan x}\,dx + \int_c^a \sqrt{\tan x}\,dx$$

$$\therefore \quad \frac{dA}{dc} = 2\sqrt{\tan c} + (2c-a)\frac{\sec^2 c}{2\sqrt{\tan c}} - \sqrt{\tan c} - \sqrt{\tan c}$$

$$= (2c-a)\frac{\sec^2 c}{2\sqrt{\tan c}}$$

$$\frac{dA}{dc} = 0 \Rightarrow c = \frac{a}{2}$$

At $c = \frac{a}{2}$, $\frac{dA}{dc}$ changes sign from negative to positive.

Hence A is minimum when $c = \frac{a}{2}$.

AREA BOUNDED BY INEQUALITIES

ILLUSTRATION 9.31

Find the area of the region R which is enclosed by the curves $y \geq \sqrt{1-x^2}$ and $\max\{|x|, |y|\} \leq 4$.

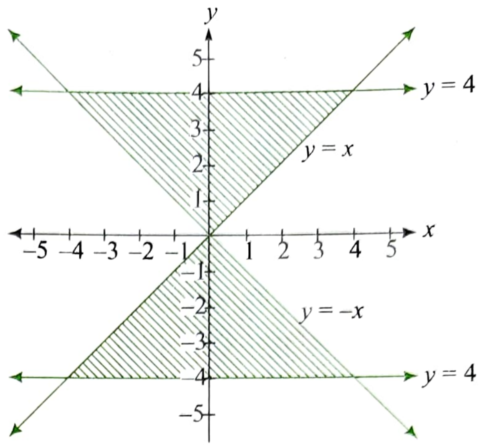
Sol. For $y \geq \sqrt{1-x^2}$ (1)

Points lie outside circle $x^2 + y^2 = 1$ for $y \geq 0$ and $-1 \leq x \leq 1$.

For $\max\{|x|, |y|\} \leq 4$, we have

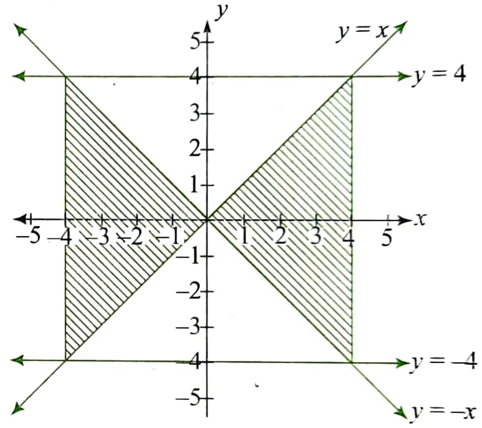
$|y| \leq 4$, when $|x| < |y|$ (2)

The points satisfying above inequalities from region as shown in the figure.

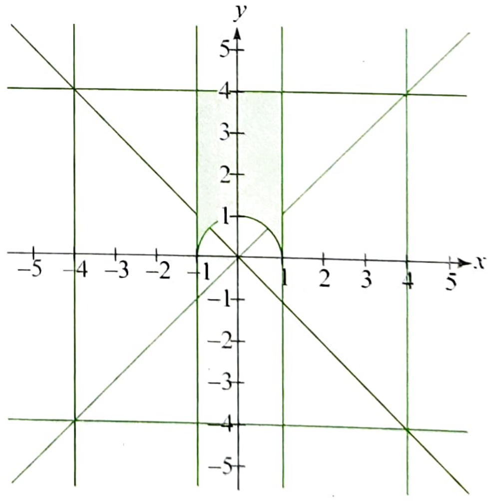


$|x| \leq 4$, when $|y| < |x|$ (3)

The points satisfying above inequalities from region as shown in the figure.



The region (1) ∩ ((2) ∪ (3)) is as shown in the figure.



From the figure, required area is

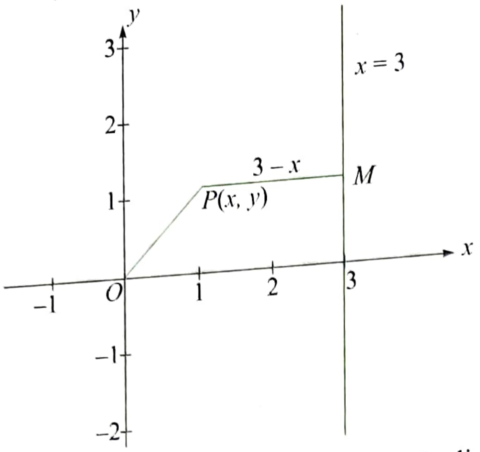
= Area of rectangle – Area of semicircle having radius 1

$= 8 - \frac{\pi}{2}$

ILLUSTRATION 9.32

Plot the region in the first quadrant in which points are nearer to the origin than to the line $x = 3$.

Sol. Point $P(x, y)$ lies in the first quadrant. Therefore, $x, y > 0$



Also point $P(x, y)$ is nearer to the origin than to the line $x = 3$

Now $OP = \sqrt{x^2 + y^2}$

Distance of P from line $x - 3 = 0$ is $PM = 3 - x$

According to the question, $OP < PM$

$\Rightarrow \quad \sqrt{x^2 + y^2} < (3 - x)$

or $\quad x^2 + y^2 < x^2 - 6x + 9$

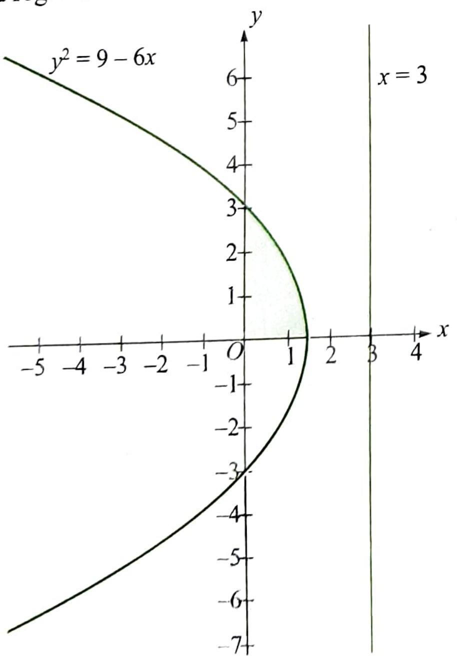
or $\quad y^2 < -6x + 9$

or $\quad y^2 < -6\left(x - \frac{3}{2}\right)$

Points satisfying above inequality lie inside parabola $y^2 = 9 - 6x$ in the first quadrant.

Parabola $y^2 = -6\left(x - \frac{3}{2}\right)$ is concave to the left, having axis as x-axis and vertex at $\left(\frac{3}{2}, 0\right)$ and directrix $x = 3$

The required region as shown in the figure.



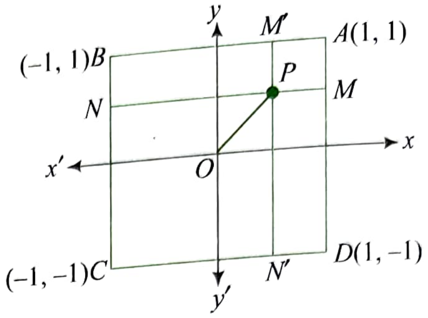
Area of the region $= \int_0^3 \frac{9 - y^2}{6} dy = \frac{1}{6}\left(9y - \frac{y^3}{3}\right)_0^3$

$= \frac{1}{6}(27 - 9) = 3$ sq. units

ILLUSTRATION 9.33

Consider a square with vertices at $(1, 1), (-1, 1), (-1, -1)$, and $(1, -1)$. Let S be the region consisting of all points inside the square which are nearer to the origin than to any of the edges. Sketch the region S and find its area.

Sol.



Let us consider any point (x, y) inside the square such that its distance from origin is less than its distance from any of the edges.

Consider edge AD. Therefore,

$$OP < PM$$

$$\Rightarrow \quad \sqrt{(x^2 + y^2)} < 1 - x \text{ or } y^2 < -2\left(x - \frac{1}{2}\right) \qquad (1)$$

Above represents all points within the parabola $y^2 = 1 - 2x$.

If we consider the edge BC, then $OP < PN$ will imply

$$y^2 < 2\left(x + \frac{1}{2}\right) \qquad (2)$$

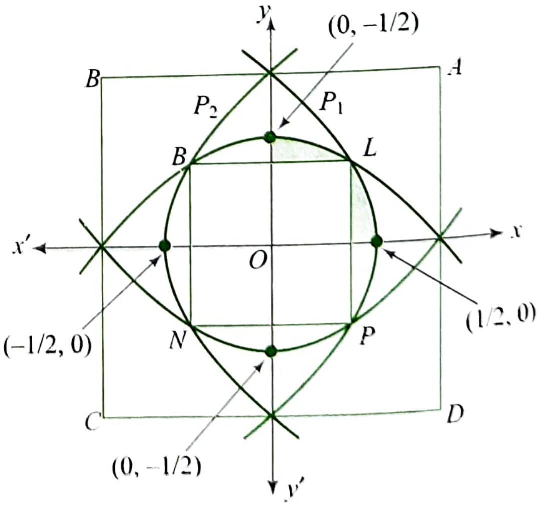
Similarly, if we consider the edges AB and CD, we will have

$$x^2 < -2\left(y - \frac{1}{2}\right) \qquad (3)$$

$$x^2 < 2\left(y + \frac{1}{2}\right) \qquad (4)$$

Hence, S consists of the region bounded by four parabolas meeting the axes at $\left(\pm\frac{1}{2}, 0\right)$ and $\left(0, \pm\frac{1}{2}\right)$.

The point L is intersection of P_1 and P_3 given by (1) and (3).



Solving, we get

$$y^2 - x^2 = -2(x - y) = 2(y - x)$$

or $\quad y - x = 0$

or $\quad y = x$

$x^2 + 2x - 1 = 0$

$\Rightarrow$ $(x+1)^2 = 2$

or $x = \sqrt{2} - 1$ as x is +ve

or L is $(\sqrt{2} - 1, \sqrt{2} - 1)$.

$\therefore$ Total area $= 4\left[\text{Square of side } (\sqrt{2} - 1) + 2\int_{\sqrt{2}-1}^{1/2} \sqrt{1-2x}\,dx\right]$

$$= 4\left\{(\sqrt{2}-1)^2 + 2\int_{\sqrt{2}-1}^{1/2} \sqrt{(1-2x)}\,dx\right\}$$

$$= 4\left[3 - 2\sqrt{2} - \frac{2}{3}\left\{(1-2x)^{3/2}\right\}_{\sqrt{2}-1}^{1/2}\right]$$

$$= 4\left[3 - 2\sqrt{2} - \frac{2}{3}\left\{0 - \left(1 - 2\sqrt{2} + 2\right)^{3/2}\right\}\right]$$

$$= 4\left[3 - 2\sqrt{2} + \frac{2}{3}\left(3 - 2\sqrt{2}\right)^{3/2}\right]$$

$$= 4\left(3 - 2\sqrt{2}\right)\left[1 + \frac{2}{3}\sqrt{\left(3 - 2\sqrt{2}\right)}\right]$$

$$= 4\left(3 - 2\sqrt{2}\right)\left[1 + \frac{2}{3}\left(\sqrt{2} - 1\right)\right]$$

$$= \frac{4}{3}\left(3 - 2\sqrt{2}\right)\left(1 + 2\sqrt{2}\right) = \frac{4}{3}\left[\left(4\sqrt{2} - 5\right)\right]$$

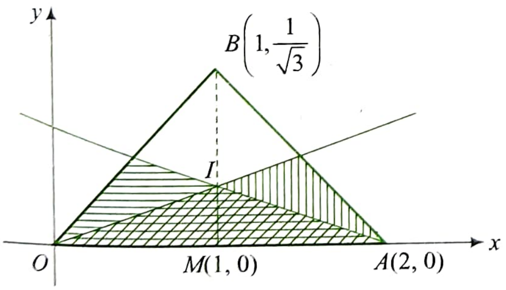
$$= \frac{16\sqrt{2} - 20}{3} \text{ sq. units.}$$

ILLUSTRATION 9.34

Let $O(0, 0)$, $A(2, 0)$, and $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle.

Let R be the region consisting of all those points P inside ΔOAB which satisfy $d(P, OA) \le \min\,[d(P, OB), d(P, AB)]$, where d denotes the distance from the point to the corresponding line. Sketch the region R and find its area.

Sol.



$d(P, OA) \le \min\,[d(P, OB), d(P, AB)]$

$\Rightarrow$ $d(P, OA) \le d(P, OB)$ and $d(P, OA) \le d(P, AB)$

When $d(P, OA) = d(P, OB)$, P is equidistant from OA and OB, or P lies on the angular bisector of lines OA and OB.

Hence, when $d(P, OA) \le d(P, OB)$, point P is nearer to OA than to OB, i.e., lies on or below the bisector of OA and OB.

Similarly, when $d(P, OA) \le d(P, AB)$, P is nearer to OA than to OB, i.e., lies on or below the bisector of OA and AB.

$\therefore$ Required area = Area of $\Delta\, OIA$.

Now, $\tan \angle BOA = \frac{1/\sqrt{3}}{1} = \frac{1}{\sqrt{3}}$

or $\angle BOA = 30° \Rightarrow \angle IOA = 15°$

$\Rightarrow$ $IM = \tan 15° = 2 - \sqrt{3}$.

Hence, Area of $\Delta\, OIA = \frac{1}{2} OA \times IM = \frac{1}{2} \times 2 \times (2 - \sqrt{3})$

$= 2 - \sqrt{3}$ sq. units

AREA BOUNDED BY INVERSE OF GIVEN FUNCTION WITHOUT GETTING INVERSE FUNCTION

In some problems, we have to find the bounded area which involves inverse function of the given function and other curves. But what if the inverse function cannot be determined? Well, in this case still we can find the area using substitution method using and integration by parts.

ILLUSTRATION 9.35

Find the area enclosed by $y = g(x)$, x-axis, $x = 1$ and $x = 37$, where $g(x)$ is inverse of $f(x) = x^3 + 3x + 1$.

Sol. Required area, $A = \int_1^{37} g(x)\,dx = \int_1^{37} f^{-1}(x)\,dx$.

Let $f^{-1}(x) = t$ or $x = f(t)$

Using intelligent guessing, $f(3) = 37$ and $f(0) = 1$

$$\therefore \quad A = \int_0^3 t f'(t)\,dt = [t f(t)]_0^3 - \int_0^3 f(t)\,dt$$

$$= 3f(3) - \int_0^3 (t^3 + 3t + 1)\,dt$$

$$= 111 - \frac{147}{4} = \frac{297}{4}$$

Alternative method:

$f(x) = x^3 + 3x + 1$.

$\therefore$ $f'(x) = 3x^2 + 3 > 0, \forall \in R$.

$\therefore$ $f(x)$ is an increasing function.

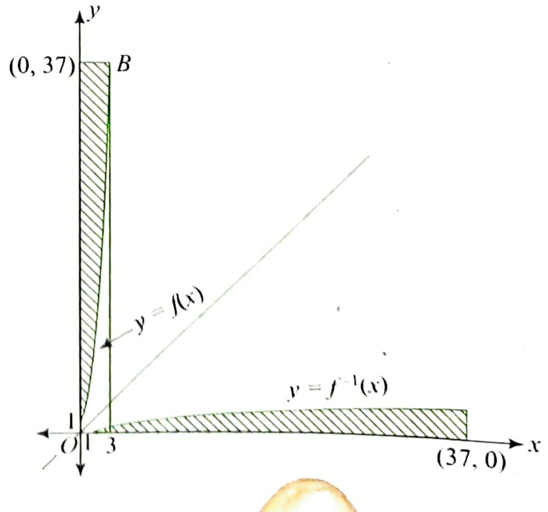
Also, $x^3 + 3x + 1 = x$ or $x^3 + 2x + 1 = 0$ has no positive root.

So, line $y = x$ never meet curve $y = f(x)$ for $x > 0$.

Graph of $y = f(x)$ and $y = f^{-1}(x)$ are as shown in the following figure.

When $y = 1$, $x^3 + 3x + 1 = 1$, $x = 0$.

When $y = 37$, $x^3 + 3x = 36$, $x = 3$



Area bounded by curves $y = f^{-1}(x)$, x-axis, $x = 1$ and $x = 37$ is same as area bounded by curves $y = f(x)$, y-axis, $y = 1$ and $y = 37$.

$\therefore$ Required area $= \int_0^3 (37 - x^3 - 3x - 1)dx$

$$= \left[36x - \frac{x^4}{4} - \frac{3x^2}{2}\right]_0^3 = \frac{297}{4} \text{ sq. units.}$$

ILLUSTRATION 9.36

Find the area bounded by the curve $f(x) = x + \sin x$ and its inverse function between the ordinates $x = 0$ to $x = 2\pi$.

Sol. $y = x + \sin x$

$\therefore \quad \frac{dy}{dx} = 1 + \cos x \geq 0 \; \forall \; x$

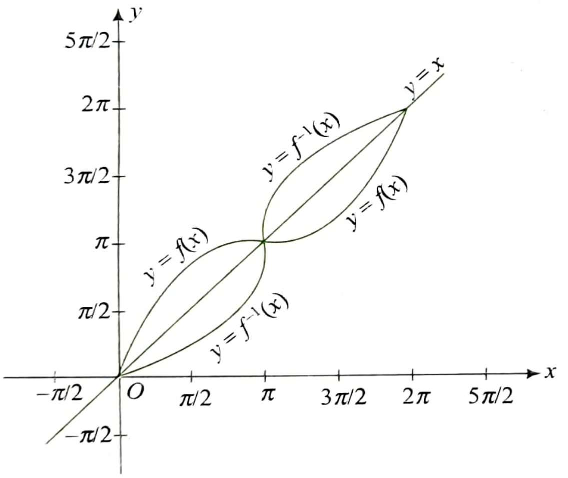
Also $\frac{d^2y}{dx^2} = -\sin x = 0$ when $x = n\pi$, $n \in Z$

Hence $x = n\pi$ are points of inflection, where curve changes its concavity.

Also, for $x \in (0, \pi)$, $\sin x > 0 \Rightarrow x + \sin x > x$

And for $x \in (\pi, 2\pi)$, $\sin x < 0 \Rightarrow x + \sin x < x$

Thus, we have following graphs of function $y = f(x)$ and its inverse function.



Required area $= 4\int_0^{\pi} (f(x) - x)dx$

$$= 4\int_0^{\pi} \sin x dx = 4 \times 2 = 8 \text{ sq. units.}$$

CONCEPT APPLICATION EXERCISE 9.3

1. Find a continuous function f, where $(x^4 - 4x^2) \leq f(x) \leq (2x^2 - x^3)$ such that the area bounded by $y = f(x)$, $y = x^4 - 4x^2$, the y-axis, and the line $x = t$, where $(0 \leq t \leq 2)$ is k times the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, y-axis, and line $x = t$(where $0 \leq t \leq 2$).
2. If the area bounded by the x-axis, the curve $y = f(x)$, $(f(x) > 0)$ and the lines $x = 1$, $x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$, then find $f(x)$.
3. The area bounded by the graph of $y = f(x)$, $f(x) > 0$ on $[0, a]$ and x-axis is $\frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$ then find the value of $f\left(\frac{\pi}{2}\right)$.
4. A curve $y = f(x)$ is such that $f(x) \geq 0$ and $f(0) = 0$ and bounds a curvilinear triangle with the base $[0, x]$ whose area is proportional to $(n + 1)^{\text{th}}$ power of $f(x)$. If $f(1) = 1$ then find $f(x)$.
5. Find the area of curve enclosed by $|x + y| + |x - y| \leq 4$, $|x| \leq 1$, $y \geq \sqrt{x^2 - 2x + 1}$.
6. Consider two regions
   R_1 : points P are nearer to (1, 0) than to $x = -1$.
   R_2 : Points P are nearer to (0, 0) than to (8, 0)
   Find the area of the region common to R_1 and R_2.
7. If $f : [-1, 1] \to \left[-\frac{1}{2}, \frac{1}{2}\right]$, $f(x) = \frac{x}{1 + x^2}$, then find the area bounded by $y = f^{-1}(x)$, x axis and lines $x = \frac{1}{2}$, $x = -\frac{1}{2}$.

ANSWERS

1. $\frac{1}{k+1}\left(x^4 - kx^3 + 2(k-2)x^2\right)$
2. $\frac{x}{\sqrt{x^2 + 1}}$
3. $\frac{1}{2}$
4. $x^{1/n}$
5. 2 sq. units
6. $\frac{64}{3}$ sq. units
7. $(1 - \log 2)$ sq. units

Solved Examples

EXAMPLE 9.1

Find the area bounded by the curve $x^2 = y$, $x^2 = -y$, and $y^2 = 4x - 3$.

Sol. The given curves are

$x^2 = y$ (1)

$x^2 = -y$ (2)

$y^2 = 4x - 3$ (3)

Clearly (1) and (2) meet at (0, 0).

Solving (1) and (3), we get $x^4 - 4x + 3 = 0$

or $(x - 1)(x^3 + x^2 + x - 3) = 0$

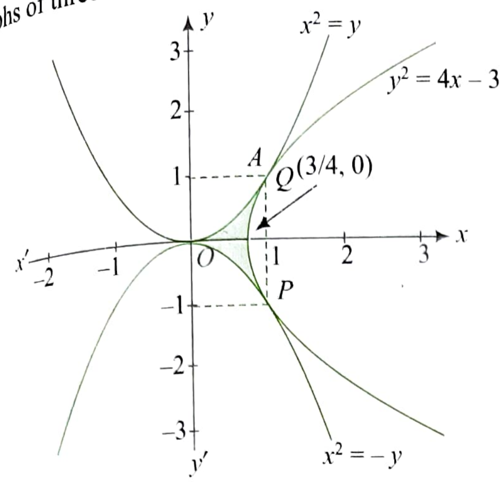
or $(x - 1)^2(x^2 + 2x + 3) = 0$

$\Rightarrow \quad x = 1 \Rightarrow y = 1$

Thus, point of intersection is (1, 1).

Similarly, point of intersection of (2) and (3) is (1, −1).

The graphs of three curves are as shown in the figure.



We also observe that at $x = 1$ and $y = 1$, $\frac{dy}{dx}$ for (1) and (3) is same and hence the two curves touch each other at (1, 1).

Same is the case with (2) and (3) at (1, – 1).

Required area = Shaded region in the figure

$$= 2\ (\text{Ar } OPQA)$$

$$= 2\left[\int_0^1 x^2\,dx - \int_{3/4}^1 \sqrt{4x-3}\,dx\right]$$

$$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{2(4x-3)^{3/2}}{4\times 3}\right)_{3/4}^1\right] = 2\left[\frac{1}{3} - \frac{1}{6}\right]$$

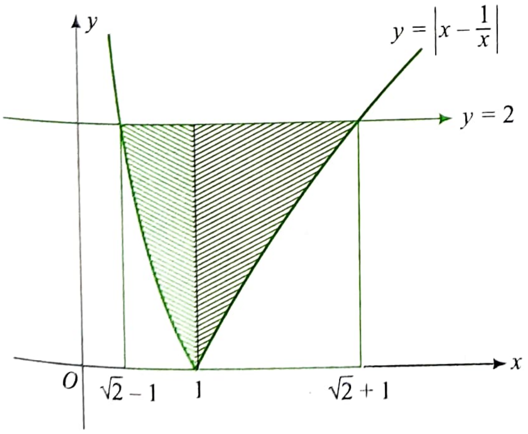
$$= \frac{1}{3} \text{ sq. units.}$$

EXAMPLE 9.2

Find the area of the region enclosed by the curve $y = \left|x - \frac{1}{x}\right|$ $(x > 0)$ and the line $y = 2$

Sol. We have $y = \begin{cases} \frac{1}{x} - x, & if\ x \in (0,1) \\ x - \frac{1}{x}, & if\ x > 1 \end{cases}$

Graph of the function is as shown in the following figure.



Solving $\frac{1}{x} - x = 2\ ; x < 1$, we get

$$x^2 + 2x - 1 = 0$$

$$\therefore \quad x = \sqrt{2} - 1$$

Solving $x - \frac{1}{x} = 2\ ; x > 1$, we get

$$x^2 - 2x - 1 = 0$$

$$\therefore \quad x = \sqrt{2} + 1$$

$\therefore$ Required area,

$$A = \int_{\sqrt{2}-1}^{1} \left(2 - \left(\frac{1}{x} - x\right)\right) dx + \int_{1}^{\sqrt{2}+1} \left(2 - \left(x - \frac{1}{x}\right)\right) dx$$

$$= \int_{\sqrt{2}-1}^{1} \left(2 + x - \frac{1}{x}\right) dx + \int_{1}^{\sqrt{2}+1} \left(2 + \frac{1}{x} - x\right) dx$$

$$= \left[2x + \frac{x^2}{2} - \log_e x\right]_{\sqrt{2}-1}^{1} + \left[2x - \frac{x^2}{2} + \log_e x\right]_{1}^{\sqrt{2}+1}$$

$$= \frac{5}{2} - \left(2\sqrt{2} - 2 + \frac{3 - 2\sqrt{2}}{2} - \log_e(\sqrt{2} - 1)\right)$$

$$+\ 2\sqrt{2} + 2 - \frac{3 + 2\sqrt{2}}{2} + \log_e(\sqrt{2} + 1) + \frac{3}{2}$$

$$= 5 + \log_e(\sqrt{2} + 1) + \log_e(\sqrt{2} - 1) = 2$$

EXAMPLE 9.3

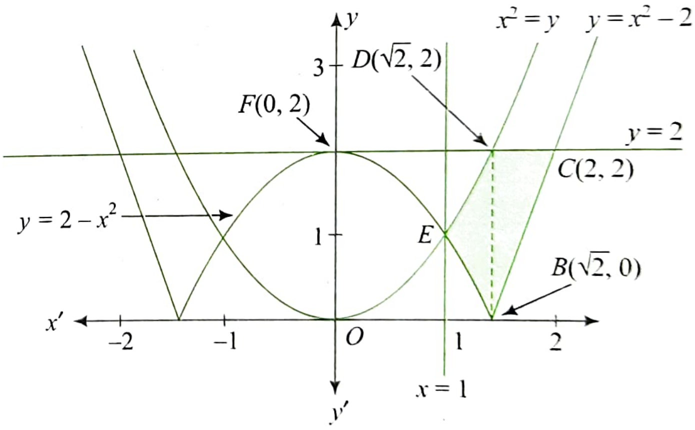
Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$, and $y = 2$, which lies to the right of the line $x = 1$.

Sol. The given curves are

$$y = x^2 \tag{1}$$

$$y = |2 - x^2| \tag{2}$$

The graphs of these curve are as follows:



$\therefore$ Required area = BCDEB

$$= \int_1^{\sqrt{2}} \left[x^2 - \left(2 - x^2\right)\right] dx + \int_{\sqrt{2}}^{2} \left[2 - (x^2 - 2)\right] dx$$

$$= \int_1^{\sqrt{2}} \left(2x^2 - 2\right) dx + \int_{\sqrt{2}}^{2} \left(4 - x^2\right) dx$$

$$= \left[\frac{2x^3}{3} - 2x\right]_1^{\sqrt{2}} + \left[4x - \frac{x^3}{3}\right]_{\sqrt{2}}^{2}$$

$$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2} - \frac{2}{3} + 2\right) + \left(8 - \frac{8}{3} - 4\sqrt{2} + \frac{2\sqrt{2}}{3}\right)$$

$$= \left(\frac{20}{3} - 4\sqrt{2}\right) \text{ sq. units.}$$

EXAMPLE 9.4

The ratio in which the line $x-1=0$ divides the area bounded by the curves $2x+1=\sqrt{4y+1}$, $y=x$ and $y=2$ is

Sol. We have $2x+1=\sqrt{4y+1}$ ...(1)

or $(2x+1)^2=4y+1$

or $y=x^2+x$ ...(2)

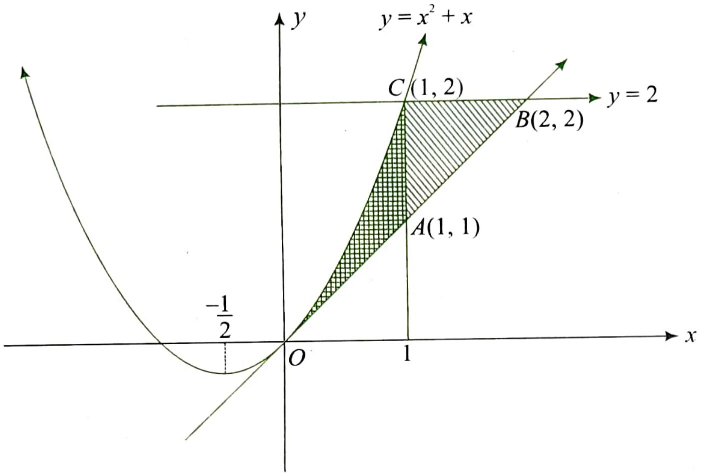
$\therefore$ Curve (1) is part of the parabola (2), for $x\ge -1/2$.

Differentiating (2), w.r.t. x we get

$$\frac{dy}{dx}=2x+1$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=0}=1.$$

$\therefore$ line $y=x$ is tangent to curve (2) at (0, 0).



From the figure,

$$\text{Area of the region } OACO \text{ is } = \int_0^1 (x^2+x-x)dx$$

$$=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}$$

$$\text{Area of the triangle } ACB \text{ is } =\frac{1}{2}\times 1\times 1=\frac{1}{2}$$

$$\therefore \text{Required ratio}=\frac{1/3}{1/2}=\frac{2}{3}$$

EXAMPLE 9.5

If $S_0, S_1, S_2, \ldots$ are areas bounded by the x-axis and half-wave of the curve $y=\sin\pi\sqrt{x}$, then prove that $S_0, S_1, S_2, \cdots$ are in A.P..

Sol. $y=\sin\pi\sqrt{x}$ meets x-axis when $\pi\sqrt{x}=n\pi$ or $x=n^2$, $n\in N$.

Therefore, area of half-wave between $x=n^2$ and $x=(n+1)^2$ is

$$S_n=\left|\int_{n^2}^{(n+1)^2}\sin\pi\sqrt{x}dx\right|$$

Putting $\pi\sqrt{x}=y$ and $\pi^2\,dx=2y\,dy$, we get

$$\therefore S_n=\left|\frac{2}{\pi^2}\int_{n\pi}^{(n+1)\pi} y\sin y dy\right|$$

$$=\left|\frac{2}{\pi^2}[-y\cos y+\sin y]_{n\pi}^{(n+1)\pi}\right|$$

$$=\left|\frac{2}{\pi^2}[-(n+1)\pi\cos(n+1)\pi+n\pi\cos n\pi]\right|$$

$$=\frac{2(2n+1)}{\pi},\ n\in N$$

Hence, $S_0, S_1, S_2, \ldots$ are in A.P..

EXAMPLE 9.6

Find the area of the figure enclosed by the curve $5x^2+6xy+2y^2+7x+6y+6=0$.

Sol. Equation of curve can be re-written as

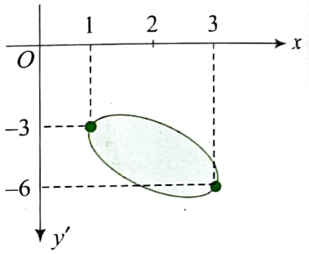
$$2y^2+6(1+x)y+5x^2+7x+6=0$$

$$\Rightarrow y_1=\frac{-3(1+x)-\sqrt{(3-x)(x-1)}}{2},$$

$$y_2=\frac{-3(1+x)+\sqrt{(3-x)(x-1)}}{2}$$

Therefore, the curves (y_1 and y_2) are defined for values of x for which $(3-x)(x-1)\ge 0$, i.e., $1\le x\le 3$.

(Actually the given equation represents an ellipse, because $\Delta\ne 0$ and $h^2<ab$.)



Required area will be given by

$$A=\int_1^3 |y_1-y_2|\,dx \Rightarrow A=\int_1^3\sqrt{(3-x)(x-1)}\,dx$$

Put $x=3\cos^2\theta+\sin^2\theta$, i.e., $dx=-2\sin 2\theta\,d\theta$

$$\therefore A=2\int_0^{\pi/2}\sin^2 2\theta\,d\theta=\frac{\pi}{2}\text{ sq. units.}$$

EXAMPLE 9.7

Find the area bounded by the curves $x^2+y^2=4$, $x^2=-\sqrt{2}y$, and $x=y$.

Sol. The given curves are

$x^2+y^2=4$ (circle) (1)

$x^2=-\sqrt{2}y$ (parabola, concave downward) (2)

$x=y$ (straight line through origin) (3)

Solving equations (1) and (2), we get

$y^2-\sqrt{2}y-4=0$

$\Rightarrow y=2\sqrt{2}$ or $-\sqrt{2}$

$\Rightarrow x^2=2$ (rejecting $y=2\sqrt{2}$ as x^2 is positive)

or $x=\pm\sqrt{2}$.

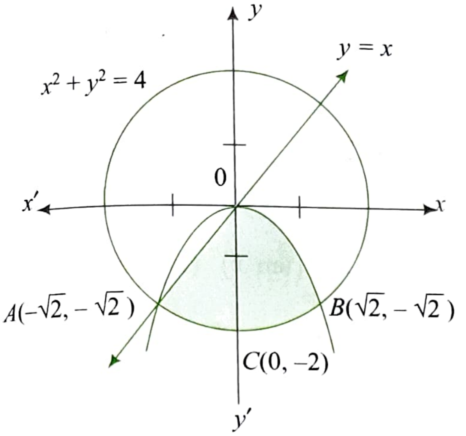
Therefore, points of intersection of (1) and (2) are $B(\sqrt{2},-\sqrt{2})$, $A(-\sqrt{2},-\sqrt{2})$.

Solving (1) and (3), we get

$2x^2=4$ or $x^2=2$ or $x=\pm\sqrt{2}$ or $y=\pm\sqrt{2}$.

Therefore, points of intersection are $(-\sqrt{2},-\sqrt{2})$, $(\sqrt{2},\sqrt{2})$.

Thus, all the three curves pass through the same point $A(-\sqrt{2}, -\sqrt{2})$.



Now, Required area = Area of the shaded region

$$= \int_{-\sqrt{2}}^{0}\left(x-\left(-\sqrt{4-x^2}\right)\right)dx + \int_0^{\sqrt{2}}\left(-\frac{x^2}{\sqrt{2}}-\left(-\sqrt{4-x^2}\right)\right)dx$$

$$= 2\int_0^{\sqrt{2}} \sqrt{4-x^2}\,dx + \int_{-\sqrt{2}}^{0} x\,dx - \int_0^{\sqrt{2}} \frac{x^2}{\sqrt{2}}dx$$

$$= 2\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{2}} + \left[\frac{x^2}{2}\right]_{-\sqrt{2}}^{0} - \left[\frac{x^3}{3\sqrt{2}}\right]_0^{\sqrt{2}}$$

$$= 2\left[\frac{\sqrt{2}}{2}\sqrt{4-2} + 2\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right] + \left[\frac{-2}{2}\right] - \left[\frac{2\sqrt{2}}{3\sqrt{2}}\right]$$

$$= 2\left[1+2\frac{\pi}{4}\right] - 1 - \frac{2}{3} = \pi + \frac{1}{3} \text{ sq. units.}$$

EXAMPLE 9.8

Find the area of the region bounded by the curve $C: y = \tan x$, tangent drawn to C at $x = \frac{\pi}{4}$, and the x-axis.

Sol. The given curve is $y = \tan x$ (1)

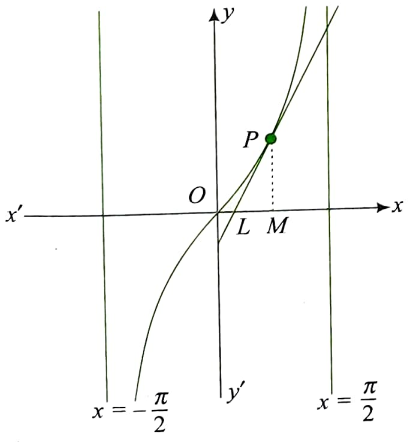
When $x = \pi/4$, $y = 1$

i.e., co-ordinates of P are $(\pi/4, 1)$

$\therefore$ equation of tangent at P is $y - 1 = \left(\sec^2\frac{\pi}{4}\right)(x - \pi/4)$

or $y = 2x + 1 - \pi/2$ (2)

The graphs of (1) and (2) are as shown in the figure.



Tangent (2) meets x-axis at $L\left(\frac{\pi-2}{4}, 0\right)$

Now, Required area = Shaded area

$= \text{Area } OPMO - \text{Ar}(\Delta PLM)$

$$= \int_0^{\pi/4} \tan x\,dx - \frac{1}{2}(OM - OL)PM$$

$$= [\log \sec x]_0^{\pi/4} - \frac{1}{2} \times \left\{\frac{\pi}{4} - \frac{\pi-2}{4}\right\} \times 1$$

$$= \frac{1}{2}\left[\log 2 - \frac{1}{2}\right] \text{ sq. units.}$$

EXAMPLE 9.9

Compute the area of the region bounded by the curves $y = ex \log_e x$ and $y = \frac{\log x}{ex}$.

Sol. The given curves are

$y = ex \log_e x$ (1)

$y = \frac{\log x}{ex}$ (2)

The two curves intersect where $ex \log x = \frac{\log x}{ex}$

$$\Rightarrow \quad \left(ex - \frac{1}{ex}\right)\log x = 0 \quad \Rightarrow \quad x = \frac{1}{e} \text{ or } x = 1$$

At $x = 1/e$, $y = -1$ (from (1))

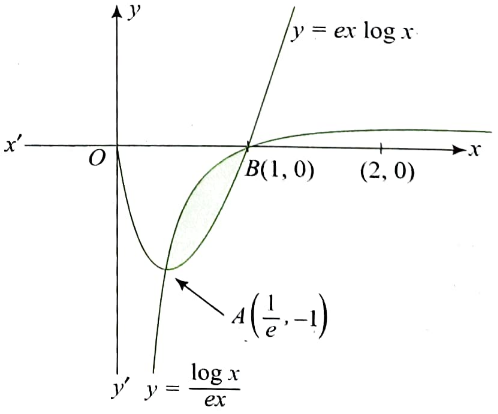
At $x = 1$, $y = 0$ (from (1))

So points of intersection are $\left(\frac{1}{e}, -1\right)$ and $(1, 0)$.

Curve Tracing

Curve 1	Curve 2
For $0 < x < 1$, $y < 0$	For $0 < x < 1$, $y < 0$
For $x > 1$, $y > 0$	For $x > 1$, $y > 0$
When $x \to 0$, $y \to -\infty$	When $x \to 0$, $y \to -\infty$
When $x \to \infty$, $y \to \infty$	When $x \to \infty$, $y \to 0$
$\frac{dy}{dx} = e(\log x + 1)$	$\frac{dy}{dx} = \frac{(1 - \log x)}{ex^2}$
$x = \frac{1}{e}$ is a point of minima.	$x = e$ is a point of maxima.

From the above information, the rough sketch of two curves is as shown in the figure and the shaded area is the required area.



$\therefore$ Required area = Area of the shaded region

$$= \left|\int_{1/e}^{1}\left[ex\log x - \frac{\log x}{ex}\right]dx\right|$$

$$= \left|e\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_{1/e}^{1} - \frac{1}{e}\left[\frac{(\log x)^2}{2}\right]_{1/e}^{1}\right|$$

$$= \left|e\left[\left(-\frac{1}{4}\right) - \left(-\frac{1}{2e^2} - \frac{1}{4e^2}\right)\right] - \frac{1}{e}\left[0 - \frac{1}{2}\right]\right|$$

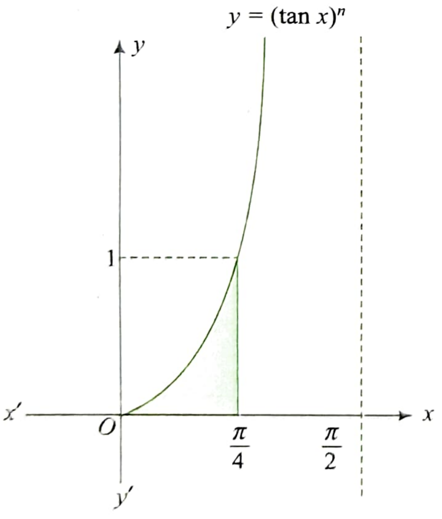
$$= \left|e\left[-\frac{1}{4} + \frac{3}{4e^2}\right] + \frac{1}{2e}\right|$$

$$= \left|\frac{5-e^2}{4e}\right| = \frac{e^2-5}{4e} \text{ sq. units.}$$

EXAMPLE 9.10

Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $y = 0$, and $x = \frac{\pi}{4}$. Prove that for $n > 2$, $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce that $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.

Sol. We have $A_n = \int_0^{\pi/4}(\tan x)^n\, dx$



Since $0 < \tan x < 1$, when $0 < x < \pi/4$, we have

$0 < (\tan x)^{n+1} < (\tan x)^n$ for each $n \in N$

$$\Rightarrow \int_0^{\pi/4}(\tan x)^{n+1}\, dx < \int_0^{\pi/4}(\tan x)^n\, dx$$

$$\Rightarrow A_{n+1} < A_n$$

Now, for $n > 2$,

$$A_n + A_{n+2} = \int_0^{\pi/4}[(\tan x)^n + (\tan x)^{n+2}]\, dx$$

$$= \int_0^{\pi/4}(\tan x)^n(1 + \tan^2 x)\, dx$$

$$= \int_0^{\pi/4}(\tan x)^n(\sec^2 x)\, dx$$

$$= \left[\frac{1}{(n+1)}(\tan x)^{n+1}\right]_0^{\pi/4}$$

$$\left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \frac{1}{(n+1)}(1-0)$$

Since $A_{n+2} < A_{n+1} < A_n$, we get

$$A_n + A_{n+2} < 2A_n$$

$$\Rightarrow \frac{1}{n+1} < 2A_n \Rightarrow \frac{1}{2n+2} < A_n \quad (1)$$

Also for $n > 2$, $A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$

or $$2A_n < \frac{1}{n-1}$$

or $$A_n < \frac{1}{2n-2} \quad (2)$$

Combining (1) and (2), we get $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.

Exercises

Single Correct Answer Type

1. Area enclosed by the curve $y = f(x)$ defined parametrically as $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ is equal to

(1) π sq. units (2) $\pi/2$ sq. units

(3) $\frac{3\pi}{4}$ sq. units (4) $\frac{3\pi}{2}$ sq. units

2. Let $f(x) = \text{minimum}(x+1, \sqrt{1-x})$ for all $x \le 1$. Then the area bounded by $y = f(x)$ and the x-axis is

(1) $\frac{7}{3}$ sq. units (2) $\frac{1}{6}$ sq. units

(3) $\frac{11}{6}$ sq. units (4) $\frac{7}{6}$ sq. units

3. The area of the closed figure bounded by $x = -1$, $y = 0$, $y = x^2 + x + 1$, and the tangent to the curve $y = x^2 + x + 1$ at $A(1, 3)$ is

(1) 4/3 sq. units (2) 7/3 sq. units

(3) 7/6 sq. units (4) None of these

4. The area bounded by the curve $a^2y = x^2(x + a)$ and the x-axis is

(1) $a^2/3$ sq. units (2) $a^2/4$ sq. units

(3) $3a^2/4$ sq. units (4) $a^2/12$ sq. units

5. The area between the curve $y = 2x^4 - x^2$, the x-axis, and the ordinates of the two minima of the curve is

(1) 11/60 sq. units (2) 7/120 sq. units

(3) 1/30 sq. units (4) 7/90 sq. units

6. The area of the closed figure bounded by $x = -1$, $x = 2$, and $y = \begin{cases} -x^2 + 2, & x \le 1 \\ 2x - 1, & x > 1 \end{cases}$ and the abscissa axis is

(1) 16/3 sq. units (2) 10/3 sq. units

(3) 13/3 sq. units (4) 7/3 sq. units

7. The value of the parameter a such that the area bounded by $y = a^2x^2 + ax + 1$, coordinate axes, and the line $x = 1$ attains its least value is equal to

(1) $-\frac{1}{4}$ sq. units (2) $-\frac{1}{2}$ sq. units

(3) $-\frac{3}{4}$ sq. units (4) -1 sq. units

8. The positive value of the parameter 'k' for which the area of the figure bounded by the curve $y = \sin(kx)$, $x = \frac{2\pi}{3k}$, $x = \frac{5\pi}{3k}$ and x-axis is less than 2 can be

(1) $\frac{1}{8} < k < \frac{3}{8}$ (2) $0 < k < \frac{1}{8}$

(3) $1 < k < 2$ (4) $\frac{3}{8} < k < \frac{5}{8}$

9. The area bounded by the curve $y = x(1 - \log_e x)$ and x-axis is

(1) $\frac{e^2}{4}$ (2) $\frac{e^2}{2}$

(3) $\frac{e^2 - e}{2}$ (4) $\frac{e^2 - e}{4}$

10. The area inside the parabola $5x^2 - y = 0$ but outside the parabola $2x^2 - y + 9 = 0$ is

(1) $12\sqrt{3}$ sq. units (2) $6\sqrt{3}$ sq. units

(3) $8\sqrt{3}$ sq. units (4) $4\sqrt{3}$ sq. units

11. Area enclosed between the curves $|y| = 1 - x^2$ and $x^2 + y^2 = 1$ is

(1) $\frac{3\pi - 8}{3}$ sq. units (2) $\frac{\pi - 8}{3}$ sq. units

(3) $\frac{2\pi - 8}{3}$ sq. units (4) None of these

12. If A_n is the area bounded by $y = x$ and $y = x^n$, $n \in N$, then $A_2 . A_3 \ldots A_n =$

(1) $\frac{1}{n(n+1)}$ (2) $\frac{1}{2^n n(n+1)}$

(3) $\frac{1}{2^{n-1} n(n+1)}$ (4) $\frac{1}{2^{n-2} n(n+1)}$

13. The area of the region in 1st quadrant bounded by the y-axis, $y = \frac{x}{4}$, $y = 1 + \sqrt{x}$, and $y = \frac{2}{\sqrt{x}}$ is

(1) 2/3 sq. units (2) 8/3 sq. units

(3) 11/3 sq. units (4) 13/6 sq. units

14. The area of the closed figure bounded by $y = \frac{x^2}{2} - 2x + 2$ and the tangents to it at (1, 1/2) and (4, 2) is

(1) 9/8 sq. units (2) 3/8 sq. units

(3) 3/2 sq. units (4) 9/4 sq. units

15. The area of the region bounded by $x^2 + y^2 - 2x - 3 = 0$ and $y = |x| + 1$ is

(1) $\frac{\pi}{2} - 1$ sq. units (2) 2π sq. units

(3) 4π sq. units (4) $\pi/2$ sq. units

16. The area enclosed by the curve $y = \sqrt{4 - x^2}$, $y \ge \sqrt{2}\sin\left(\frac{x\pi}{2\sqrt{2}}\right)$, and the x-axis is divided by the y-axis in the ratio

(1) $\frac{\pi^2 - 8}{\pi^2 + 8}$ (2) $\frac{\pi^2 - 4}{\pi^2 + 4}$

(3) $\frac{\pi - 4}{\pi - 4}$ (4) $\frac{2\pi^2}{2\pi + \pi^2 - 8}$

17. The area bounded by the curve $y^2 = 1 - x$ and the lines $y = \frac{|x|}{x}$, $x = -1$, and $x = \frac{1}{2}$ is

(1) $\frac{3}{\sqrt{2}} - \frac{11}{6}$ sq. units (2) $3\sqrt{2} - \frac{11}{4}$ sq. units

(3) $\frac{6}{\sqrt{2}} - \frac{11}{5}$ sq. units (4) None of these

18. The area bounded by the curves $y = \log_e x$ and $y = (\log_e x)^2$ is

(1) $e - 2$ sq. units (2) $3 - e$ sq. units

(3) e sq. units (4) $e - 1$ sq. units

19. The area bounded by $y = 3 - |3 - x|$ and $y = \frac{6}{|x+1|}$ is

(1) $\frac{15}{2} - 6 \ln 2$ sq. units

(2) $\frac{13}{2} - 3 \ln 2$ sq. units

(3) $\frac{13}{2} - 6 \ln 2$ sq. units

(4) None of these

20. The area enclosed between the curves $y = \log_e (x + e)$, $x = \log_e \left(\frac{1}{y}\right)$, and the x-axis is

(1) 2 sq. units (2) 1 sq. units

(3) 4 sq. units (4) None of these

21. The area enclosed between $y = \sin 2x$, $y = \sqrt{3} \sin x$ between $x = 0$ and $x = \pi$ is

(1) $\frac{7}{2}$ sq. units (2) $\frac{7}{4} + \sqrt{3}$ sq. units

(3) $\frac{7\sqrt{3}}{4}$ sq. units (4) $7 - \frac{\sqrt{3}}{4}$ sq. units

22. The area bounded by $y = x^2$, $y = [x + 1]$, $0 \le x \le 2$ and the y-axis is where [.] is greatest integer function.

(1) $\frac{1}{3}$ (2) $\frac{\sqrt{2}}{3}$

(3) 1 (4) $\frac{7}{3}$

23. The area of the figure bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to it at the point with the ordinate $x = 3$, and the x-axis is

(1) 7 sq. units (2) 6 sq. units

(3) 9 sq. units (4) None of these

24. The area bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$ is

(1) $\frac{2}{e}$ sq. units (2) $1 - \frac{2}{e}$ sq. units

(3) $\frac{1}{e}$ sq. units (4) $1 - \frac{1}{e}$ sq. units

25. The area of the region whose boundaries are defined by the curves $y = 2 \cos x$, $y = 3 \tan x$, and the y-axis is

(1) $1 + 3 \ln\left(\frac{2}{\sqrt{3}}\right)$ sq. units

(2) $1 + \frac{3}{2} \ln 3 - 3 \ln 2$ sq. units

(3) $1 + \frac{3}{2} \ln 3 - \ln 2$ sq. units

(4) $\ln 3 - \ln 2$ sq. units

26. The area bounded by $y = \sec^{-1} x$, $y = \operatorname{cosec}^{-1} x$, and line $x - 1 = 0$ is

(1) $\log (3 + 2\sqrt{2}) - \frac{\pi}{2}$ sq. units

(2) $\frac{\pi}{2} - \log (3 + 2\sqrt{2})$ sq. units

(3) $\pi - \log_e 3$ sq. units

(4) None of these

27. The area bounded by the curve $y^2 (2 - x) = x^3$ and $x = 2$ is

(1) $\frac{\pi}{2}$ (2) π (3) 2π (4) 3π

28. The area enclosed by $y = x^2 + \cos x$ and its normal at $x = \frac{\pi}{2}$ in the first quadrant is

(1) $\frac{\pi^5}{32} - \frac{\pi^4}{64} + \frac{\pi^3}{32} + 1$ (2) $\frac{\pi^5}{16} - \frac{\pi^4}{32} + \frac{\pi^3}{24} - 1$

(3) $\frac{\pi^5}{32} - \frac{\pi^4}{32} + \frac{\pi^3}{16}$ (4) $\frac{\pi^5}{32} - \frac{\pi^4}{32} + \frac{\pi^3}{24} + 1$

29. Given $f(x) = \int_0^x e^t (\log_e \sec t - \sec^2 t)\, dt$, $g(x) = -2e^x \tan x$, then the area bounded by the curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = 0$ and $x = \frac{\pi}{3}$, is (in sq. units)

(1) $\frac{1}{2} e^{\frac{\pi}{3}} \log_e 2$ (2) $e^{\frac{\pi}{3}} \log_e 2$

(3) $\frac{1}{4} e^{\frac{\pi}{3}} \log_e 2$ (4) $e^{\frac{\pi}{3}} \log_e 3$

30. The area of the loop of the curve $ay^2 = x^2 (a - x)$ is

(1) $4a^2$ sq. units (2) $\frac{8a^2}{15}$ sq. units

(3) $\frac{16a^2}{9}$ sq. units (4) None of these

31. The area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y| \sqrt{1 - y^2}$ is

(1) 1 sq. units (2) 4/3 sq. units

(3) 2/3 sq. units (4) 2 sq. units

32. The area bounded by the loop of the curve $4y^2 = x^2(4 - x^2)$ is

(1) 7/3 sq. units (2) 8/3 sq. units

(3) 11/3 sq. units (4) 16/3 sq. units

33. The area enclosed by the curves $xy^2 = a^2(a - x)$ and $(a - x)$ $y^2 = a^2x$ is

(1) $(\pi - 2)a^2$ sq. units (2) $(4 - \pi)a^2$ sq. units

(3) $\pi a^2/3$ sq. units (4) None of these

34. The area bounded by the two branches of curve $(y - x)^2 = x^3$ and the straight line $x = 1$ is

(1) 1/5 sq. units (2) 3/5 sq. units

(3) 4/5 sq. units (4) 8/4 sq. units

35. The area bounded by the curves $y = \sin^{-1}|\sin x|$ and $y = (\sin^{-1}|\sin x|)^2$, where $0 \le x \le 2\pi$, is

(1) $\frac{1}{3}+\frac{\pi^2}{4}$ sq. units (2) $\frac{1}{6}+\frac{\pi^3}{8}$ sq. units

(3) 2 sq. units (4) None of these

36. Consider two curves C_1: $y^2 = 4[\sqrt{y}]x$ and C_2: $x^2 = 4[\sqrt{x}]y$, where [.] denotes the greatest integer function. Then the area of region enclosed by these two curves within the square formed by the lines $x = 1, y = 1, x = 4, y = 4$ is

(1) 8/3 sq. units (2) 10/3 sq. units

(3) 11/3 sq. units (4) 11/4 sq. units

37. The area enclosed between the curve $y^2(2a - x) = x^3$ and the line $x = 2$ above the x-axis is

(1) πa^2 sq. units (2) $\frac{3\pi a^2}{2}$ sq. units

(3) $2\pi a^2$ sq. units (4) $3\pi a^2$ sq. units

38. The area of the region of the plane bounded by $\max(|x|, |y|) \le 1$ and $xy \le \frac{1}{2}$ is

(1) $1/2 + \ln 2$ sq. units (2) $3 + \ln 2$ sq. units

(3) 31/4 sq. units (4) $1 + 2\ln 2$ sq. units

39. The area of the region containing the points (x, y) satisfying $4 \le x^2 + y^2 \le 2(|x| + |y|)$ is

(1) 8 sq. units (2) 2 sq. units

(3) 4π sq. units (4) 2π sq. units

40. Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$ is $\beta\sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta$. Then $f'\left(\frac{\pi}{2}\right)$ is

(1) $\left(\frac{\pi}{2} - \sqrt{2} - 1\right)$ (2) $\left(\frac{\pi}{4} + \sqrt{2} - 1\right)$

(3) $-\frac{\pi}{2}$ (4) $\left(1 - \frac{\pi}{4} - \sqrt{2}\right)$

Multiple Correct Answers Type

1. Let $A(k)$ be the area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$.

(1) The range of $A(k)$ is $\left[\frac{10\sqrt{5}}{3}, \infty\right)$

(2) The range of $A(k)$ is $\left[\frac{20\sqrt{5}}{3}, \infty\right)$

(3) If function $k \to A(k)$ is defined for $k \in [-2, \infty)$, then $A(k)$ is many-one function

(4) The value of k for which area is minimum is 1

2. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4$, $y = 4$ and the coordinate axes. If S_1, S_2, S_3 are the areas of these parts numbered from top to bottom, respectively, then

(1) $S_1:S_2 \equiv 1:1$ (2) $S_2:S_3 \equiv 1:2$

(3) $S_1:S_3 \equiv 1:1$ (4) $S_1:(S_1 + S_2) = 1:2$

3. Which of the following have the same bounded area

(1) $f(x) = \sin x$, $g(x) = \sin^2 x$, where $0 \le x \le 10\pi$

(2) $f(x) = \sin x$, $g(x) = |\sin x|$, where $0 \le x \le 20\pi$

(3) $f(x) = |\sin x|$, $g(x) = \sin^3 x$, where $0 \le x \le 10\pi$

(4) $f(x) = \sin x$, $g(x) = \sin^4 x$, where $0 \le x \le 10\pi$

4. If the curve $y = ax^{1/2} + bx$ passes through the point (1, 2) and lies above the x-axis for $0 \le x \le 9$ and the area enclosed by the curve, the x-axis, and the line $x = 4$ is 8 sq. units, then

(1) $a = 1$ (2) $b = 1$ (3) $a = 3$ (4) $b = -1$

5. The area enclosed by the curves $x = a\sin^3 t$ and $y = a\cos^3 t$ is equal to

(1) $12a^2 \int_0^{\pi/2} \cos^4 t \sin^2 t\, dt$

(2) $12a \int_0^{\pi/2} \cos^2 t \sin^4 t\, dt$

(3) $2\int_{-a}^{a} \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$

(4) $4\int_0^{a} \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$

6. If A_i is the area bounded by $|x - a_i| + |y| = b_i$, $i \in N$, where $a_{i+1} = a_i + \frac{3}{2}b_i$ and $b_{i+1} = \frac{b_i}{2}$, $a_1 = 0$, $b_1 = 32$, then

(1) $A_3 = 128$

(2) $A_3 = 256$

(3) $\lim_{n\to\infty} \sum_{i=1}^{n} A_i = \frac{8}{3}(32)^2$

(4) $\lim_{n\to\infty} \sum_{i=1}^{n} A_i = \frac{4}{3}(16)^2$

7. Area of the region bounded by the curve $y = e^{x^2}$ and the line $y = e$ is

(1) $2\int_1^{e} \sqrt{\log_e y}\, dy$ (2) $2e - \int_{-1}^{1} e^{x^2} dx$

(3) $\int_{-1}^{1} (e - e^{x^2})\, dx$ (4) $2\int_0^{1} \sqrt{x}\, e^{x}\, dx$

8. Consider curves $y = \frac{1}{x^2}; y = \frac{1}{4(x-1)}$. Let α be the value of a $(a > 2)$ for which area bounded by curves between $x = 2$ and $x = a$ is $1/a$ is $e^2 + 1$ and β be the of $b \in (1, 2)$, for which the area bounded by curves between $x = b$ and $x = 2$ is $1 - \frac{1}{b}$, then
   (1) $\alpha = e^2 + 1$
   (2) $\alpha = e^2 - 2$
   (3) $\beta = 1 + e^{-1}$
   (4) $\beta = 1 + e^{-2}$

9. Consider curves S_1: $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$, S_2: $x^2 + y^2 = a^2$ and $S_3 : |x| + |y| = a$. If α is area bounded by S_1 and S_2, β is area bounded by S_1 and S_3 and γ is the area bounded by S_2 and S_3, then
   (1) $\alpha = a^2\left(\pi - \frac{2}{3}\right)$
   (2) $\beta = \frac{4a^2}{3}$
   (3) $\gamma = 2a^2(\pi - 1)$
   (4) the ratio in which S_3 divides area between S_1 and S_2 is $4 : 3(\pi - 2)$

10. Let $A(k)$ be the area bounded by the curves $y = x^2 + 2x - 3$ and $y = kx + 1$. Then
   (1) the value of k for which $A(k)$ is least is 2
   (2) the value of k for which $A(k)$ is least is 3/2
   (3) least value of $A(k)$ is 32/3
   (4) least value of $A(k)$ is 64/3

Linked Comprehension Type

For Problems 1 and 2

Let A_r be the area of the region bounded between the curves $y^2 = (e^{-kr})x$ (where $k > 0, r \in N$) and the line $y = mx$ (where $m \neq 0$), k and m are some constants.

1. $A_1, A_2, A_3, \ldots$ are in G.P. with common ratio
   (1) e^{-k}
   (2) e^{-2k}
   (3) e^{-4k}
   (4) None of these

2. $\lim_{n\to\infty} \sum_{i=1}^{n} A_i = \frac{1}{48(e^{2k}-1)}$, then the value of m is
   (1) 3
   (2) 1
   (3) 2
   (4) 4

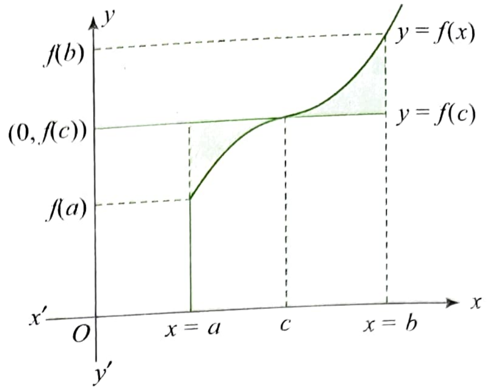
For Problems 3–5

If $y = f(x)$ is a monotonic function in (a, b), then the area bounded by the ordinates at $x = a$, $x = b$, $y = f(x)$ and $y = f(c)$ (where $c \in (a, b)$) is minimum when $c = \frac{a+b}{2}$.

Proof: $A = \int_a^c (f(c) - f(x))\,dx + \int_c^b (f(x) - f(c))\,dx$

$= f(c)\,(c - a) - \int_a^c (f(x))dx + \int_c^b (f(x))dx - f(c)\,(b - c)$

$\Rightarrow \quad A = [2c - (a + b)]f(c) + \int_c^b (f(x))\,dx - \int_a^c (f(x))dx$



Differentiating w.r.t. c, we get

$$\frac{dA}{dc} = [2c - (a+b)]f'(c) + 2f(c) + 0 - f(c) - (f(c) - 0)$$

For maxima and minima, $\frac{dA}{dc} = 0$

$\Rightarrow \quad f'(c)\,[2c - (a + b)] = 0$ (as $f'(c) \neq 0$)

Hence, $c = \frac{a+b}{2}$

Also for $c < \frac{a+b}{2}$, $\frac{dA}{dc} < 0$ and for $c > \frac{a+b}{2}$, $\frac{dA}{dc} > 0$

Hence, A is minimum when $c = \frac{a+b}{2}$.

3. If the area bounded by $f(x) = \frac{x^3}{3} - x^2 + a$ and the straight lines $x = 0$, $x = 2$, and the x-axis is minimum, then the value of a is
   (1) 1/3
   (2) 2
   (3) 1
   (4) 2/3

4. The value of the parameter a for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$, and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, which is the least, is
   (1) 2
   (2) 0
   (3) −1
   (4) 1

5. If the area enclosed by $f(x) = \sin x + \cos x$, $y = a$ between two consecutive points of extremum is minimum, then the value of a is
   (1) 0
   (2) −1
   (3) 1
   (4) 2

For Problems 6–8

Consider the areas S_0, S_1, S_2 … bounded by the x-axis and half-waves of the curve $y = e^{-x}\sin x$, where $x \geq 0$.

6. The value of S_0 is
   (1) $\frac{1}{2}\left(1+e^{\pi}\right)$ sq. units
   (2) $\frac{1}{2}\left(1+e^{-\pi}\right)$ sq. units
   (3) $\frac{1}{2}\left(1-e^{-\pi}\right)$ sq. units
   (4) $\frac{1}{2}\left(e^{\pi}-1\right)$ sq. units

7. The sequence S_0, S_1, S_2, ..., forms a G.P. with common ratio

(1) $\frac{e^{\pi}}{2}$ (2) $e^{-\pi}$ (3) e^{π} (4) $\frac{e^{-\pi}}{2}$

8. $\sum_{n=0}^{\infty} S_n$ is equal to

(1) $\frac{1+e^{\pi}}{1-e^{-\pi}}$ (2) $\frac{\frac{1}{2}(1+e^{\pi})}{1-e^{-\pi}}$

(3) $\frac{1}{2(1-e^{-\pi})}$ (4) None of these

For Problems 9–11

Two curves $C_1 \equiv [f(y)]^{2/3} + [f(x)]^{1/3} = 0$ and $C_2 \equiv [f(y)]^{2/3} + [f(x)]^{2/3} = 12$, satisfying the relation $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2-y^2)$.

9. The area bounded by C_1 and C_2 is

(1) $2\pi - \sqrt{3}$ sq. units (2) $2\pi + \sqrt{3}$ sq. units
(3) $\pi + \sqrt{6}$ sq. units (4) $2\sqrt{3} - \pi$ sq. units

10. The area bounded by the curve C_2 and $|x| + |y| = \sqrt{12}$ is

(1) $12\pi - 24$ sq. units (2) $6 - \sqrt{12}$ sq. units
(3) $2\sqrt{12} - 6$ sq. units (4) None of these

11. The area bounded by C_1 and $x + y + 2 = 0$ is

(1) 5/2 sq. units (2) 7/2 sq. units
(3) 9/2 sq. units (4) None of these

For Problems 12–13

Consider the two curves C_1: $y = 1 + \cos x$ and C_2 : $y = 1 + \cos(x - \alpha)$ for $\alpha \in \left(0, \frac{\pi}{2}\right)$, where $x \in [0, \pi]$. Also the area of the figure bounded by the curves C_1, C_2, and $x = 0$ is same as that of the figure bounded by C_2, $y = 1$, and $x = \pi$.

12. The value of α is

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{8}$

13. For the values of α, the area bounded by C_1, C_2, $x = 0$, and $x = \pi$ is

(1) 1 sq. units (2) 2 sq. units
(3) $2 + \sqrt{3}$ sq. units (4) None of these

For Problems 14–16

Consider the function defined implicitly by the equation $y^2 - 2ye^{\sin^{-1}x} + x^2 - 1 + [x] + e^{2\sin^{-1}x} = 0$ (where $[x]$ denotes the greatest integer function).

14. The area of the region bounded by the curve and the line $x = -1$ is

(1) $\pi + 1$ sq. units (2) $\pi - 1$ sq. units

(3) $\frac{\pi}{2} + 1$ sq. units (4) $\frac{\pi}{2} - 1$ sq. units

15. Line $x = 0$ divides the region mentioned above in two parts. The ratio of area of left-hand side of line to that of right-hand side of line is

(1) $1 + \pi : \pi$ (2) $2 - \pi : \pi$
(3) 1:1 (4) $\pi + 2 : \pi$

16. The area of the region of curve and line $x = 0$ and $x = \frac{1}{2}$ is

(1) $\frac{\sqrt{3}}{4} + \frac{\pi}{6}$ sq. units (2) $\frac{\sqrt{3}}{2} + \frac{\pi}{6}$ sq. units

(3) $\frac{\sqrt{3}}{4} - \frac{\pi}{6}$ sq. units (4) $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$ sq. units

For Problems 17–19

Computing area with parametrically represented boundaries: If the boundary of a figure is represented by parametric equation, i.e., $x = x(t)$, $y = y(t)$, then the area of the figure is evaluated by one of the three formulas:

$$S = -\int_{\alpha}^{\beta} y(t)x'(t)\,dt,$$

$$S = \int_{\alpha}^{\beta} x(t)y'(t)\,dt,$$

$$S = \frac{1}{2}\int_{\alpha}^{\beta} (xy' - yx')\,dt,$$

where α and β are the values of the parameter t corresponding respectively to the beginning and the end of the traversal of the curve corresponding to increasing t.

17. The area of the region bounded by an arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ and the x-axis is

(1) $6\pi a^2$ sq. units (2) $3\pi a^2$ sq. units
(3) $4\pi a^2$ sq. units (4) None of these

18. The area of the loop described as

$x = \frac{t}{3}(6 - t)$, $y = \frac{t^2}{8}(6 - t)$ is

(1) $\frac{27}{5}$ sq. units (2) $\frac{24}{5}$ sq. units

(3) $\frac{27}{6}$ sq. units (4) $\frac{21}{5}$ sq. units

19. If the curve given by parametric equation $x = t - t^3$, $y = 1 - t^4$ forms a loop for all values of $t \in [-1, 1]$, then the area of the loop is

(1) $\frac{1}{7}$ sq. units (2) $\frac{3}{5}$ sq. units

(3) $\frac{16}{35}$ sq. units (4) $\frac{8}{35}$ sq. units

For Problems 20 and 21

Let $f(x)$ be a continuous function given by

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}, \text{ then}$$

20. If $f(x)$ is continuous for all real x then the value of $a^2 + b^2$ is

(1) 3 (2) 4 (3) 5 (4) 11

21. The area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$ is

(1) sq. units (2) $\frac{257}{192}$ sq. units

(3) $\frac{257}{96}$ sq. units (4) $\frac{289}{192}$ sq. units

Matrix Match Type

1. Match the following lists:

List I	List II
a. The area bounded by the curve $y = x\,\lvert x\rvert$, x-axis and the ordinates $x = 1, x = -1$	**p.** 10/3 sq. units
b. The area of the region lying between the lines $x - y + 2 = 0$, $x = 0$, and the curve $x = \sqrt{y}$	**q.** 64/3 sq. units
c. The area enclosed between the curves $y^2 = x$ and $y = \lvert x\rvert$	**r.** 2/3 sq. units
d. The area bounded by parabola $y^2 = x$, straight line $y = 4$, and the y-axis	**s.** 1/6 sq. units

2. Match the following lists:

List I	List II
a. Area enclosed by $y = [x]$ and $y = \{x\}$, where $[\cdot]$ and $\{\cdot\}$ represent greatest integer and fractional part functions, respectively	**p.** 32/5 sq. units
b. The area bounded by the curves $y^2 = x^3$ and $\lvert y\rvert = 2x$.	**q.** 1 sq. units
c. The smaller area included between the curves $\sqrt{x} + \sqrt{\lvert y\rvert} = 1$ and $\lvert x\rvert + \lvert y\rvert = 1$.	**r.** 4 sq. units
d. Area bounded by the curves $y = \left[\frac{x^2}{64} + 2\right]$ (where [.] denotes the greatest integer function), $y = x - 1$ and $x = 0$ above the x-axis.	**s.** 2/3 sq. units

3. Match the following lists:

List I: [·] represents greatest integer	List II
a. Area enclosed by $[x]^2 = [y]^2$ for $1 \le x \le 4$	**p.** 8 sq. units
b. Area enclosed by $[\lvert x\rvert] + [\lvert y\rvert] = 2$	**q.** 6 sq. units
c. Area enclosed by $[\lvert x\rvert]\,[\lvert y\rvert] = 2$	**r.** 4 sq. units
d. Area enclosed by $\frac{[\lvert x\rvert]}{[\lvert y\rvert]} = 2, -5 \le x \le 5$	**s.** 12 sq. units

4. Match the following lists:

List I	List II
a. Area bounded by curve $y^2 = x$ and $x = 4$ is divided into 4 equal parts by the lines $x = a$ and $y = b$. Then the value of $[a^2]$ is , where [.] represents greatest integer part function	**p.** 1
b. The area bounded by the curves $f(x) = \begin{cases} x^{\frac{1}{\log_e x}}, & x \ne 1 \\ e, & x = 1 \end{cases}$ and $y = \lvert x - e\rvert$ is A then the value of $[A]$ is, where [.] represents greatest integer part function	**q.** 6
c. Area bounded by the curves $y = e^x$, $y = \log_e x$ and the lines $x = 0, y = 0, y = 1$ is A, then the value of $[A]$ is, where [.] represents greatest integer part function	**r.** 8
d. The area of the region whose boundaries are defined by the curves $y = 2\cos x$, $y = 3\tan x$ and the y-axis, is $1 + \frac{3}{2}\log_e 3 - \log_e k$, then the value of k is	**s.** 7

Codes:

	a	b	c	d
(1)	q	p	s	r
(2)	s	p	q	r
(3)	q	s	p	r
(4)	q	p	r	s

Numerical Value Type

1. The area enclosed by the curve $C: y = x\sqrt{9 - x^2}$ $(x \ge 0)$ and the x-axis is_____.

2. Let S be the area bounded by the curve $y = \sin x$ $(0 \le x \le \pi)$ and the x-axis and T be the area bounded by the curves $y = \sin x\left(0 \le x \le \frac{\pi}{2}\right)$, $y = a\cos x\left(0 \le x \le \frac{\pi}{2}\right)$, and the x-axis (where $a \in R^+$). The value of $(3a)$ such that $S : T = 1 : \frac{1}{3}$ is_____.

3. Let C be a curve passing through $M(2, 2)$ such that the slope of the tangent at any point to the curve is reciprocal of the ordinate of the point. If the area bounded by curve C and line $x = 2$ is A, then the value of $\frac{3A}{2}$ is_____.

4. The area enclosed by $f(x) = 12 + ax + -x^2$ coordinates axes and the ordinates at $x = 3$ $(f(3) > 0)$ is 45 sq. units. If m and n are the x-axis intercepts of the graph of $y = f(x)$, then the value of $(m + n + a)$ is_____.

5. If the area bounded by the curve $f(x) = x^{1/3}(x - 1)$ and the x-axis is A, then the value of $28A$ is_____.

6. If the area bounded by the curve $y = x^2 + 1$ and the tangents to it drawn from the origin is A, then the value of $3A$ is_____.

7. If the area enclosed by the curve $y = \sqrt{x}$ and $x = -\sqrt{y}$, the circle $x^2 + y^2 = 2$ above the x-axis is A, then the value of $\frac{16}{\pi}A$ is____.

8. The value of a ($a > 0$) for which the area bounded by the curves $y = \frac{x}{6} + \frac{1}{x^2}$, $y = 0$, $x = a$, and $x = 2a$ has the least value is____.

9. Area bounded by the relation $[2x] + [y] = 5$, $x, y > 0$ is____ (where [·] represents greatest integer function).

10. The area bounded by the curves $y = x(x - 3)^2$ and $y = x$ is____ (in sq. units).

11. If the area of the region $\{(x, y): 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$ is A, then the value of $3A - 17$ is____.

12. If S is the sum of possible values of c for which the area of the figure bounded by the curves $y = \sin 2x$, the straight lines $x = \pi/6$, $x = c$, and the abscissa axis is equal to 1/2, then the value of π/S is____.

13. If A is the area bounded by the curves $y = \sqrt{1-x^2}$ and $y = x^3 - x$, then the value of π/A is ____.

14. Consider two curves C_1: $y = \frac{1}{x}$ and C_2: $y = \ln x$ on the xy plane. Let D_1 denotes the region surrounded by C_1, C_2, and the line $x = 1$ and D_2 denotes the region surrounded by C_1, C_2, and the line $x = a$. If $D_1 = D_2$, then the sum of logarithm of possible values of a is____.

15. If 'a' ($a > 0$) is the value of parameter for each of which the area of the figure bounded by the straight line $y = \frac{a^2 - ax}{1 + a^4}$ and the parabola $y = \frac{x^2 + 2ax + 3a^2}{1 + a^4}$ is the greatest, then the value of a^4 is____.

16. If S is the sum of cubes of possible value of c for which the area of the figure bounded by the curve $y = 8x^2 - x^5$, then straight lines $x = 1$ and $x = c$ and the abscissa axis is equal to 16/3, then the value of [S], where [·] denotes the greatest integer function, is____.

Archives

JEE Main

Single Correct Answer Type

1. The area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to the parabola at the point (2, 3), and the x-axis is

(1) 3 (2) 6 (3) 9 (4) 12

(AIEEE 2009)

2. The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = 3\pi/2$ is

(1) $4\sqrt{2} + 1$ (2) $4\sqrt{2} - 1$

(3) $4\sqrt{2} + 2$ (4) $4\sqrt{2} - 2$

(AIEEE 2010)

3. The area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive x-axis is

(1) $\frac{5}{2}$ square units (2) $\frac{1}{2}$ square units

(3) 1 square units (4) $\frac{3}{2}$ square units

(AIEEE 2011)

4. The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line $y = 2$ is

(1) $20\sqrt{2}$ (2) $\frac{10\sqrt{2}}{3}$ (3) $\frac{20\sqrt{2}}{3}$ (4) $10\sqrt{2}$

(AIEEE 2012)

5. The area (in square units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis, and lying in the first quadrant is

(1) 9 (2) 36 (3) 18 (4) $\frac{27}{4}$

(JEE Main 2013)

6. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1$ and $y^2 \le 1 - x\}$ is

(1) $\frac{\pi}{2} + \frac{4}{3}$ (2) $\frac{\pi}{2} - \frac{4}{3}$

(3) $\frac{\pi}{2} - \frac{2}{3}$ (4) $\frac{\pi}{2} + \frac{2}{3}$

(JEE Main 2014)

7. The area (in sq. units) of the region described by $\{(x, y); y^2 \le 2x$ and $y \ge 4x - 1\}$ is

(1) $\frac{7}{32}$ (2) $\frac{5}{64}$ (3) $\frac{15}{64}$ (4) $\frac{9}{32}$

(JEE Main 2015)

8. The area (in sq. units) of the region $\{(x, y) : y^2 \ge 2x$ and $x^2 + y^2 \le 4x, x \ge 0, y \ge 0\}$ is

(1) $\pi - \frac{8}{3}$ (2) $\pi - \frac{4\sqrt{2}}{3}$

(3) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$ (4) $\pi - \frac{4}{3}$

(JEE Main 2016)

9. The area (in sq. units) of the region $\{(x, y): x \geq 0, x + y \leq 3, x^2 \leq 4y, y \geq 1 + \sqrt{x}\}$ is

(1) $\frac{5}{2}$ (2) $\frac{59}{12}$ (3) $\frac{3}{2}$ (4) $\frac{7}{3}$

(JEE Main 2017)

10. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$, and α, β $(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq. units) bounded by the curve $y = (gof)(x)$ and the lines $x = \alpha$, $x = \beta$ and $y = 0$, is

(1) $\frac{1}{2}(\sqrt{2} - 1)$ (2) $\frac{1}{2}(\sqrt{3} - 1)$

(3) $\frac{1}{2}(\sqrt{3} + 1)$ (4) $\frac{1}{2}(\sqrt{3} - \sqrt{2})$

(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. Let the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$, and $x = 0$ into two parts R_1 $(0 \leq x \leq b)$ and R_2 $(b \leq x \leq 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals

(1) 3/4 (2) 1/2 (3) 1/3 (4) 1/4

(IIT-JEE 2011)

2. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $[0, \pi/2]$ is

(1) $4(\sqrt{2} - 1)$ (2) $2\sqrt{2}(\sqrt{2} - 1)$

(3) $2(\sqrt{2} + 1)$ (4) $2\sqrt{2}(\sqrt{2} + 1)$

(JEE Advanced 2013)

3. Area of the region

$\left\{(x, y) \in R^2 : y \geq \sqrt{|x + 3|}, 5y \leq x + 9 \leq 15\right\}$ is equal to

(1) $\frac{1}{6}$ (2) $\frac{4}{3}$ (3) $\frac{3}{2}$ (4) $\frac{5}{3}$

(JEE Advanced 2016)

Multiple Correct Answers Type

1. The area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is

(1) $e - 1$ (2) $\int_1^e \ln(e + 1 - y)\,dy$

(3) $e - \int_0^1 e^x\,dx$ (4) $\int_1^e \ln y\,dy$

(IIT-JEE 2009)

2. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in R^2 : x^3 \leq y \leq x, 0 \leq x \leq 1\}$ into two equal parts, then

(1) $\frac{1}{2} < \alpha < 1$ (2) $\alpha^4 + 4\alpha^2 - 1 = 0$

(3) $0 < \alpha \leq \frac{1}{2}$ (4) $2\alpha^4 - 4\alpha^2 + 1 = 0$

(JEE Advanced 2017)

3. Let $f : [0, \infty) \to R$ be a continuous function such that

$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)\,dt$ for all $x \in [0, \infty)$. Then, which of the following statement(s) is (are)) **TRUE**?

(1) The curve $y = f(x)$ passes through the point (1, 2)

(2) The curve $y = f(x)$ passes through the point (2, –1)

(3) The area of the region

$\{(x, y) \in [0, 1] \times R : f(x) \leq y \leq \sqrt{1 - x^2}\}$ is $\frac{\pi - 2}{4}$

(4) The area of the region

$\{(x, y) \in [0,1] \times R : f(x) \leq y \leq \sqrt{1 - x^2}\}$ is $\frac{\pi - 1}{4}$

(JEE Advanced 2018)

Matrix Match Type

1. Match the statements given in List I with the values given in List II.

List I	List II
a. In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X - Y)}{\sin Z}$ then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are)	**p.** 1
b. In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	**q.** 2
c. In R^2, let $\sqrt{3}\hat{i} + \hat{j}$, $\hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1 - \beta)\hat{j}$ be the position vectors of X, Y and Z with respect of the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overline{OX}$ and $\overline{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $\lvert\beta\rvert$ is (are)	**r.** 3
d. Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = \lvert\alpha x - 1\rvert + \lvert\alpha x - 2\rvert + \alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are)	**s.** 5
	t. 6

(JEE Advanced 2015)

Numerical Value Type

1. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$, is

(JEE Advanced 2014)

2. A farmer F_1 has a land in the shape of a triangle with vertices at $P(0, 0)$, $Q(1, 1)$ and $R(2, 0)$. From this land, a neighboring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n$ $(n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is ________.

(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	2. (4)	3. (3)	4. (4)	5. (2)
6. (1)	7. (3)	8. (3)	9. (1)	10. (1)
11. (1)	12. (4)	13. (3)	14. (1)	15. (1)
16. (4)	17. (1)	18. (2)	19. (3)	20. (1)
21. (1)	22. (2)	23. (3)	24. (1)	25. (2)
26. (1)	27. (4)	28. (4)	29. (2)	30. (2)
31. (4)	32. (4)	33. (1)	34. (3)	35. (4)
36. (3)	37. (2)	38. (2)	39. (1)	40. (3)

Multiple Correct Answers Type

1. (2), (3)	2. (1), (3), (4)
3. (1), (3), (4)	4. (3), (4)
5. (1), (3), (4)	6. (1), (3)
7. (1), (2), (3), (4)	8. (1), (4)
9. (1), (2), (4)	10. (1), (3)

Linked Comprehension Type

1. (2)	2. (3)	3. (4)	4. (3)	5. (1)
6. (1)	7. (3)	8. (2)	9. (2)	10. (1)
11. (3)	12. (3)	13. (2)	14. (1)	15. (4)
16. (1)	17. (2)	18. (1)	19. (3)	20. (3)
21. (2)				

Matrix Match Type

1. a. $\rightarrow$ r; b. $\rightarrow$ p; c. $\rightarrow$ s; d. $\rightarrow$ q
2. a. $\rightarrow$ q; b. $\rightarrow$ p; c. $\rightarrow$ s; d. $\rightarrow$ r
3. a. $\rightarrow$ q; b. $\rightarrow$ s; c. $\rightarrow$ p; d. $\rightarrow$ p
4. (3)

Numerical Value Type

1. (9)	2. (4)	3. (8)	4. (8)	5. (9)
6. (2)	7. (8)	8. (1)	9. (3)	10. (8)
11. (6)	12. (6)	13. (2)	14. (1)	15. (3)
16. (2)				

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (3)	2. (4)	3. (4)	4. (3)	5. (1)
6. (4)	7. (4)	8. (1)	9. (1)	10. (2)

JEE ADVANCED

Single Correct Answer Type

1. (2) 2. (2) 3. (3)

Multiple Correct Answers Type

1. (2), (3), (4) 2. (1), (4)
3. (2), (3)

Matrix Match Type

1. d $\rightarrow$ s, t

Numerical Value Type

1. (6) 2. (4)

10 Differential Equations

DIFFERENTIAL EQUATION

An equation that involves independent and dependent variables and the derivatives of the dependent variable w.r.t. independent variable is called a Differential Equations.

e.g., $y\frac{dy}{dx} = x^2 \log x$, $dy = \sin x\, dx$, $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$ are differential equations.

A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation. All above differential equations are ordinary differential equations.

There are differential equations involving derivatives with respect to more than one independent variables. Such equations are called partial differential equations. But in this chapter we shall confine ourselves to the study of ordinary differential equations only.

ORDER AND DEGREE OF DIFFERENTIAL EQUATION

Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation. Consider the following differential equations and their orders.

Differential Equation	Order
$\frac{dy}{dx} = \sin x$	1
$\frac{d^2y}{dx^2} + x = 0$	2
$\frac{d^3y}{dx^3} + 3\left(\frac{d^2y}{dx^2}\right)^3 - xy^2 = 0$	3

The key point in studying the differential equation is that it must be a polynomial equation in derivatives, i.e., y', y'', y''' etc. Consider the following differential equations:

$$\frac{d^3y}{dx^3} + 3\left(\frac{d^2y}{dx^2}\right)^3 - \frac{dy}{dx} + xy^2 = 0 \quad (1)$$

$$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - x\sin y = 0 \quad (2)$$

$$\frac{d^2y}{dx^2} + \cos\frac{dy}{dx} = 0 \quad (3)$$

We observe that equation (1) is a polynomial equation in y''', y'' and y'. Equation (2) is a polynomial in y'. So, in these equations, degree of differential equation can be defined. But equation (3) is not a polynomial in y'. So, degree of this equation cannot be defined.

Now consider differential equation, $\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{2}} = \left(y + \left(\frac{dy}{dx}\right)^4\right)^{\frac{1}{3}}$.

Here derivatives have no integral powers. We can rewrite the equation as $\left(\frac{d^2y}{dx^2}\right)^3 = \left(y + \left(\frac{dy}{dx}\right)^4\right)^2$. Now this is in polynomial of derivatives. So, its degree is defined.

The degree of a differential equation, when it is a polynomial equation in derivatives, is the highest power (positive integral index) of the highest order derivative involved in the given differential equation.

So, degree of equation (1) is 1 and that of equation (2) is 2.

ILLUSTRATION 10.1

Find the order and degree of the following differential equations.

(i) $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}}$ **(ii)** $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$

(iii) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$ **(iv)** $\log_e\left(\frac{dy}{dx}\right) = ax + by$

(v) $y\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - 4y\frac{dy}{dx} = 0$

Sol.

(i) $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}}$

$\therefore \left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right) = 1$

Hence order is 1 and degree is 2.

(ii) $e^{\frac{d^3y}{dx^3}} - x\dfrac{d^2y}{dx^2} + y = 0$

Clearly order is 3, but degree is not defined as it cannot be expressed as a polynomial equation in derivatives.

(iii) $\sin^{-1}\left(\dfrac{dy}{dx}\right) = x + y$

$\therefore \quad \dfrac{dy}{dx} = \sin(x + y)$

Hence order is 1 and degree is 1.

(iv) $\log_e\left(\dfrac{dy}{dx}\right) = ax + by$

$\therefore \quad \dfrac{dy}{dx} = e^{ax+by}$

Hence order is one and degree is also 1.

(v) $y\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - 4y\dfrac{dy}{dx} = 0$

Clearly, order is 2 and degree is 1.

CONCEPT APPLICATION EXERCISE 10.1

Find the order and degree (if defined) of the following differential equations:

1. $\dfrac{d^2y}{dx^2} = \left\{1 + \left(\dfrac{dy}{dx}\right)^4\right\}^{5/3}$
2. $\dfrac{d^3y}{dx^3} = x\ln\left(\dfrac{dy}{dx}\right)$
3. $\left(\dfrac{d^4y}{dx^4}\right)^3 + 3\left(\dfrac{d^2y}{dx^2}\right)^6 + \sin x = 2\cos x$
4. $\left(\dfrac{d^3y}{dx^3}\right)^{2/3} + 4 - 3\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} = 0$
5. $a = \dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$, where a is constant
6. $\dfrac{d^4y}{dx^4} - \sin\left(\dfrac{d^3y}{dx^3}\right) = 0$

ANSWERS

1. Order-2, degree-3
2. Order-3, degree not defined
3. Order-4, degree-3
4. Order-3, degree-2
5. Order-2, degree-2
6. Order-4, degree not defined

FORMATION OF DIFFERENTIAL EQUATION

We know that the equation $y^2 = 4x$ (1)

represents a parabola having vertex at (0, 0).

Differentiating equation (1) with respect to x, we get

$$2y\frac{dy}{dx} = 4 \text{ or } y\frac{dy}{dx} = 2 \quad \ldots(2)$$

which is differential equation.

Rewriting the equation (2) as $ydy = 2dx$ and then integrating both sides, we get

$$\int ydy = \int 2dx$$

or $\dfrac{y^2}{2} = 2x + c$ or $y^2 = 4x + k$ (where $k = 2c$)

This is called **general solution** of the differential equation (2), which represents the family of parabolas and one member of the family is the parabola given in equation (1). Also, equation (1) is called **particular solution** of the differential equation (2).

Consider the equation

$$x^2 + y^2 = a^2 \quad (3)$$

By giving different values to a, we get different members of the family e.g., $x^2 + y^2 = 1$, $x^2 + y^2 = 9$, $x^2 + y^2 = 20$ etc.

Thus, equation (3) represents a family of concentric circles having centre at origin and having different radii. We are interested in finding a differential equation that is satisfied by each member of the family. The differential equation must be free from 'a' because 'a' is different for different members of the family. This equation is obtained by differentiating equation (3) with respect to x, i.e.,

$$2x + 2y\frac{dy}{dx} = 0 \text{ or } x + y\frac{dy}{dx} = 0$$

The aim of differentiating the equation is to get another equation to eliminate arbitrary constant a. Equation (3) has one arbitrary constant. So, we need to get one more equation.

Now consider equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \ldots(4)$

This equation represents family of ellipses having centre at origin and axis along coordinate axis. Giving different values to 'a' and 'b' we get different ellipse on the plane. So, here 'a' and 'b' are two arbitrary constants. To find the differential equation which will be satisfied by each member of the family, we need two more equations. We get these equations by differentiating equation (4) twice successively.

Differentiating (4), w.r.t. x, we get

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \text{ or } \frac{x}{a^2} = -\frac{yy'}{b^2} \quad \ldots(5)$$

Differentiating above equation w.r.t. x, we get

$$\frac{1}{a^2} = -\frac{yy'' + (y')^2}{b^2} \quad \ldots(6)$$

Eliminating 'a' and 'b' from (5) and (6), we have

$$yy' = xyy'' + x(y')^2$$

This is required differential equation of equation (4).

From the above two examples it is clear that order of differential equation of the given family of curves is equal to number of arbitrary constants in the equation of the family of the curve.

ILLUSTRATION 10.2

Form the differential equation of family of lines concurrent at the origin.

Sol. Equation of family of line concurrent at origin is given by

$$y = mx \quad \text{...(1)}$$

$$\therefore \quad \frac{dy}{dx} = m$$

Putting the value of m in (1), we get

$$y = \frac{dy}{dx} \cdot x$$

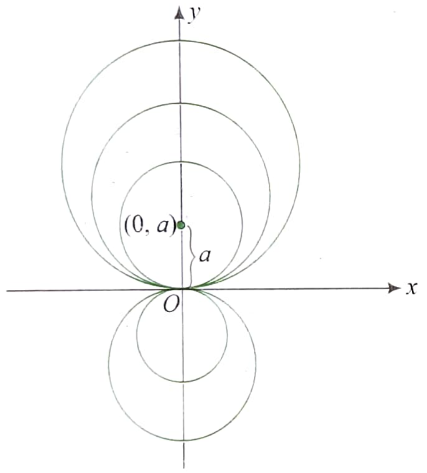
or $xdy - ydx = 0$, which is required differential equation of order 1.

Here order is same as number of effective constants in the equation of family of curve.

ILLUSTRATION 10.3

Form the differential equation of all circles touching the x-axis at the origin and centre on y-axis.

Sol.



If centre of the circle is $(0, a)$ then its radius is $|a|$.

$\therefore$ equation of such family of circles is given by

$$x^2 + (y - a)^2 = a^2$$

or $$x^2 + y^2 - 2ay = 0 \quad (1)$$

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx} \quad (2)$$

From (1), $2a = \dfrac{x^2 + y^2}{y}$

Putting the value of $2a$ in (2), we get

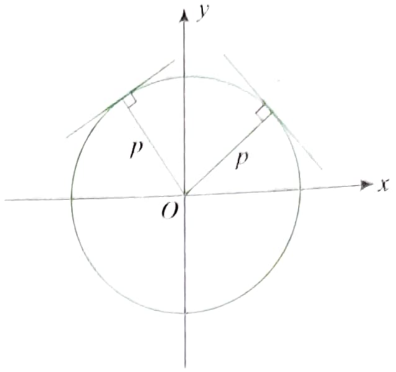
$$2x + 2y\frac{dy}{dx} = \frac{x^2 + y^2}{y}\frac{dy}{dx}$$

or $(x^2 - y^2)\dfrac{dy}{dx} = 2xy$, which is required differential equation

ILLUSTRATION 10.4

From the differential equation of family of lines situated at a constant distance p from the origin.

Sol. All such lines are tangent to circle of radius p.



Equation of tangent to circle $x^2 + y^2 = p^2$, having slope m is

$$y = mx + p\sqrt{1 + m^2}, \text{ where } m = \frac{dy}{dx}$$

$$\therefore \quad y = \frac{dy}{dx}x + p\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

or $$\left(y - x\frac{dy}{dx}\right)^2 = p^2\left(1 + \left(\frac{dy}{dx}\right)^2\right),$$

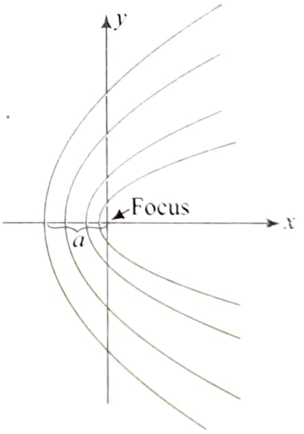
which is required differential equation.

Here, order is 1 and degree is 2.

ILLUSTRATION 10.5

Form the differential equation of the family of parabolas with focus at the origin and axis of symmetry along the x-axis. Find the order and degree of the differential equation.

Sol.



From the figure, equation of family of parabolas is

$$y^2 = 4a(a + x) \quad (1)$$

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a \text{ or } y\frac{dy}{dx} = 2a \quad (2)$$

Eliminating a from (1) and (2), we get

$$y^2 = \left(y\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx}x$$

or $$y^2 = y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

Which has order 1 and degree 2

ILLUSTRATION 10.6

Find the differential equation of all parabolas whose axis is parallel to the y-axis.

Sol. Parabola having axis parallel to y-axis, represent the quadratic function,

$$y = ax^2 + bx + c \qquad (1)$$

Here, we have three effective constants; a, b and c.

Differentiating (1), successively three times, we get

$$y' = 2ax + b$$

$$y'' = 2a$$

and $y''' = 0$, which is required differential equation.

Order of differential equation is three.

ILLUSTRATION 10.7

From the differential equation of the family curves having equation $y = (\sin^{-1}x)^2 + A\cos^{-1}x + B$.

Sol. $y = (\sin^{-1}x)^2 + A\cos^{-1}x + B$

$$= (\sin^{-1}x)^2 - A\sin^{-1}x + \frac{\pi A}{2} + B$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} - \frac{A}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\sin^{-1}x)^2 - 4A\sin^{-1}x + A^2$$

$$= 4y - 2\pi A - 4B + A^2$$

Differentiating again w.r.t. x, we get

$$2(1-x^2)\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) - 2x\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2,$$ which is required differential equation.

ILLUSTRATION 10.8

What is the order of the differential equation whose general solution is $y = c_1\cos 2x + c_2\sin^2 x + c_3\cos^2 x + c_4e^{2x} + c_5e^{2x+c_6}$?

Sol. $y = c_1\cos 2x + c_2\sin^2 x + c_3\cos^2 x + c_4e^{2x} + c_5e^{2x+c_6}$

$$= c_1\cos 2x + c_2\left(\frac{1-\cos 2x}{2}\right) + c_3\left(\frac{\cos 2x + 1}{2}\right) + c_4e^{2x} + c_5e^{c_6}e^{2x}$$

$$= \left(c_1 - \frac{c_2}{2} + \frac{c_3}{2}\right)\cos 2x + \left(\frac{c_2}{2} + \frac{c_3}{2}\right) + (c_4 + c_5e^{c_6})e^{2x}$$

$$= \lambda_1\cos 2x + \lambda_2 e^{2x} + \lambda_3$$

So, number of arbitrary constants in the equation is 3.

Therefore, order of the differential equation will be 3.

CONCEPT APPLICATION EXERCISE 10.2

1. Find the differential equation of all non-vertical lines in a plane.
2. Form the differential equation of family of circles having centre at origin.
3. Find the differential equation of all parabolas whose axes are parallel to the x-axis and having latus rectum a.
4. Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants.
5. Find the differential equation satisfying the relation $\sqrt{1+x^2} + \sqrt{1+y^2} = \lambda\left(x\sqrt{1+y^2} - y\sqrt{1+x^2}\right)$.
6. Find the differential equation whose general solution is given by $y = (c_1 + c_2)\cos(x + c_3) - c_4e^{x+c_5}$ where c_1, c_2, c_3, c_4, c_5 are arbitrary constants.

ANSWERS

1. $\frac{d^2y}{dx^2} = 0$ 2. $x + y\frac{dy}{dx} = 0$ 3. $a\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0$

4. $\frac{d^2y}{dx^2} = 4y$ 5. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

6. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

SOLVING DIFFERENTIAL EQUATION OF VARIABLE SEPARABLE TYPE

In previous section we learned how to form a differential equation of the given family of curves by eliminating effective constants. Now let us see how to find the equation of family of curve for the given differential equation. This process is called solving differential equation. There are different methods to solve the differential equation depending upon its type. One of the simplest forms of differential equation is 'variable separable type'.

The differential equation has the form

$$\frac{dy}{dx} = f(x)g(y),$$

where $f(x)$ is function of x and $g(y)$ is function of y.

We can rewrite the equation as

$$\frac{dy}{g(y)} = f(x)dx \qquad \text{(where } g(y) \neq 0)$$

This process is called separating the variables.

Now integrating both sides, we get

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

If $G'(y) = \frac{1}{g(y)}$ and $F'(x) = f(x)$, then we have $G(y) = F(x) + C$, which is required solution or equation of the family of the curves.

ILLUSTRATION 10.9

Find the particular solution of the differential equation $(1 + e^{2x})\,dy + (1 + y^2)\,e^x\,dx = 0$, given that $y = 1$ when $x = 0$.

$(1+e^{2x})\,dy + (1+y^2)\,e^x\,dx = 0$

Sol. $(1+e^{2x})\,dy + (1+y^2)\,e^x\,dx = 0$

or $\dfrac{dy}{1+y^2} + \dfrac{e^x dx}{1+e^{2x}} = 0$

Integrating both sides, we get

$$\int \frac{dy}{1+y^2} + \int \frac{e^x dx}{1+e^{2x}} = C$$

or $\tan^{-1} y + \tan^{-1}(e^x) = C$ (1)

Now, $y = 1$ at $x = 0$

$\therefore$ $\tan^{-1} 1 + \tan^{-1} 1 = C$

or $C = \dfrac{\pi}{2}$

Substituting $C = \dfrac{\pi}{2}$ in equation (1), we get

$$\tan^{-1} y + \tan^{-1}(e^x) = \frac{\pi}{2}$$

This is the required particular solution of the given differential equation.

ILLUSTRATION 10.10

Solve: $\log_e \dfrac{dy}{dx} = 4x - 2y - 2$, given that $y = 1$ when $x = 1$.

Sol. $\log_e \dfrac{dy}{dx} = 4x - 2y - 2$

$\Rightarrow$ $\dfrac{dy}{dx} = e^{4x-2y-2}$

$\Rightarrow$ $e^{2y+2} dy = e^{4x} dx$

Integrating both sides,

$$\int e^{2y+2} dy = \int e^{4x} dx$$

$\Rightarrow$ $\dfrac{e^{2y+2}}{2} = \dfrac{e^{4x}}{4} + c$

Putting $x = 1$ and $y = 1$, we get

$$\frac{e^4}{2} = \frac{e^4}{4} + c \text{ or } c = \frac{e^4}{4}$$

So, particular solution is $\dfrac{e^{2y+2}}{2} = \dfrac{e^{4x} + e^4}{4}$

or $2(e^{2y+2}) = e^{4x} + e^4$

ILLUSTRATION 10.11

Solve the differential equation $xy\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}(1+x+x^2)$

Sol. We have $\dfrac{y\,dy}{1+y^2} = \dfrac{1+x+x^2}{x(1+x^2)}dx$

Integrating both sides,

$$\int \frac{y\,dy}{1+y^2} = \int \left(\frac{1}{x} + \frac{1}{1+x^2}\right) dx$$

$\therefore$ $\dfrac{1}{2}\log_e(1+y^2) = \log_e x + \tan^{-1} x + \log_e c$

$\Rightarrow$ $\log_e \dfrac{\sqrt{1+y^2}}{cx} = \tan^{-1} x$

$\Rightarrow$ $\sqrt{1+y^2} = cxe^{\tan^{-1} x}$

ILLUSTRATION 10.12

Solve: $\sqrt{1+x^2+y^2+x^2y^2} + xy\dfrac{dy}{dx} = 0$

Sol. We have $\sqrt{(1+x^2)(1+y^2)} = -xy\dfrac{dy}{dx}$

$\therefore$ $\dfrac{\sqrt{1+x^2}}{x}dx = -\dfrac{y}{\sqrt{1+y^2}}dy$

Integrating both sides,

$$\int \frac{1+x^2}{x\sqrt{1+x^2}}dx = -\int \frac{y}{\sqrt{1+y^2}}dy$$

$\Rightarrow$ $\underbrace{\int \dfrac{x}{x^2\sqrt{1+x^2}}dx}_{\text{Putting } 1+x^2=z^2} + \int \dfrac{x}{\sqrt{1+x^2}}dx = -\int \dfrac{y}{\sqrt{1+y^2}}dy$

$\Rightarrow$ $\int \dfrac{dz}{z^2-1} + \sqrt{1+x^2} = -\sqrt{1+y^2} + C$

$\Rightarrow$ $\dfrac{1}{2}\log_e\left|\dfrac{z-1}{z+1}\right| + \sqrt{1+x^2} = -\sqrt{1+y^2} + C$

$\Rightarrow$ $\dfrac{1}{2}\log_e\left|\dfrac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + \sqrt{1+x^2} = -\sqrt{1+y^2} + C$

ILLUSTRATION 10.13

Solve $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

Sol. $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

or $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \left(\dfrac{dy}{dx}\right)^2$

or $d\left(\dfrac{dy}{dx}\right) = \left(\dfrac{dy}{dx}\right)^2 dx$

or $\dfrac{d\left(\dfrac{dy}{dx}\right)}{\left(\dfrac{dy}{dx}\right)} = \left(\dfrac{dy}{dx}\right)dx$

or $\int \dfrac{d\left(\dfrac{dy}{dx}\right)}{\dfrac{dy}{dx}} = \int dy$

$\therefore$ $\log\left(\dfrac{dy}{dx}\right) = \log y + \log C_1$

$\therefore$ $\dfrac{dy}{dx} = C_1 y$

$\therefore \quad \dfrac{dy}{y} = C_1\, dx$

$\therefore \quad \displaystyle\int \frac{dy}{y} = C_1 \int dx$

$\therefore \quad \log y = C_1 x + k$

$\therefore \quad y = e^{C_1 x + k} = C_2 e^{C_1 x}$

Differential Equations Reducible to the Variable Separable Type

Sometimes differential equation cannot be put in the form $\dfrac{dy}{dx} = f(x)g(y)$, i.e., variable separation is not possible.

But using proper substitution, such equation can be reduced to variable separable type.

Substitution depends upon the terms (expressions) of the differential equation.

Common differential equation which can be reduced to variable separable type is of the form $\dfrac{dy}{dx} = f(ax + by + c)$.

To solve this equation, we substitute $ax + by + c = t$

Illustration 10.14

Solve : $\dfrac{dy}{dx} = (x + y)^2$

Sol. $\dfrac{dy}{dx} = (x + y)^2 \qquad \ldots(1)$

Here direct variables separation is not possible.

So, we choose substitution method.

Let $x + y = v$.

$\therefore \quad 1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$

So, equation (1) reduces to

$$\frac{dv}{dx} = v^2 + 1$$

or $\quad \displaystyle\int \frac{dv}{v^2 + 1} = \int dx$

$\therefore \quad \tan^{-1} v = x + c$

$\Rightarrow \quad x + y = \tan (x + c)$, which is required solution.

Illustration 10.15

Solve $\dfrac{dy}{dx}\sqrt{1 + x + y} = x + y - 1$.

Sol. Putting $\sqrt{1 + x + y} = v$, we have

$x + y - 1 = v^2 - 2$

or $\quad 1 + \dfrac{dy}{dx} = 2v\dfrac{dv}{dx}$

Then the given equation transforms to

$$\left(2v\frac{dv}{dx} - 1\right)v = v^2 - 2$$

or $\quad \dfrac{dv}{dx} = \dfrac{v^2 + v - 2}{2v^2}$

or $\quad \displaystyle\int \frac{2v^2}{v^2 + v - 2}\, dv = \int dx$

or $\quad 2\displaystyle\int \left[1 + \frac{1}{3(v - 1)} - \frac{4}{3(v + 2)}\right] dv = \int dx$

or $\quad 2\left[v + \dfrac{1}{3}\log|v - 1| - \dfrac{4}{3}\log|v + 2|\right] = x + c$

where $v = \sqrt{1 + x + y}$

Concept Application Exercise 10.3

1. Solve $e^{\frac{dy}{dx}} = x + 1$, given that when $x = 0$, $y = 3$.
2. Solve $(x - y^2x)dx = (y - x^2y)dy$.
3. Solve $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$.
4. Solve $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$.
5. Solve $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$.
6. Solve $(x - y)(dx + dy) = dx - dy$, given that $y = -1$, where $x = 0$.
7. Solve $\dfrac{dy}{dx} + yf'(x) = f(x)\, f'(x)$, where $f(x)$ is a given integrable function of x.
8. Solve $\dfrac{dy}{dx} = \cos(x + y) - \sin(x + y)$.
9. If a function 'f' satisfies the relation $f(x)f''(x) - f(x)f'(x) - (f'(x))^2 = 0\ \forall x \in R$ and $f(0) = 1 = f'(0)$. Then find $f(x)$.

Answers

1. $y = (x + 1)(\log_e(x + 1)) - x + 3$
2. $(x^2 - 1) = C(y^2 - 1)$
3. $(\tan x)(\tan y) = c$
4. $\tan y = C(1 - e^x)$
5. $\dfrac{y}{1 - ay} = c(a + y)$
6. $\log_e|x - y| = x + y + 1$
7. $\log_e(1 + y - f(x)) + f(x) + c = 0$
8. $-\log_e\left|1 - \tan\dfrac{x + y}{2}\right| = x + c$
9. $f(x) = e^{e^x - 1}$

Homogeneous Differential Equation

Consider the following functions in x and y

$F_1(x, y) = y^2 - 3xy,$

$F_2(x, y) = x - 2y,$

$F_3(x, y) = \tan\dfrac{y}{x},$

$F_4(x, y) = \cos x + \sin y$

If we replace x and y by λx and λy respectively in the above functions, for any nonzero constant λ, we get

$F_1(\lambda x, \lambda y) = \lambda^2(y^2 - 3xy) = \lambda^2 F_1(x, y)$

$F_2(\lambda x, \lambda y) = \lambda(x - 2y) = \lambda F_2(x, y)$

$F_3(\lambda x, \lambda y) = \tan\dfrac{\lambda y}{\lambda x} = \tan\dfrac{y}{x} = \lambda^0 F_3(x, y)$

$F_4(\lambda x, \lambda y) = \cos \lambda x + \sin \lambda y \neq \lambda^n F_4(x, y)$ for any $n \in N$

We observe that it is possible to write functions F_1, F_2, F_3 in the form $F(\lambda x, \lambda y) = \lambda^n F(x, y)$.

But F_4 cannot be written in this form. From this we have following definition.

A function $F(x, y)$ is said to be homogeneous function of degree n, if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any nonzero constant λ.

Here, F_1, F_2, F_3 are homogeneous functions of degree 2, 1, 0 respectively but F_4 is not a homogeneous function.

Also, we have

$$F_1(x, y) = x^2\left(\frac{y^2}{x^2} - \frac{3y}{x}\right) = x^2 G_1\left(\frac{y}{x}\right)$$

or $$F_1(x, y) = y^2\left(1 - \frac{3x}{y}\right) = y^2 G_2\left(\frac{x}{y}\right)$$

$$F_2(x, y) = x^1\left(1 - \frac{2y}{x}\right) = x^1 G_3\left(\frac{y}{x}\right)$$

or $$F_2(x, y) = y^1\left(\frac{x}{y} - 2\right) = y^1 G_4\left(\frac{x}{y}\right)$$

$$F_3(x, y) = x^0 \tan\frac{y}{x} = x^0 G_5\left(\frac{y}{x}\right)$$

or $$F_3(x, y) = y^0 \tan\frac{y}{x} = y^0 G_5\left(\frac{y}{x}\right)$$

$$F_4(x, y) \neq x^n G_6\left(\frac{y}{x}\right), \text{ for any } n \in N$$

or $$F_4(x, y) \neq y^n G_7\left(\frac{x}{y}\right), \text{ for any } n \in N$$

From this we can also say that, a function F(x, y) is a homogeneous function of degree n if

$$F(x, y) = x^n G\left(\frac{y}{x}\right) \text{ or } y^n G\left(\frac{x}{y}\right)$$

A differential equation of the form $\dfrac{dy}{dx} = F(x, y)$ is said to be homogenous if $F(x, y)$ is a homogenous function of degree zero.

If $F(x, y) = G\left(\dfrac{y}{x}\right)$, then we have $\dfrac{dy}{dx} = G\left(\dfrac{y}{x}\right)$ ... (1)

To, solve such equation we substitute, $\dfrac{y}{x} = v$ or $y = vx$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

So, equation (1) reduces to,

$$v + x\frac{dv}{dx} = G(v)$$

or $$x\frac{dv}{dx} = G(v) - v$$

$$\Rightarrow \quad \frac{dv}{G(v) - v} = \frac{dx}{x}$$

Now, integrating both sides, we get

$$\int \frac{dv}{G(v) - v} = \int \frac{1}{x} dx + C$$

This is solution of the differential equation (1).

If homogeneous differential equation is in the form $\dfrac{dx}{dy} = F(x, y)$, we substitute $\dfrac{x}{y} = v$ i.e., $x = vy$ and then we proceed to solve the equation as discussed above by writing $\dfrac{dx}{dy} = F(x, y) = G\left(\dfrac{x}{y}\right)$.

ILLUSTRATION 10.16

Show that the differential equation $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$ is homogeneous and solve it.

Sol. The given differential equation is

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} \quad \text{...(1)}$$

Let $F(x, y) = \dfrac{x^2 + y^2}{x^2 + xy}$.

Now, $F(\lambda x, \lambda y) = \dfrac{(\lambda x)^2 + (\lambda y)^2}{(\lambda x)^2 + (\lambda x)(\lambda y)} = \dfrac{x^2 + y^2}{x^2 + xy} = \lambda^0 \,.\, F(x, y)$

This shows that equation (1) is a homogeneous equation.

To solve it, we make the substitution as

$$y = vx$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of v and $\dfrac{dy}{dx}$ in equation (1), we get

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{1 + v}$$

or $$x\frac{dv}{dx} = \frac{1 + v^2}{1 + v} - v = \frac{1 - v}{1 + v}$$

or $$\left(\frac{1 + v}{1 - v}\right) dv = \frac{dx}{x}$$

or $$\left(\frac{2 - 1 + v}{1 - v}\right) dv = \frac{dx}{x}$$

or $$\left(\frac{2}{1 - v} - 1\right) dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(\frac{2}{1 - v} - 1\right) dv = \int\frac{dx}{x}$$

or $\quad -2\log(1 - v) - v = \log x - \log C$

or $\quad v = -2\log(1 - v) - \log x + \log C$

or $\quad v = \log\left[\dfrac{C}{x(1-v)^2}\right]$

$$= \log\left[\dfrac{C}{x\left(1-\dfrac{y}{x}\right)^2}\right]$$

$$= \log\left[\dfrac{Cx}{(x-y)^2}\right]$$

or $\quad \dfrac{Cx}{(x-y)^2} = e^{\frac{y}{x}}$

or $\quad (x-y)^2 = Cxe^{-\frac{y}{x}}$

ILLUSTRATION 10.17

Show that the given differential equation $xdy - ydx = \sqrt{x^2+y^2}\,dx$ is homogeneous and solve it.

Sol. $xdy - ydx = \sqrt{x^2+y^2}\,dx$

or $\quad xdy = \left[y + \sqrt{x^2+y^2}\right]dx$

or $\quad \dfrac{dy}{dx} = \dfrac{y+\sqrt{x^2+y^2}}{x}$ ...(1)

Let $\quad F(x, y) = \dfrac{y+\sqrt{x^2+y^2}}{x}$

$\therefore \quad F(\lambda x, \lambda y) = \dfrac{\lambda x + \sqrt{(\lambda x)^2 + (\lambda y)^2}}{\lambda x} = \dfrac{y+\sqrt{x^2+y^2}}{x} = \lambda^0 F(x, y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as

$y = vx$

or $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substituting the values of v and $\dfrac{dy}{dx}$ in equation (1), we get

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+(vx)^2}}{x}$$

or $\quad v + x\dfrac{dv}{dx} = v + \sqrt{1+v^2}$

or $\quad \dfrac{dv}{\sqrt{1+v^2}} = \dfrac{dx}{x}$

Integrating both sides, we get

$$\log|v + \sqrt{1+v^2}| = \log|x| + \log C$$

or $\quad \log\left|\dfrac{y}{x} + \sqrt{1+\dfrac{y^2}{x^2}}\right| = \log|Cx|$

or $\quad \log\left|\dfrac{y+\sqrt{x^2+y^2}}{x}\right| = \log|Cx|$

or $\quad y + \sqrt{x^2+y^2} = Cx^2$

ILLUSTRATION 10.18

Solve $\left(x\sin\dfrac{y}{x}\right)dy = \left(y\sin\dfrac{y}{x} - x\right)dx.$

Sol. $\left(\sin\dfrac{y}{x}\right)\dfrac{dy}{dx} = \left(\dfrac{y}{x}\cdot\sin\dfrac{y}{x} - 1\right)$

Put $y = vx$

$\Rightarrow \quad \sin v\left(v + x\dfrac{dv}{dx}\right) = (v\sin v - 1)$

or $\quad \sin v\,\dfrac{xdv}{dx} = -1$

or $\quad \displaystyle\int \sin v\,dv = -\int\frac{dx}{x}$

$\Rightarrow \quad \cos v = \log_e x + c$

$\Rightarrow \quad \cos\dfrac{y}{x} = \log_e x + c$

ILLUSTRATION 10.19

Solve $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\dfrac{x}{y}\right)dy = 0$

Sol. $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\dfrac{x}{y}\right)dy = 0$

or $\quad \left(1+e^{\frac{x}{y}}\right)dx = -e^{\frac{x}{y}}\left(1-\dfrac{x}{y}\right)dy$

or $\quad \dfrac{dx}{dy} = \dfrac{-e^{\frac{x}{y}}\left(1-\dfrac{x}{y}\right)}{1+e^{\frac{x}{y}}}$ ...(1)

The given differential equation is a homogeneous equation.

Put $x = vy$

$\Rightarrow \quad \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

Substituting the values of x and $\dfrac{dx}{dy}$ in equation (1), we get

$$v + y\frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v}$$

or $\quad y\dfrac{dv}{dy} = \dfrac{-e^v + ve^v}{1+e^v} - v$

or $\quad y\dfrac{dv}{dy}=-\left[\dfrac{v+e^v}{1+e^v}\right]$

or $\quad \left[\dfrac{1+e^v}{v+e^v}\right]dv=-\dfrac{dy}{y}$

Integrating both sides, we get

$$\log(v+e^v)=-\log y+\log C=\log\left(\frac{C}{y}\right)$$

or $\quad \left[\dfrac{x}{y}+e^{\frac{x}{y}}\right]=\dfrac{C}{y}$

or $\quad x+ye^{\frac{x}{y}}=C$

ILLUSTRATION 10.20

Solve $xdy=\left(y+x\dfrac{f(y/x)}{f'(y/x)}\right)dx$.

Sol. $xdy=\left(y+x\dfrac{f(y/x)}{f'(y/x)}\right)dx$

or $\quad \dfrac{dy}{dx}=\dfrac{y}{x}+\dfrac{f(y/x)}{f'(y/x)}$

Putting $y/x=v$, $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

The given equation transforms to

$$v+x\frac{dv}{dx}=v+\frac{f(v)}{f'(v)}$$

or $\quad \displaystyle\int\frac{f'(v)}{f(v)}dv=\int\frac{dx}{x}$

or $\quad \log|f(v)|=\log|x|+\log c$

$\Rightarrow \quad |f(v)|=c|x|$

$\Rightarrow \quad f\left(\dfrac{y}{x}\right)=cx$

ILLUSTRATION 10.21

Find the real value of m for which the substitution $y=u^m$ will transform the differential equation $2x^4y\dfrac{dy}{dx}+y^4=4x^6$ in to a homogeneous equation.

Sol. $y=u^m$

or $\quad dy/dx=mu^{m-1}\dfrac{du}{dx}$

The given differential equation becomes

$$2x^4\cdot u^m\cdot mu^{m-1}\frac{du}{dx}+u^{4m}=4x^6$$

or $\quad \dfrac{du}{dx}=\dfrac{4x^6-u^{4m}}{2mx^4x^{2m-1}}$

For homogeneous equation, degree of each term should be same in the numerator and the denominator. Hence,

$$6=4m=4+2m-1 \text{ or } m=3/2$$

ILLUSTRATION 10.22

Solve $\dfrac{dy}{dx}=\dfrac{x+2y+3}{2x+3y+4}$.

Sol. The given differential equation is not homogeneous. So, we use following solution technique.

Put $x=X+h$, $y=Y+k$

We have $\dfrac{dY}{dX}=\dfrac{X+2Y+(h+2k+3)}{2X+3Y+(2h+3k+4)}$

To determine h and k, we write

$$h+2k+3=0,\ 2h+3k+4=0$$

$\Rightarrow \quad h=1, k=-2$

so that $\dfrac{dY}{dX}=\dfrac{X+2Y}{2X+3Y}$

Putting $Y=VX$, we get

$$V+X\frac{dV}{dX}=\frac{1+2V}{2+3V} \text{ or } \frac{2+3V}{3V^2-1}dV=-\frac{dX}{X}$$

or $\quad \left[\dfrac{2+\sqrt{3}}{2(\sqrt{3}V-1)}-\dfrac{2-\sqrt{3}}{2(\sqrt{3}V+1)}\right]dV=-\dfrac{dX}{X}$

or $\quad \dfrac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V-1)-\dfrac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V+1)$

$$=-\log X+c$$

$$\frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y-X)-\frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y+X)=A,$$

where A is another constant and $X=x-1$, $Y=y+2$

CONCEPT APPLICATION EXERCISE 10.4

1. Solve $x\dfrac{dy}{dx}=y+2\sqrt{y^2-x^2}$.
2. Solve $[2\sqrt{xy}-x]\,dy+y\,dx=0$
3. Solve $x(dy/dx)=y(\log y-\log x+1)$.
4. Solve $\left[x\sin^2\left(\dfrac{y}{x}\right)-y\right]dx+xdy=0$; $y=\dfrac{\pi}{4}$ when $x=1$.
5. Show that the differential equation $y^3dy-(x+y^2)\,dx=0$ can be reduced to a homogeneous equation.
6. Solve $\dfrac{dy}{dx}=\dfrac{2x-y+1}{x+2y-3}$.

ANSWERS

1. $\dfrac{1}{2}\log_e\left(\dfrac{y+\sqrt{y^2-x^2}}{x}\right)=\log_e cx$
2. $\log_e y+\sqrt{\dfrac{x}{y}}=C$
3. $y=xe^{cx}$, $c>0$
4. $\cot\left(\dfrac{y}{x}\right)=\log_e|ex|$
6. $xy+y^2-3y=x^2+x+c$

SOLVING DIFFERENTIAL EQUATION USING GENERAL FORM OF VARIABLE SEPARATION

If we can write the differential equation in the form $(f_1(x, y))$ $d(f_1(x, y)) + f_2(x, y)d(f_2(x, y)) + \ldots = 0$, then each term can be easily integrated separately. In solving such equation following differentiation helps.

(i) $\dfrac{d(xy)}{dx} = x\dfrac{dy}{dx} + y$ i.e., $d(xy) = xdy + ydx$

(ii) $\dfrac{d}{dx}\left(\dfrac{y}{x}\right) = \dfrac{x\dfrac{dy}{dx} - y}{x^2}$, i.e., $d\left(\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{x^2}$

(iii) $\dfrac{d}{dx}\left(\dfrac{x}{y}\right) = \dfrac{y - x\dfrac{dy}{dx}}{y^2}$, i.e., $d\left(\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{y^2}$

Using above differentiation formula, we have

$$d(x^2y^3) = y^3d(x^2) + x^2d(y^3)$$

$$= y^3(2x)dx + x^2(3y^2)dy \quad \ldots(1)$$

$$d((\sin x)(\log_e y)) = \sin x\, d(\log_e y) + \log_e y\, d(\sin x)$$

$$= \sin x \cdot \frac{1}{y}dy + \cos x \cdot \log_e y dx \quad \ldots(2)$$

$$d\left(\frac{x^3}{y^2}\right) = \frac{y^2d(x^3) - x^3d(y^2)}{y^4}$$

$$= \frac{y^2(3x^2)dx - x^3(2y)dy}{y^4} \quad \ldots(3)$$

In differential equations we find terms of R.H.S. of (1), (2) and (3). Grouping these terms, we create differential coefficient of L.H.S..

ILLUSTRATION 10.23

Solve: $xdy + ydx = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$.

Sol. We have,

$$xdy + ydx = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

$$\therefore \quad d(xy) = \frac{\dfrac{x\,dy - y\,dx}{x^2}}{1 + \left(\dfrac{y}{x}\right)^2}$$

$$\Rightarrow \quad \int d(xy) = \int \frac{d\left(\dfrac{y}{x}\right)}{1 + \left(\dfrac{y}{x}\right)^2}$$

$$\Rightarrow \quad xy = \tan^{-1}\frac{y}{x} + c$$

ILLUSTRATION 10.24

Solve: $\left[(x+1)\dfrac{y}{x} + \sin y\right]dx + \left[x + \log_e x + x\cos y\right]dy = 0$.

Sol. We can rewrite the given equation as

$$(y\,dx + x\,dy) + \left(\frac{y}{x}dx + \log_e x\,dy\right) + (\sin y\,dx + x\cos y\,dy) = 0$$

$$\Rightarrow \quad d(xy) + \left(yd(\log_e x) + \log_e x\,dy\right) + (\sin y\,dx + xd(\sin y)) = 0$$

$$\Rightarrow \quad d(xy) + d(y\log_e x) + d(x\sin y) = 0$$

Integrating both the sides, we get

$$xy + y\log_e x + x\sin y = c$$

ILLUSTRATION 10.25

Solve: $(x\cos x - \sin x)dx = \dfrac{x}{y}\sin x dy$.

Sol. We have, $(x\cos x - \sin x)\,dx = \dfrac{x}{y}\sin x\,dy$

$\therefore \quad (xy\cos x - y\sin x)\,dx = x\sin x\,dy$

$\Rightarrow \quad xy\cos x = (y\,dx + x\,dy)\sin x$

$\Rightarrow \quad \cot x\,dx = \dfrac{d(xy)}{xy}$

On integrating, we get

$\log_e \sin x = \log_e xy + \log_e c$

or $\quad \sin x = cxy$

ILLUSTRATION 10.26

Solve: $y^4\,dx + 2xy^3\,dy = \dfrac{y\,dx - x\,dy}{x^3\,y^3}$.

Sol. We can rewrite the given equation as

$$y^4dx + 2xy^3\,dy + \frac{1}{xy^3}\frac{(x\,dy - y\,dx)}{x^2} = 0$$

$$\Rightarrow \quad xy^7dx + 2x^2y^6\,dy + d\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad x^2y^6\,dx + 2x^3\,y^5\,dy + \frac{x}{y}\,d\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad \frac{1}{3}(3x^2y^6\,dx + 6x^3y^5\,dy) + \frac{d(y/x)}{y/x} = 0$$

$$\Rightarrow \quad \frac{1}{3}(y^6\,d(x^3) + x^3d(y^6)) + \frac{d(y/x)}{y/x} = 0$$

$$\Rightarrow \quad \frac{1}{3}\int d(x^3y^6) + \int\frac{d(y/x)}{y/x} = 0$$

$$\Rightarrow \quad \frac{x^3y^6}{3} + \log_e\frac{y}{x} = c$$

ILLUSTRATION 10.27

Solve: $\dfrac{dy}{dx} = \dfrac{yf'(x) - y^2}{f(x)}$.

Sol. We have $\frac{dy}{dx} = \frac{yf'(x) - y^2}{f(x)}$

$\therefore \quad yf'(x)dx - f(x)\,dy = y^2\,dx$

$\Rightarrow \quad \frac{yf'(x)dx - f(x)dy}{y^2} = dx$

$\Rightarrow \quad d\left[\frac{f(x)}{y}\right] = dx$

On integrating, we get

$\frac{f(x)}{y} = x + c$ or $f(x) = (x + c)y$

ILLUSTRATION 10.28

Solve : $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y^2\right)dy\ (y \neq 0)$.

Sol. We have $ye^{\frac{x}{y}}dx - xe^{\frac{x}{y}}dy = y^2dy$

$\therefore \quad e^{\frac{x}{y}}\frac{ydx - xdy}{y^2} = dy$

$\Rightarrow \quad \int e^{\frac{x}{y}}d\left(\frac{x}{y}\right) = \int dy$

$\Rightarrow \quad e^{\frac{x}{y}} = y + c$

ILLUSTRATION 10.29

Solve: $\frac{dy}{dx} = \frac{x - 2y + 5}{2x + y - 1}$.

Sol. We have, $2xdy + ydy - dy = xdx - 2ydx + 5dx$.

$\Rightarrow \quad 2(xdy + y\,dx) + ydy - dy = xdx + 5dx$

$\Rightarrow \quad 2d(xy) + ydy - dy = xdx + 5dx$

On integrating, we get

$$2(xy) + \frac{y^2}{2} - y = \frac{x^2}{2} + 5x + c$$

ILLUSTRATION 10.30

Solve: $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \frac{y^4}{x^2}$

Sol. We have $\frac{xdx + ydy}{ydx - xdy} = \left(x + \frac{y^2}{x}\right)^2$

$\Rightarrow \quad \dfrac{\frac{1}{2}\,d(x^2) + d(y^2)}{x^2\,\dfrac{xdy - ydx}{x^2}} = \left(\frac{x^2 + y^2}{x}\right)^2$

$\Rightarrow \quad -\frac{1}{2}\frac{d(x^2 + y^2)}{(x^2 + y^2)^2} = d\left(\frac{y}{x}\right)$

Integrating both sides, we get

$\Rightarrow \quad \frac{1}{2(x^2 + y^2)} = \frac{y}{x} + C$

ILLUSTRATION 10.31

Solve:

$$\left(\frac{1}{y}\sin\frac{x}{y} - \frac{y}{x^2}\cos\frac{y}{x} + 1\right)dx + \left(\frac{1}{x}\cos\frac{y}{x} - \frac{x}{y^2}\sin\frac{x}{y} + \frac{1}{y^2}\right)dy = 0$$

Sol. We have

$$\left(\frac{dx}{y} - \frac{xdy}{y^2}\right)\sin\frac{x}{y} + \left(\frac{dy}{x} - \frac{ydx}{x^2}\right)\cos\frac{y}{x} + dx + \frac{dy}{y^2} = 0$$

$\therefore \quad \frac{ydx - xdy}{y^2}\sin\frac{x}{y} + \frac{xdy - ydx}{x^2}\cos\frac{y}{x} + dx + \frac{dy}{y^2} = 0$

$\Rightarrow \quad \sin\frac{x}{y}\cdot d\left(\frac{x}{y}\right) + \cos\frac{y}{x}\cdot d\left(\frac{y}{x}\right) + dx + \frac{dy}{y^2} = 0$

Integrating, we get

$$-\cos\frac{x}{y} + \sin\frac{y}{x} + x - \frac{1}{y} = C$$

CONCEPT APPLICATION EXERCISE 10.5

Solve the following differential equations:

1. $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$
2. $\frac{dy}{dx} = \frac{2xy}{x^2 - 1 - 2y}$
3. $y\,dx + (x + x^2y)dy = 0$
4. $(xy^4 + y)\,dx - x\,dy = 0$
5. $y\,(x^2y + e^x)dx - e^x\,dy = 0$
6. $\frac{dy}{dx} = -\left(\frac{y + \sin x}{x}\right)$ satisfying condition $y(0) = 1$
7. $y(xy + 1)dx + x(1 + xy + x^2y^2)dy = 0$

ANSWERS

1. $\frac{x}{y} + e^{x^3} = c$
2. $\frac{x^2}{y} = \frac{1}{y} - 2\log_e y + c$
3. $-\frac{1}{xy} + \log_e y = c$
4. $3x^4y^3 + 4x^3 = cy^3$
5. $x^3y + 3e^x = cy$
6. $\cos x = xy + 1$
7. $-\frac{1}{2x^2y^2} - \frac{1}{xy} + 2\log_e y = c$

LINEAR DIFFERENTIAL EQUATIONS

Equation of the form $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x alone, is called linear differential equation.

For solving such equation, we multiply both sides by

Integrating factor = I.F. = $e^{\int Pdx}$

Here in $\int P dx$ we don't put constant of integration as this is factor which help to create differential coefficient in L.H.S..

So, we have now equation $e^{\int P dx}\left(\frac{dy}{dx}+Py\right)=Qe^{\int P dx}$

$$\Rightarrow \quad \frac{dy}{dx}e^{\int P dx}+Py\,e^{\int P dx}=Qe^{\int P dx}$$

$$\Rightarrow \quad \frac{dy}{dx}e^{\int P dx}+y\frac{d}{dx}\left(e^{\int P dx}\right)=Qe^{\int P dx}$$

$$\Rightarrow \quad \frac{d}{dx}\left(ye^{\int P dx}\right)=Qe^{\int P dx}$$

$$\Rightarrow \quad \int d\left(ye^{\int P dx}\right)=\int Qe^{\int P dx}dx$$

$$\Rightarrow \quad ye^{\int P dx}=\int Qe^{\int P dx}dx+C$$

which is the required solution of the given differential equation

Another form of linear differential equation is $\frac{dx}{dy}+P_1x=Q_1$, where P_1 and Q_1 are function of y alone.

In such equation, integrating factor, I.F. = $e^{\int P_1 dy}$.

Solutions of the equation is given by,

$$xe^{\int P dy}=\int Qe^{\int P dy}dy+C$$

ILLUSTRATION 10.32

Solve $x^2(dy/dx)+y=1$.

Sol. The given differential equation can be written as

$$\frac{dy}{dx}+\frac{1}{x^2}y=\frac{1}{x^2}, \text{ which is linear}$$

Here $P=1/x^2$ and $Q=1/x^2$

$$\text{I.F.}=e^{\int(1/x^2)dx}=e^{-1/x}$$

Therefore, the solution is

$$ye^{-1/x}=\int e^{-1/x}(1/x^2)dx+c=e^{-1/x}+c$$

or $\quad y=1+ce^{1/x}$

ILLUSTRATION 10.33

Solve $y\,dx-x\,dy+\log x\,dx=0$.

Sol. The given equation can be written as

$$\frac{dy}{dx}-\frac{1}{x}y=\frac{1}{x}\log x$$

$$\text{I.F.}=e^{\int-\frac{1}{x}}=e^{-\log x}=\frac{1}{x}$$

Therefore, the solution is

$$\frac{y}{x}=\int\frac{1}{x^2}\log x\,dx+c \qquad ...(1)$$

$$\therefore \quad \frac{y}{x}=\log_e x\cdot\left(\frac{-1}{x}\right)-\int\left(\frac{1}{x}\right)\left(\frac{-1}{x}\right)dx+C$$

$$\therefore \quad \frac{y}{x}=\frac{-\log_e x}{x}-\frac{1}{x}+C$$

Hence, the required solution is $y+1+\log x$ ...

ILLUSTRATION 10.34

Solve: $(1+y+x^2y)\,dx+(x+x^3)\,dy=0$.

Sol. We have $(1+y+x^2y)\,dx+(x+x^3)\,dy=0$

$$\Rightarrow \quad 1+y(1+x^2)+x(1+x^2)\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{x}=-\frac{1}{x(1+x^2)}, \text{ which is linear differential equation}$$

$$\text{I.F.}=e^{\int\frac{1}{x}dx}=e^{\log_e x}=x$$

$$\therefore \quad \text{Solution is } y\times x=-\int\frac{1}{1+x^2}dx$$

$$\therefore \quad y\times x=-\tan^{-1}x+c$$

ILLUSTRATION 10.35

Solve $(x+2y^3)\frac{dy}{dx}=y$.

Sol. This equation can be rewritten as

$$\frac{dx}{dy}=\frac{1}{y}x+2y^2$$

or $\quad \frac{dx}{dy}-\frac{1}{y}.\,x=2y^2$

This is linear differential equation of the form $\frac{dx}{dy}+Px=Q$, where P and Q are functions of y.

$$\text{Here} \quad \text{I.F.}=e^{-\int\frac{1}{y}dy}=e^{-\log y}=\frac{1}{y}$$

$$\therefore \quad \text{Solution is } x\frac{1}{y}=\int\frac{1}{y}2y^2dy+C$$

$$\Rightarrow \quad \frac{x}{y}=y^2+C$$

$$\Rightarrow \quad x=y^3+cy$$

ILLUSTRATION 10.36

Solve: $(1+y^2)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$.

Sol. Differential equation can be rewritten as

$$(1+y^2)\frac{dx}{dy}+x=e^{\tan^{-1}y}$$

or $\quad \frac{dx}{dy}+\frac{1}{1+y^2}.\,x=\frac{e^{\tan^{-1}y}}{1+y^2}$

$$\text{Here} \quad \text{I.F.}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

$$\therefore \quad \text{Solution is } xe^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}}{1+y^2}e^{\tan^{-1}y}dy$$

$$\Rightarrow \quad xe^{\tan^{-1}y}=e^{2\tan^{-1}y}\ldots+C$$

ILLUSTRATION 10.37

Let $u(x)$ and $v(x)$ be two continuous functions satisfying the differential equations $\frac{du}{dx}+p(x)u=f(x)$ and $\frac{dv}{dx}+p(x)v=g(x)$, respectively. If $u(x_1)>v(x_1)$ for some x_1 and $f(x)>g(x)$ for all $x>x_1$, prove that any point (x, y), where $x>x_1$, does not satisfy the equations $y=u(x)$ and $y=v(x)$ simultaneously.

Sol. We have following information:

a. $y=u(x)$ and $y=v(x)$ are solutions of given differential equations.

b. $u(x_1)>v(x_1)$ for some x_1

c. $f(x)>g(x)\ \forall\ x>x_1$

$$\frac{du}{dx}+p(x)u=f(x) \text{ and } \frac{dv}{dx}+p(x)v=g(x)$$

or $$\frac{d(u-v)}{dx}+p(x)(u-v)=f(x)-g(x)$$

or $$e^{\int pdx}\frac{d(u-v)}{dx}+e^{\int pdx}p(x)(u-v)=e^{\int pdx}(f(x)-g(x))$$

or $$\frac{d}{dx}\left[(u-v).e^{\int pdx}\right]=[f(x)-g(x)]e^{\int pdx}$$

Given $f(x)>g(x)\ \forall\ x>x_1$ and exponential function is always positive. Then R.H.S. is positive.

$$\therefore \quad \frac{d}{dx}\left[(u-v).e^{\int pdx}\right]>0$$

Hence, the function $F(x)=(u-v)\,e^{\int pdx}$ is an increasing function.

Again, $u(x_1)>v(x_1)$ for some x_1.

$\therefore \quad F(x)=(u-v)\,e^{\int pdx}$ is +ve at $x=x_1$

or $\quad F(x)=(u-v)\,e^{\int pdx}$ is +ve $\forall\ x>x_1$

($\because$ F being increasing function)

$\therefore \quad u(x)>u(x_1)\ \forall\ x>x_1$

Hence, there is no point (x, y) such that $x>x_1$ which can satisfy the equations, $y=u(x)$ and $y=v(x)$ simultaneously.

CONCEPT APPLICATION EXERCISE 10.6

1. What is the integrating factor of the differential equation $(1-y^2)\frac{dx}{dy}+yx=ay\ (-1<y<1)$?
2. Solve $\frac{dy}{dx}+y\cot x=\sin x$.
3. Solve $(x+y+1)(dy/dx)=1$.
4. Solve $(1-x^2)(dy/dx)+2xy=x\sqrt{1-x^2}$.
5. Solve $\frac{dy}{dx}=\frac{y}{2y\ln y+y-x}$.
6. Solve $y\,dx+(x-y^2)\,dy=0$.
7. Find the equation of a curve passing through (0, 1) and having gradient $\frac{-(y+y^3)}{1+x+xy^2}$ at (x, y).

ANSWERS

1. $\frac{1}{\sqrt{1-y^2}}$
2. $y\sin x=\frac{1}{4}[2x-\sin 2x]+c$
3. $x=ce^y-y-2$
4. $y=\sqrt{1-x^2}+c(1-x^2)$
5. $xy=y^2\log_e y+c$
6. $x=\frac{y^2}{3}+\frac{C}{y}$
7. $xy+\tan^{-1}y=\frac{\pi}{4}$

DIFFERENTIAL EQUATIONS REDUCIBLE TO LINEAR DIFFERENTIAL EQUATIONS

Some differential equation can be reduced to linear differential equation using proper substitution.

General form of such differential equation is

$$f'(y)\frac{dy}{dx}+f(y)P(x)=Q(x) \qquad \ldots(1)$$

Now substituting, $f(y)=u$, we have $f'(y)\frac{dy}{dx}=\frac{du}{dx}$

$\therefore$ Equation (1) reduces to $\frac{du}{dx}+uP(x)=Q(x)$, which is linear differential equation.

In a particular case we have $\frac{dy}{dx}+P(x)y=Q(x)y^n \qquad \ldots(2)$

This is called **Bernoulli's Equation.**

To solve the equation we divide, both sides by y^n

So, we get $\frac{1}{y^n}\frac{dy}{dx}+\frac{P}{y^{n-1}}=Q$

Now this equation takes the form of equation (1).

We can reduce this equation to linear differential equation by substituting $\frac{1}{y^{n-1}}=v$

ILLUSTRATION 10.38

Solve $(dy/dx)+(y/x)=y^3$.

Sol. Dividing the given equation by y^3, we get

$$\frac{1}{y^3}\frac{dy}{dx}+\frac{1}{y^2}\frac{1}{x}=1 \qquad (1)$$

Putting $1/y^2=v$, we have

$$(-2/y^3)\,dy/dx=dv/dx$$

Therefore, equation (1) becomes

$$-\frac{1}{2}\frac{dv}{dx}+\frac{1}{x}v=1 \text{ or } \frac{dv}{dx}-\frac{2}{x}v=-2$$

This is a linear equation with v as the dependent variable.

$$\text{I.F.}=e^{-\int(2/x)dx}=e^{-2\log x}=1/x^2$$

Therefore, the solution is

$$v(1/x^2)=-2\int(1/x^2)\,dx+c=2/x+c$$

or $$2xy^2+cx^2y^2=1$$

ILLUSTRATION 10.39

Solve $(dy/dx) = e^{x-y}(e^x - e^y)$.

Sol. Multiplying the given equation by e^y, we get

$$e^y\frac{dy}{dx} + e^x e^y = e^{2x} \quad (1)$$

Putting $e^y = v$, so that $e^y\frac{dy}{dx} = \frac{dv}{dx}$,

and equation (1) transform to $\frac{dv}{dx} + e^x v = e^{2x}$

$$\text{I.F.} = e^{\int e^x dx} = e^{e^x}$$

Hence, solution is $ve^{e^x} = \int e^{2x}e^{e^x}dx + c$

Let $e^x = t$ or $e^x dx = dt$

Hence, solution is $v.e^{e^x} = \int te^t dt + c$

or $e^y e^{e^x} = te^t - e^t + c$

or $e^y e^{e^x} = e^x e^{e^x} - e^{e^x} + c$

ILLUSTRATION 10.40

Solve $(x-1)dy + y\,dx = x(x-1)\,y^{1/3}\,dx$.

Sol. Dividing by $dx\,y^{1/3}(x-1)$, the given equation reduces to

$$y^{-1/3}\frac{dy}{dx} + \frac{1}{x-1}y^{2/3} = x$$

put $y^{2/3} = z$, so that $\frac{2}{3}y^{-1/3}\frac{dy}{dx} = \frac{dz}{dx}$

Then given equation reduces to

$$\frac{dz}{dx} + \frac{2}{3(x-1)}z = \frac{2}{3}x \quad \text{(linear form)}$$

$$\text{I.F.} = e^{\frac{2}{3}\int\frac{1}{(x-1)}dx} = e^{\frac{2}{3}\log(x-1)} = (x-1)^{2/3}$$

Hence, solution is given by

$$z(x-1)^{\frac{2}{3}} = \frac{2}{3}\int x(x-1)^{2/3}dx + c$$

Putting $(x-1) = t^3$ in the R.H.S., we get

$$\int x(x-1)^{2/3}dx = \int (t^3+1)\,t^2\,3t^2dt$$

$$= 3\int (t^7 + t^4)\,dt$$

$$= 3[(1/8)t^8 + (1/5)t^5]$$

$$= (3/8)(x-1)^{8/3} + (3/5)(x-1)^{5/3}$$

Hence, the solution is $y^{2/3} = \frac{1}{4}(x-1)^2 + \frac{2}{5}(x-1) + c(x-1)^{-2/3}$.

ILLUSTRATION 10.41

Solve: $e^x(x+1)dx + (ye^y - xe^x)dy = 0$, such that $f(0) = 0$.

Sol. We have $e^x(x+1)dx + (ye^y - xe^x)dy = 0$

or $e^x(x+1)\frac{dx}{dy} - xe^x = -ye^y \quad \ldots(1)$

Put $xe^x = t$

$\Rightarrow e^x(x+1)\frac{dx}{dy} = \frac{dt}{dy}$

$\therefore$ Equation (1), reduces to

$$\Rightarrow \frac{dt}{dy} - t = -ye^y$$

$$\text{I.F.} = e^{-\int dy} = e^{-y}$$

$\therefore$ Solution is $t \times e^{-y} = -\int ye^y e^{-y}dy$

$\Rightarrow xe^x e^{-y} = -\frac{y^2}{2} + C$

Since $f(0) = 0$, $C = 0$

$\therefore$ particular solution is $2x\,e^x\,e^{-y} + y^2 = 0$

CONCEPT APPLICATION EXERCISE 10.7

Solve the following differential equations:

1. $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$
2. $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$
3. $\frac{dy}{dx} + \frac{xy}{(1-x^2)} = x\sqrt{y}$
4. $\frac{dy}{dx} = (x^3 - 2x\tan^{-1}y)(1+y^2)$
5. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x\sec y$

ANSWERS

1. $\frac{e^{-y}}{x} = \frac{1}{2x^2} + c$
2. $e^{x^2}\tan y = \frac{1}{2}e^{x^2}(x^2-1) + c$
3. $\frac{\sqrt{y}}{(1-x^2)^{1/4}} = -\frac{1}{3}(1-x^2)^{3/4} + c$
4. $2\tan^{-1}y = x^2 - 1 + 2Ce^{-x^2}$
5. $\sin y = (e^x + c)(x+1)$

APPLICATION OF DIFFERENTIAL EQUATION IN CURVE

Till now we have been solving given differential equations. Now we will have situation in which some information is given about the family of curves in the form of its slope, equation of tangent and normal, sub-tangent, sub-normal, length of tangent or length of normal. Using this we can form differential equation, solving which we can get the equation of family of curves. Let us first see how to write equations of tangent and normal.

We write equation of the tangent at a point $P(x, y)$ on the curve $y = f(x)$ as $Y - y = \frac{dy}{dx}(X - x)$.

Here we are considering curve $y = f(x)$ drawn on XY-plane.

For X-intercept we put $Y = 0$. So, X-intercept is $X = -y\frac{dx}{dy} + x$

Similarly, Y-intercept is $Y = -x\frac{dy}{dx} + y$

Also, equation of normal at point $P(x, y)$ is $Y - y = -\frac{dx}{dy}(X - x)$

ILLUSTRATION 10.42

The slope of a curve, passing through (3, 4) at any point is the reciprocal of twice the ordinate of that point. Show that it is a parabola.

Sol. It is given that $\frac{dy}{dx} = \frac{1}{2y}$.

or $2y\,dy = dx$

Integrating, we get $y^2 = x + c$.

Now when $x = 3$, $y = 4$, which gives $c = 13$

Hence, the equation of the required curve is $y^2 = x + 13$, which is a parabola.

ILLUSTRATION 10.43

Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

Sol. The slope of the tangent to the curve $y = f(x)$ at $P(x, y)$ is $\frac{dy}{dx}$.

According to the given information,

$$\frac{dy}{dx} + 5 = x + y$$

or $$\frac{dy}{dx} - y = x - 5$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -1 \text{ and } Q = x - 5$$

Now, I.F $= e^{\int P dx} = e^{\int(-1)dx} = e^{-x}$.

The general equation of the curve is given by the relation,

$$y\,(\text{I.F.}) = \int (Q \times \text{I.F.})\,dx + C$$

or $$y.e^{-x} = \int (x-5)e^{-x}\,dx + C \quad \ldots(1)$$

Now, $\int (x-5)e^{-x}dx$

$$= (x-5)\int e^{-x}dx - \int\left[\frac{d}{dx}(x-5)\cdot\int e^{-x}dx\right]dx$$

$$= (x-5)(-e^{-x}) - \int(-e^{-x})dx$$

$$= (5-x)e^{-x} + (-e^{-x})$$

$$= (4-x)e^{-x}$$

Therefore, equation (1) becomes

$$ye^{-x} = (4-x)e^{-x} + C$$

or $$y = 4 - x + Ce^x$$

or $$x + y - 4 = Ce^x \quad \ldots(2)$$

The curve passes through point (0, 2).

Therefore, equation (2) becomes

$$0 + 2 - 4 = Ce^0$$

or $$C = -2$$

Substituting $C = -2$ in equation (2), we get

$$x + y - 4 = -2e^x$$

or $y = 4 - x - 2e^x$

ILLUSTRATION 10.44

Find the equation of the curve passing through (2, 1) which has constant sub-tangent.

Sol. We are given that

$$\text{sub-tangent} = \frac{y}{\frac{dy}{dx}} = k \text{ (constant)}$$

or $$k\frac{dy}{y} = dx$$

Integrating we get, $k \log y = x + c$

Given that curve passes through (2, 1) $\Rightarrow c = -2$

Hence, the equation of such curve is $k \log y = x - 2$.

ILLUSTRATION 10.45

If length of tangent at any point on the curve $y = f(x)$ intercepted between the point and the x-axis is of length 1. Find the equation of the curve.

Sol. Given length of tangent to curve $y = f(x)$ is 1

$$\therefore \left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)}\right| = 1$$

or $$y^2\left(1+\left(\frac{dy}{dx}\right)^2\right) = \left(\frac{dy}{dx}\right)^2$$

or $$\left(\frac{dy}{dx}\right)^2 = \frac{y^2}{1-y^2}$$

or $$\frac{dy}{dx} = \pm\frac{y}{\sqrt{1-y^2}}$$

or $$\int\frac{\sqrt{1-y^2}}{y}dy = \int \pm dx$$

Put $y = \sin\theta$ so that $dy = \cos\theta\,d\theta$

$$\therefore \int\frac{\cos\theta}{\sin\theta}\cos\theta\,d\theta = \pm x + c$$

or $$\int(\operatorname{cosec}\theta - \sin\theta)\,d\theta = \pm x + c$$

or $$\log|\operatorname{cosec}\theta - \cot\theta| + \cos\theta = \pm x + c$$

or $$\log\left|\frac{1-\sqrt{1-y^2}}{y}\right| + \sqrt{1-y^2} = \pm x + c$$

ILLUSTRATION 10.46

A curve passing through the point (1, 1) has the property that the perpendicular distance form origin to the normal at any point P of the curve is equal to the distance of P from the x-axis. Determine the equation of the curve.

Sol. Equation of normal is $\frac{dx}{dy}(X - x) + Y - y = 0$

Given that perpendicular distance form origin to the normal at P = distance of P from the x-axis

or $$\frac{\left|x\frac{dx}{dy}+y\right|}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}}=|y|$$

or $$x^2\left(\frac{dx}{dy}\right)^2+y^2+2xy\frac{dx}{dy}=y^2+y^2\left(\frac{dx}{dy}\right)^2$$

i.e., $$\left(\frac{dx}{dy}\right)=0 \text{ or } \frac{dx}{dy}=\left(\frac{2xy}{y^2-x^2}\right)$$

If $\frac{dx}{dy}=0$, then $x=c$. When $x=1, y=1$, then $c=1$.

$$\therefore \quad x=1 \qquad (1)$$

$$\frac{dx}{dy}=\frac{2xy}{y^2-x^2} \quad \text{(homogeneous)} \qquad (2)$$

Putting, $x=vy$ and $\frac{dx}{dy}=v+y\frac{dv}{dy}$,

equation (2) transforms to

$$v+y\frac{dv}{dy}=\frac{2v}{1-v^2}$$

or $$y\frac{dv}{dy}=\frac{2v}{1-v^2}-v=\frac{2v-v+v^3}{1-v^2}=\frac{v+v^3}{1-v^2}$$

or $$\int\frac{(1-v^2)dv}{v(1+v^2)}=\int\frac{dy}{y}$$

or $$\int\left(\frac{1}{v}-\frac{2v}{1+v^2}\right)dy=\int\frac{dy}{y}$$

or $$\log v-\log(1+v^2)=\log y+C$$

or $$\frac{v}{1+v^2}=cy$$

or $$\frac{x}{x^2+y^2}=c$$

when $x=1, y=1, c=1/2$.

Thus, solution is $x^2+y^2-2x=0$.

Hence, the solutions are $x^2+y^2-2x=0,\ x-1=0$.

ILLUSTRATION 10.47

Find the curve passing through (1, 2) such that the intercept on the x-axis cut off between the origin, and the tangent at a point is twice the abscissa.

Sol. The equation of the tangent at any point $P(x, y)$ is

$$Y-y=\frac{dy}{dx}(X-x) \qquad (1)$$

Given that intercept on X-axis (putting $Y=0$) = 2 (x-coordinates of P). Thus,

$$x-y\frac{dx}{dy}=2x$$

or $$-\frac{dy}{y}=\frac{dx}{x}$$

Integrating we get $xy=c$

Since the curve passes through (1, 2), $c=2$.

Hence, the equation of the required curve is $xy=2$.

ILLUSTRATION 10.48

Find the equation of the curve passing through the point (5, 4) if the sum of reciprocal of the intercepts of the normal drawn at any point $P(x, y)$ on it is 1.

Sol. The equation of the normal at (x, y) is

$$(X-x)+(Y-y)\frac{dy}{dx}=0$$

or $$\frac{X}{x+y\frac{dy}{dx}}+\frac{Y}{\frac{(x+y\,dy/dx)}{dy/dx}}=1$$

x-intercept $OA=x+y\frac{dy}{dx}$,

x-intercept $OB=\dfrac{\left(x+y\frac{dy}{dx}\right)}{\frac{dy}{dx}}$

Given, $$\frac{1}{OA}+\frac{1}{OB}=1$$

or $$1+\frac{dy}{dx}=x+y\frac{dy}{dx}$$

or $$(y-1)\frac{dy}{dx}+(x-1)=0$$

or $$(y-1)dy+(x-1)dx=0$$

Integrating, we get

$$(y-1)^2+(x-1)^2=c$$

Since the curve passes through (5, 4), $c=25$.

Hence, the curve is $(x-1)^2+(y-1)^2=25$.

ILLUSTRATION 10.49

Find the equation of the curve passing through the origin if the middle point of the segment of its normal from any point of the curve to the x-axis lies on the parabola $2y^2=x$.

Sol. Equation of normal at any point $P(x, y)$ is

$$\frac{dy}{dx}(Y-y)+(X-x)=0$$

This meets the x-axis at $A\left(x+y\frac{dy}{dx}, 0\right)$.

Mid-point of AP is $\left(x+\frac{1}{2}y\frac{dy}{dx}, \frac{y}{2}\right)$ which lies on the parabola $2y^2=x$. Therefore,

$$2\times\frac{y^2}{4}=x+\frac{1}{2}y\frac{dy}{dx} \text{ or } y^2=2x+y\frac{dy}{dx}$$

Putting $y^2 = t$, so that $2y\frac{dy}{dx} = \frac{dt}{dx}$,

we get $\frac{dt}{dx} - 2t = -4x$ (linear)

$$\text{I.F.} = e^{-2\int dx} = e^{-2x}$$

Therefore, solution is given by

$$te^{-2x} = -4\int xe^{-2x}dx + c$$

$$= -4\left[-\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}1dx\right] + c$$

$\Rightarrow$ $y^2e^{-2x} = 2xe^{-2x} + e^{-2x} + c$

Since curve passes through (0, 0), $c = -1$. Therefore,

$$y^2e^{-2x} = 2xe^{-2x} + e^{-2x} - 1$$

or $y^2 = 2x + 1 - e^{2x}$ is the equation of the required curve.

ISOGONAL TRAJECTORY

A curve that crosses every member of a family of curves with the same crossing angle is called an isogonal trajectory of the family. If two families of curves, all cross at the same angle, then each family is called a family of isogonal trajectories with respect to the other family. If the crossing angle is a right angle, the curves or families of curves are orthogonal trajectories.

If one given family is described by the differential equation $\frac{dy}{dx} = f(x, y)$, then its isogonal family with crossing angle α satisfies the differential equation $\frac{dy}{dx} = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}$.

To find the equation of the orthogonal trajectories of the family of curves $f(x, y, C) = 0$, we first find its differential equation $F(x, y, y') = 0$. Then the differential equation of the orthogonal trajectory will be $F\left(x, y, -\frac{1}{y'}\right) = 0$. Solving this differential equation, we get equation of orthogonal trajectories of $f(x, y, C) = 0$.

ILLUSTRATION 10.50

Find the equation of family of curves which intersect the family of curves $xy = c$ at an angle 45°.

Sol. We have family of curves xy = c

$\therefore$ $y + x\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\frac{y}{x} = m_1$ (say)

Let slope of the required family of curves be $\frac{dy}{dx} = m_2$.

According to the question,

$$\tan\frac{\pi}{4} = \pm\frac{m_1 - m_2}{1 + m_1m_2}$$

$\Rightarrow$ $1 + m_1m_2 = \pm(m_1 - m_2)$

$\Rightarrow$ $1 - \frac{y}{x}\frac{dy}{dx} = -\frac{y}{x} - \frac{dy}{dx}$ (considering + sign)

$\Rightarrow$ $1 + \frac{y}{x} = \frac{dy}{dx}\left(\frac{y}{x} - 1\right)$

$\Rightarrow$ $(x + y)dx = (y - x)dy$

$\Rightarrow$ $xdy + ydx = ydy - xdx$

$\Rightarrow$ $d(xy) = ydy - xdx$

Integrating both sides, we get

$xy = \frac{y^2}{2} - \frac{x^2}{2} + c$, which is required family of curves.

ILLUSTRATION 10.51

Find the orthogonal trajectory of $y^2 = 4ax$ (a being the parameter).

Sol. $y^2 = 4ax$ ...(1)

$\therefore$ $2y\frac{dy}{dx} = 4a$ ...(2)

Eliminating a from equation (1) and (2), we get

$$y^2 = 2y\frac{dy}{dx}x$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$$y = 2\left(-\frac{dx}{dy}\right)x$$

$$2x\,dx + y\,dy = 0$$

Integrating both sides, we get

$$x^2 + \frac{y^2}{2} = c$$

$$2x^2 + y^2 = 2c$$

which is the required orthogonal trajectory.

CONCEPT APPLICATION EXERCISE 10.8

1. Find the equation of the curve in which the subnormal varies as the square of the ordinate.
2. Find the curve for which the length of normal is equal to the radius vector.
3. Find the curve for which the perpendicular from the foot of the ordinate to the tangent is of constant length.
4. A curve $y = f(x)$ passes through the origin. Through any point (x, y) on the curve, lines are drawn parallel to the co-ordinate axes. If the curve divides the area formed by these lines and co-ordinates axes in the ratio $m:n$, find the curve.
5. A normal is drawn at a point $P(x, y)$ of a curve. It meets the x-axis at Q. If PQ has constant length k, then show that the differential equation describing such curves is $y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$. Find the equation of such a curve passing through $(0, k)$.
6. Find the orthogonal trajectories of family of curves $x^2 + y^2 = cx$.
7. A curve C has the property that if the tangent drawn at any point P on C Meets the co-ordinate axis at A and B, then P is the mid-point of AB. The curve passes through the point (1, 1). Determine the equation of the curve.

ANSWERS

1. $y = ce^{kx}$ 2. $y^2 \pm x^2 = c$
3. $y + \sqrt{y^2 - k^2} = ce^{\pm x/k}$ 4. $y = cx^{m/n}$
5. $x^2 + y^2 = k^2$ 6. $x^2 + y^2 = ky$
7. $xy = 1$

MISCELLANEOUS APPLICATIONS OF DIFFERENTIAL EQUATION

Differential equations are used in variety of disciplines, like biology, economics, physics, chemistry and engineering. They can describe exponential growth and decay, the population growth of species or the change in investment return over time.

In physics differential equations are used with Newton's Second Law of Motion and the Law of Cooling. In chemistry they are used for modelling chemical reactions.

ILLUSTRATION 10.52

The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled, and after three years the population is 20,000, estimate the number of people initially living in the country.

Sol. Let N denote the number of people living in the country at any time t, and let N_0 denote the number of people initially living in the country.

Then, from $\frac{dN}{dt} \propto N$, $\frac{dN}{dt} - kN = 0$

which has the solution $N = ce^{kt}$ (1)

At $t = 0$, $N = N_0$; hence, equation (1) states that $N_0 = ce^{k(0)}$,

or $c = N_0$.

Thus, $N = N_0e^{kt}$ (2)

At $t = 2$, $N = 2\,N_0$.

Substituting these values into equation (2), we have

$$2\,N_0 = N_0e^{2k} \text{ or } e^k = \sqrt{2}$$

Substituting this value into equation (1) gives

$$N = N_0\left(\sqrt{2}\right)^t \quad (3)$$

At $t = 3$, $N = 20{,}000$.

Substituting these values into equation (3), we obtain

$$N_0 = 20{,}000/2\sqrt{2} \approx 7071.$$

ILLUSTRATION 10.53

What constant interest rate is required if an initial deposit placed into an account accrues interest compounded continuously is to double its value in six years? ($\ln|2| = 0.6930$)

Sol. The balance $N(t)$ in the account at any time t,

$\frac{dN}{dt} - kN = 0$, its solution is $N(t) = ce^{kt}$ (1)

Let initial deposit be N_0.

At $t = 0$, $N(0) = N_0$, which when substituted into equation (1) yields

$$N_0 = ce^{k(0)} = c$$

and equation (1) becomes $N(t) = N_0e^{kt}$ (2)

We seek the value of k for which $N = 2N_0$ when $t = 6$, Substituting these values into (2) and solving for k we find $2N_0 = N_0e^{k(6)}$

or $e^{6k} = 2$ or $k = \frac{1}{6}\ln|2| = 0.1155$

An interest rate of 11.55 percent is required.

ILLUSTRATION 10.54

Suppose that a mothball loses volume by evaporation at a rate proportional to its instantaneous area. If the diameter of the ball decreases from 2 cm to 1 cm in 3 months, how long will it take until the ball has practically gone?

Sol. Let at any instance (t), the radius of moth ball be r and v be its volume. Then

$$v = \frac{4}{3}\pi r^3$$

or $\frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}$

Thus, as per the information

$$4\pi r^2\frac{dr}{dt} = -k4\pi r^2, \text{ where } k \in R^+$$

or $\frac{dr}{dt} = -k$

or $r = -kt + c$

at $t = 0$, $r = 2$ cm; $t = 3$ month, $r = 1$ cm

$\Rightarrow$ $c = 2$, $k = \frac{1}{3}$

$\Rightarrow$ $r = -\frac{1}{3}t + 2$

now for $r \to 0$, $t \to 6$

Hence, it will take six months until the ball is practically gone.

ILLUSTRATION 10.55

A body at a temperature of 50 °F is placed outdoors where the temperature is 100 °F. If the rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium. If after 5 min the temperature of the body is 60 °F, find (a) how long it will take the body to reach a temperature of 75 °F and (b) the temperature of the body after 20 min.

Sol. Let T be the temperature of the body at time t and $T_m = 100$ (the temperature of the surrounding medium). We have

$$\frac{dT}{dt} = -k(T - T_m)$$

or $\frac{dT}{dt} + kT = kT_m$,

where k is constant of proportionality. Thus,

$$\frac{dT}{dt} + kT = 100k$$

The solution of differential equation is

$$T = ce^{-kt} + 100$$

Since $T = 50$ when $t = 0$, (1)

from equation (1), $50 = ce^{-k(0)} + 100$, or $c = -50$.

Substituting this value in equation (1), we obtain

$$T = -50e^{-kt} + 100 \quad (2)$$

At $t = 5$, we are given that $T = 60$; hence, from equation (2),

$60 = -50e^{-5k} + 100$.

Solving for k, we obtain $-40 = -50e^{-5k}$ or $k = -\frac{1}{5}\ln\frac{40}{50}$

Substituting this value in equation (2), we obtain the temperature of the body at any time t as

$$T = -50e^{(1/5)\ln(4/5)t} + 100 \quad (3)$$

(a) We require t when $T = 75$. Substituting $T = 75$ in equation (3), we have

$75 = -50e^{(1/5)\ln(4/5)t} + 100$, from which we get $t = 15.53$ min

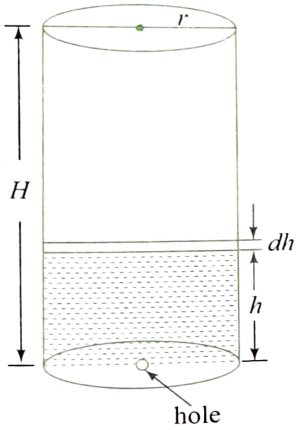
(b) We require T when $t = 20$. Substituting $t = 20$ in equation (3) and then solving for T, we find

$T = -50e^{(1/5)(\ln 4/5)(20)} + 100 = 79.52°$

ILLUSTRATION 10.56

Find the time required for a cylindrical tank of radius r and height H to empty through a round hole of area a at the bottom. The flow through the hole is according to the law $v(t) = k\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$, are respectively, the velocity of flow through the hole and the height of the water level above the hole at time t, and g is the acceleration due to gravity.

Sol.



Let at time t the depth of water is h and radius of water surface is r.
If in time dt the decrease of water level is dh, then

$$-\pi r^2\, dh = ak\sqrt{(2gh)}\; dt$$

or $$\frac{-\pi r^2}{ak\sqrt{2g}\sqrt{h}}\, dh = dt$$

or $$-\frac{\pi r^2}{ak\sqrt{2g}}\frac{dh}{\sqrt{h}} = dt$$

Now when $t = 0$, $h = H$ and when $t = t$, $h = 0$, then

$$-\frac{\pi r^2}{\ldots}\int^{0}\frac{dh}{\ldots} = \int^{t} dt$$

or $$-\frac{\pi r^2}{ak\sqrt{2g}}\left\{2\sqrt{h}\right\}_H^0 = t$$

or $$t = \frac{\pi r^2\, 2\sqrt{H}}{ak\sqrt{2g}} = \frac{\pi r^2}{ak}\sqrt{\left(\frac{2H}{g}\right)}$$

ILLUSTRATION 10.57

A country has a food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after n years, where n is the smallest integer bigger than or equal to $\frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$.

Sol. Let P_0 be the initial population of country and P be the population of country in year t. Then,

$$\frac{dP}{dt} = \text{rate of change of population} = \frac{3}{100}P = 0.03\,P$$

$\therefore$ Population of P at the end of n years is given by

$$\int_{P_0}^{P}\frac{dP}{P} = \int_0^n 0.03\,dt$$

or $\ln P - \ln P_0 = (0.03)\,n$

or $\ln P = \ln P_0 + (0.03)\,n$ (1)

If F_0 is its initial food production and F is the food production in year n, then

$F_0 = 0.9\,P_0$

and $F = (1.04)^n F_0$

or $\ln F = n\ln(1.04) + \ln F_0$ (2)

The country will be self-sufficient if $F \ge P$

or $\ln F \ge \ln P$

or $n\ln(1.04) + \ln F_0 \ge \ln P_0 + (0.03)\,n$

or $$n \ge \frac{\ln P_0 - \ln F_0}{\ln(1.04) - (.03)} = \frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$$

Hence, $$n \ge \frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$$

Thus, the least integral values of the year n, when the country becomes self-sufficient, is the smallest integer greater than or equal to $\frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$.

ILLUSTRATION 10.58

A hemi-spherical tank of radius 2 m is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t) = 0.6\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$ are, respectively, the velocity of the flow through the outlet and the height of water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to empty the tank.

Sol. Let the water level be at a height h after time t, and water level falls by dh in time dt, and the corresponding volume of water gone out be dV.

$\therefore \quad dV = -\pi r^2\, dh$

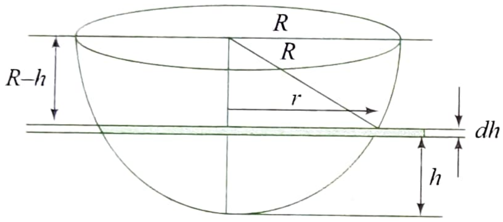
$\therefore \quad \dfrac{dV}{dt} = -\pi r^2 \dfrac{dh}{dt}$ (∵ as t increases, h decreases)

Now, velocity of water, $v = \dfrac{3}{5}\sqrt{2gh}$

Rate of flow of water $= Av$ $(A = 12\ \text{cm}^2)$

$\therefore \quad \dfrac{dV}{dt} = \left(\dfrac{3}{5}\sqrt{2gh}A\right)$

$= -\pi r^2 \dfrac{dh}{dt}$



Also from the figure,

$R^2 = (R-h)^2 + r^2$

or $\quad r^2 = 2hR - h^2$

So, $\quad \dfrac{3}{5}\sqrt{2g}\ \sqrt{h}A = -\pi(2hR - h^2) \times \dfrac{dh}{dt}$

or $\quad \dfrac{2hR - h^2}{\sqrt{h}}\, dh = -\dfrac{3}{5\pi}\sqrt{2g}\, A dt$

Integrating, we get

$$\int_R^0 (2R\sqrt{h} - h^{3/2})dh = -\frac{3\sqrt{2g}}{5\pi} A \cdot \int_0^T dt$$

or $\quad T = -\dfrac{5\pi}{3A\sqrt{2g}}\left(2R\dfrac{h^{3/2}}{3/2} - \dfrac{h^{5/2}}{5/2}\right)\Bigg|_R^0$

$= \dfrac{5\pi}{3A\sqrt{2g}}\left(\dfrac{4R}{3}R^{3/2} - \dfrac{2}{5}R^{5/2}\right)$

$= \dfrac{5\pi}{3A\sqrt{2g}}\dfrac{14}{15}R^{5/2}$

For $R = 200$ cm,

$T = \dfrac{56\pi}{9A\sqrt{g}}(10)^5$

$= \dfrac{56\pi}{9\times 12\sqrt{g}}(10)^5$

$= \dfrac{14\pi}{27\sqrt{g}}(10)^5$ units

Concept Application Exercise 10.9

1. A person places ₹5000 in an account that interest compounded continuously. Assuming no additional deposits or withdrawals, how much will be in the account after seven years if the interest rate is a constant 8.5 percent for the first four years and a constant 9.25 percent for the last thee years ($e^{0.340} = 1.404948$, $e^{0.37} = 1.447735$, $e^{0.6457} = 1.910758$)?
2. Find the time required for a cylindrical tank of radius 2.5 m and height 3 m to empty through a round hole of 2.5 cm with a velocity $2.5\sqrt{h}$ m/s, h being the depth of the water in the tank.
3. If the population of country doubles in 50 years, in how many years will it triple under the assumption that the rate of increase is proportional to the number of inhabitants?
4. The rate at which a substance cools in moving air is proportional to the difference between the temperatures of the substance and that of the air. If the temperature of the air is 290 K and the substance cools from 370 K to 330 K in 10 min, when will the temperature be 295 K?
5. A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant $= k > 0$). Find the time after which the cone will be empty.

Answers

1. ₹9271.44 2. $8000\sqrt{3}$ s
3. $50 \log_2 3$ years 4. 40 min. 5. H/k

Solved Examples

Example 10.1

Solve $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$.

Sol. $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$

or $\quad \dfrac{xdx + ydy}{(x^2+y^2)^2} = \dfrac{ydx - xdy}{y^2}\dfrac{y^2}{x^2}$

or $\quad \displaystyle\int \frac{d(x^2+y^2)}{(x^2+y^2)^2} = -2\int \frac{1}{x^2/y^2}d\left(\frac{x}{y}\right)$

or $\quad -\dfrac{1}{(x^2+y^2)} = \dfrac{-2}{x/y} + c$

or $\quad \dfrac{2y}{x} - \dfrac{1}{(x^2+y^2)} = c$

EXAMPLE 10.2

Solve

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\} ydx = \left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\} xdy$$

Sol. The given differential equation is

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\} ydx = \left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\} xdy$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x} = \frac{\left\{\cos\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)\right\}}{\left\{\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)-\cos\left(\frac{y}{x}\right)\right\}}\left(\frac{y}{x}\right) \qquad (1)$$

This is a homogeneous equation.

To solve it, we make the substitution as

$$y = vx$$

or $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v}$$

or $$x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

or $$x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

or $$\int\left[\frac{v\sin v - \cos v}{v\cos v}\right]dv = \int\frac{2dx}{x}$$

or $$-\log(v\cos v) = 2\log x + \text{loc } C$$

or $$\frac{1}{v\cos v} = Cx^2$$

or $$\cos\left(\frac{y}{x}\right) = \frac{1}{Cxy} = \frac{1}{C}\cdot\frac{1}{xy}$$

or $$xy\cos\left(\frac{y}{x}\right) = k \qquad \left(k = \frac{1}{C}\right)$$

EXAMPLE 10.3

Solve $\frac{dy}{dx} = \frac{(x+y)^2}{(x+2)(y-2)}$.

Sol. $$\frac{dy}{dx} = \frac{(x+y)^2}{(x+2)(y-2)} = \frac{(x+2+y-2)^2}{(x+2)(y-2)}$$

On putting $X = x + 2$ and $Y = y - 2$, the given differential equation reduces to

$$\frac{dY}{dX} = \frac{(X+Y)^2}{XY}$$

Put $Y = VX$, i.e., $\frac{dY}{dX} = V + X\frac{dV}{dX}$

$$\therefore \quad V + X\frac{dV}{dX} = \frac{(1+V)^2}{V}$$

or $$\frac{V}{2V+1}dV = \frac{dX}{X}$$

or $$\int\left(1 - \frac{1}{1+2V}\right)dV = 2\int\frac{dX}{X}$$

or $$V - \frac{1}{2}\ln(1+2V) = 2\ln X + C$$

or $$X^4\left(1 + \frac{2Y}{X}\right) = Ce^{2Y/X}$$

where $X = x + 2$ and $Y = y - 2$.

EXAMPLE 10.4

Solve $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ given that $y(0) = \sqrt{5}$.

Sol. $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$

Solving quadratic in $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y}$$

$$= \frac{-x \pm \sqrt{x^2 + y^2}}{y} \quad \text{which is homogeneous.}$$

Put $y = vx$, i.e., $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then given equation transforms to

$$v + x\frac{dv}{dx} = \frac{-1 \pm \sqrt{1+v^2}}{v}$$

or $$v^2 + xv\frac{dv}{dx} = -1 \pm \sqrt{1+v^2}$$

or $$(v^2 + 1) \pm \sqrt{1+v^2} = -xv\frac{dv}{dx}$$

or $$\int\frac{vdv}{(1+v^2) \pm \sqrt{v^2+1}} = -\int\frac{dx}{x}$$

or $$\int\frac{vdv}{\sqrt{v^2+1}\left(\sqrt{1+v^2} \pm 1\right)} = -\int\frac{dx}{x} \qquad (1)$$

Put $\sqrt{1+v^2} \pm 1 = t$, i.e., $\frac{v}{\sqrt{1+v^2}}dv = dt$

Then equation (1) transforms to $\int\frac{dt}{t} = -\int\frac{dx}{x}$

or $\ln t = -\ln x + \ln c$

or $tx = c$

or $\left(\sqrt{1+v^2} \pm 1\right)x = c$

or $\sqrt{x^2 + y^2} \pm x = c$

Given when $x = 0, y = \sqrt{5}$

or $[\sqrt{5} - 0] = c$ or $c = \sqrt{5}$

$\therefore$ $\sqrt{x^2 + y^2} = \sqrt{5} \pm x$

or $x^2 + y^2 = 5 + x^2 \pm 2\sqrt{5}\, x$

or $y^2 = 5 \pm 2\sqrt{5}\, x$

EXAMPLE 10.5

If $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$, find $y(x)$.

Sol. The given differential equation is

$$y + \frac{d}{dx}(xy) = x(\sin x + \log x)$$

i.e., $x\frac{dy}{dx} + 2y = x(\sin x + \log x)$

or $\frac{dy}{dx} + \frac{2}{x}y = \sin x + \log x$ (1)

This is a linear different equation

$$\text{I.F.} = e^{2\int \frac{1}{x}dx} = e^{2\log x} = x^2 \quad (2)$$

Thus, solution is given by

$$yx^2 = \int x^2(\sin x + \log x)\,dx + c$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + \frac{x^3}{3}\log x - \frac{x^3}{9} + c$$

or $y = -\cos x + \frac{2}{x}\sin x + \frac{2}{x^2}\cos x + \frac{x}{3}\log x - \frac{x}{3} + \frac{c}{x^2}$

EXAMPLE 10.6

If $\int_a^x t\,y(t)dt = x^2 + y(x)$, then find $y(x)$.

Sol. $\int_a^x t\,y(t)dt = x^2 + y(x)$

Differentiating both sides w.r.t. x, we get

$$x\,y(x) = 2x + y'(x)$$

Hence, $\frac{dy}{dx} - xy = -2x$ (linear)

$$\text{I.F.} = e^{\int -x\,dx} = e^{-x^2/2}$$

Thus, solution is $y e^{-x^2/2} = \int -2x e^{-x^2/2}dx$

or $y = 2 + ce^{-x^2/2}$

If $x = a$, then $a^2 + y = 0$ or $y = -a^2$

Hence, $-a^2 = 2 + ce^{-a^2/2}$

or $ce^{-a^2/2} = -(2 + a^2)$

or $c = -(2 + a^2)\,e^{a^2/2}$

or $y = 2 - (2 + a^2)\,e^{(a^2 - x^2)/2}$

EXAMPLE 10.7

Given a function g which has a derivative $g'(x)$ for every real x and which satisfies $g'(0) = 2$ and $g(x + y) = e^y g(x) + e^x g(y)$ for all x and y. Find $g(x)$ and determine the range of the function.

Sol. Put $x = 0, y = 0$. Then $g(0) = 0$

and $g'(0) = \lim_{h\to 0}\frac{g(h)}{h} = 2$

Now, $g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$

$$= \lim_{h\to 0}\frac{e^h g(x) + e^x g(h) - g(x)}{h}$$

$$= g(x)\lim_{h\to 0}\left(\frac{e^h - 1}{h}\right) + e^x\lim_{h\to 0}\frac{g(h)}{h}$$

$$= g(x) + 2e^x$$

Let $g(x) = y$. Then $\frac{dy}{dx} = y + 2e^x$

or $e^{-x}\frac{dy}{dx} - y e^{-x} = 2$

or $\frac{d}{dx}(y e^{-x}) = 2$

or $ye^{-x} = 2x + c$

Given if $x = 0, y = 0$, then $c = 0$.

Then $ye^{-x} = 2x$ or $y = 2xe^x$.

Now, $\frac{dy}{dx} = 2[e^x + x e^x] = 0$ or $x = -1$

Thus, minima is at $x = -1$.

Thus, range is $\left[-\frac{2}{e}, \infty\right)$.

EXAMPLE 10.8

Solve: $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$

Sol. We have $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$

$\therefore$ $\cos y\frac{dy}{dx} = \frac{\sin y + x}{2\sin y - x}$ ...(1)

Put $\sin y = vx$

$\therefore$ $\cos y\frac{dy}{dx} = v + x\frac{dv}{dx}$

So, from (1), we have

$$v + x\frac{dv}{dx} = \frac{vx + x}{2vx - x}$$

$\Rightarrow$ $x\frac{dv}{dx} = \frac{v+1}{2v-1} - v$

$$\int \frac{2v-1}{-2v^2+2v+1}dv = \int \frac{dx}{x}$$

$\Rightarrow$ $$-\frac{1}{2}\log_e(-2v^2+2v+1) = \log_e x + \log_e C$$

$\Rightarrow$ $$\log_e \frac{1}{\sqrt{-2v^2+2v+1}} = \log_e Cx$$

$\Rightarrow$ $$\frac{1}{\sqrt{-2v^2+2v+1}} = Cx$$

$\Rightarrow$ $$\frac{1}{\sqrt{-2\frac{\sin^2 y}{x^2}+2\frac{\sin y}{x}+1}} = Cx$$

$\Rightarrow$ $$C_1 = -2\sin^2 y + 2x\sin y + x^2$$

EXAMPLE 10.9

Solve: $\frac{dy}{dx}+\frac{3y}{x} = g(x)$ where $g(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}$,

$y\left(\frac{1}{2}\right) = \frac{1}{8}$ and $y(x)$ is continuous on $[0, \infty)$.

Sol. For $0 \le x < 1$

$$\frac{dy}{dx}+\frac{3y}{x} = 1$$

$$\text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\log_e x} = x^3$$

$\therefore$ Solution is $yx^3 = \int x^3 dx = \frac{1}{4}x^4 + C$

$\Rightarrow$ $$y = \frac{1}{4}x + \frac{C}{x^3}, 0 \le x < 1$$

Using $y\left(\frac{1}{2}\right) = \frac{1}{8}$, we get $C = 0$

So, $y = \frac{x}{4}, 0 \le x < 1$

For $x > 1$

$$\frac{dy}{dx}+\frac{3}{x}y = \frac{1}{x}$$

$$\text{I.F.} = x^3$$

$\therefore$ Solution is, $yx^3 = \int x^2 dx = \frac{1}{3}x^3 + C.$

So, $y = \frac{1}{3}+\frac{C}{x^3}, x > 1$

Thus, $y = \begin{cases} \frac{x}{4}, & 0 \le x < 1 \\ \frac{1}{3}+\frac{C}{x^3}, & x > 1 \end{cases}$

Since $y(x)$ is continuous, we have

$$\lim_{x\to 1^-}\frac{x}{4} = \lim_{x\to 1^+}\left(\frac{1}{3}+\frac{C}{x^3}\right)$$

$\Rightarrow$ $$C = -\frac{1}{12}$$

So, $y = \begin{cases} \frac{x}{4}, & 0 \le x \le 1 \\ \frac{1}{3}-\frac{1}{12x^3}, & x > 1 \end{cases}$

EXAMPLE 10.10

Solve: $(x\cos y - y\sin y)\,dy + (x\sin y + y\cos y)\,dx = 0$

Sol. We have

$$(x\cos y - y\sin y)\frac{dy}{dx} + (x\sin y + y\cos y) = 0 \quad \ldots(1)$$

Let $x\sin y + y\cos y = t$

Differentiating w.r.t. x, we get

$$x\cos y\frac{dy}{dx} + \sin y + \cos y\frac{dy}{dx} - y\sin y\frac{dy}{dx} = \frac{dt}{dx}$$

$\therefore$ $$(x\cos y - y\sin y)\frac{dy}{dx} + \sin y + \cos y\frac{dy}{dx} = \frac{dt}{dx}$$

So, from (1), we have

$$\frac{dt}{dx} - \left(\sin y + \frac{d(\sin y)}{dx}\right) + t = 0$$

$\Rightarrow$ $$\frac{d(t-\sin y)}{dx} = -(t-\sin y)$$

$\Rightarrow$ $$\frac{d(t-\sin y)}{t-\sin y} = -dx$$

$\Rightarrow$ $$\int \frac{d(t-\sin y)}{t-\sin y} = -\int dx$$

$\Rightarrow$ $\log_e(t-\sin y) = -x + \log_e C$

$\Rightarrow$ $t - \sin y = Ce^{-x}$

$\Rightarrow$ $x\sin y + y\cos y - \sin y = Ce^{-x}$

EXAMPLE 10.11

If y_1 and y_2 are the solutions of the differential equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone and $y_2 = y_1 z$, then prove that $z = 1 + c.\, e^{-\int \frac{Q}{y_1}dx}$, where c is an arbitrary constant.

Sol. Given $\frac{dy_1}{dx} + Py_1 = Q \quad (1)$

$$\frac{dy_2}{dx} + Py_2 = Q \quad (2)$$

Clearly, we have to eliminate P and y_2.

In equation (2), put $y_2 = zy_1$

or $$\frac{d(zy_1)}{dx} + Pzy_1 = Q$$

or $$y_1\frac{dz}{dx} + z\frac{dy_1}{dx} + Pzy_1 = Q$$

From equation (1), put the value of Py_1

We have $y_1\frac{dz}{dx}+z\frac{dy_1}{dx}+z\left(Q-\frac{dy_1}{dx}\right)=Q$

or $\quad y_1\frac{dz}{dx}=Q(1-z)$

or $\quad \int\frac{dz}{z-1}=-\int\frac{Q}{y_1}dx$

or $\quad \log(z-1)=-\int\frac{Q}{y_1}dx+\log c$

or $\quad \log\frac{z-1}{c}=-\int\frac{Q}{y_1}dx$

or $\quad z=1+c\,e^{\int-\frac{Q}{y_1}dx}$

EXAMPLE 10.12

If y_1 and y_2 are two solutions to the differential equation $\frac{dy}{dx}+P(x)$ $y=Q(x)$. Then prove that $y=y_1+c(y_1-y_2)$ is the general solution to the equation where c is any constant.

Sol. y_1, y_2 are the solutions of the differential equation

$$\frac{dy}{dx}+P(x)\,y=Q(x) \quad (1)$$

Then $\quad \frac{dy_1}{dx}+P(x)\,y_1=Q(x) \quad (2)$

and $\quad \frac{dy_2}{dx}+P(x)\,y_2=Q(x) \quad (3)$

From equations (1) and (2), we get

$$\frac{d(y-y_1)}{dx}+P(x)(y-y_1)=0 \quad (4)$$

and from equations (2) and (3), we get

$$\frac{d(y_1-y_2)}{dx}+P(x)(y_1-y_2)=0 \quad (5)$$

Now, from equations (4) and (5), we get

$$\frac{\frac{d}{dx}(y-y_1)}{\frac{d}{dx}(y_1-y_2)}=\frac{(y-y_1)}{(y_1-y_2)}$$

or $\quad \int\frac{d(y-y_1)}{(y-y_1)}=\int\frac{d(y_1-y_2)}{(y_1-y_2)}$

or $\quad \ln(y-y_1)=\ln(y_1-y_2)+\ln c$

or $\quad \ln(y-y_1)=\ln(c(y_1-y_2))$

or $\quad y-y_1=c(y_1-y_2)$

or $\quad y=y_1+c(y_1-y_2)$

EXAMPLE 10.13

Let $f(x)$, $x\ge 0$, be a non-negative continuous function, and let $F(x)=\int_0^x f(t)\,dt$, $x\ge 0$, if for some $c>0$, $f(x)\le c\,F(x)$ for all $x\ge 0$, then show that $f(x)=0$ for all $x\ge 0$.

Sol. Given that $F(x)=\int_0^x f(t)\,dt$

$\therefore \quad F'(x)=f(x)$

Also, given that $f(x)\le c\,F(x)\ \forall\ x\ge 0$

$\therefore \quad f(0)\le cF(0)=0$

$\therefore \quad f(0)\le 0$

But given that $f(x)$ is non-negative function on $[0,\infty)$.

$\therefore \quad f(x)\ge 0$

$\therefore \quad f(0)\ge 0$

Thus, from equations (2) and (3), $f(0)=0$.

Again, $f(x)\le cF(x)\ \forall\ x\ge 0$

$\therefore \quad f(x)-cF(x)\le 0$

or $\quad F'(x)-cF(x)\le 0\ \forall\ x\ge 0$ [Using equation (1)]

or $\quad e^{-cx}F'(x)-ce^{-cx}F(x)\le 0$

[Multiplying both sides by e^{-cx} (I.F.)]

or $\quad \frac{d}{dx}\left[e^{-cx}F(x)\right]\le 0$

Thus, $g(x)=e^{-cx}F(x)$ is a decreasing function on $[0,\infty)$, i.e.,

$g(x)\le g(0)$ for all $x\ge 0$

But $g(0)=F(0)=0$

$\therefore \quad g(x)\le 0\ \forall\ x\ge 0$

or $\quad e^{-cx}F(x)\le 0\ \forall\ x\ge 0$

or $\quad F(x)\le 0\ \forall\ x\ge 0$

$\therefore \quad f(x)\le cF(x)\le 0\ \forall\ x\ge 0$ [$\because c>0$ and using $f(x)\le cF(x)$]

or $\quad f(x)\le 0\ \forall\ x\ge 0$

But given $f(x)\ge 0$

$\therefore \quad f(x)=0\ \forall\ x\ge 0$

EXAMPLE 10.14

Find a pair of curves such that

(a) the tangents drawn at points with equal abscissas intersect on the y-axis.

(b) the normal drawn at points with equal abscissas intersect on the x-axis.

(c) one curve passes through (1, 1) and other passes through (2, 3).

Sol. Let the curve be $y=f_1(x)$ and $y=f_2(x)$. Equations of tangents with equal abscissa x are

$$Y-f_1'(x)=f'(x)(X-x) \text{ and } Y-f_2(x)=f_2'(x)(X-x)$$

These tangent intersect at y-axis.

So, their y-intercept are same.

$\therefore \quad -xf_1'(x)+f_1(x)=-xf_2'(x)+f_2(x)$

or $\quad f_1(x)-f_2(x)=x(f_1'(x)-f_2'(x))$

or $\quad \int\frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)}=\int\frac{dx}{x}$

or $\quad \ln|f_1(x)-f_2(x)|=\ln|x|+\ln C_1$

or $\quad f_1(x)-f_2(x)=\pm C_1x \quad (1)$

Now, equations of normal with equal abscissa x are

$$(Y-f_1(x))=-\frac{1}{f_1'(x)}(X-x)$$

and $\quad (Y-f_2(x))=-\frac{1}{f_2'(x)}(X-x)$

As these normals intersect on the x-axis,

$$x+f_1(x)f_1'(x)=x+f_2(x)\cdot f_2'(x)$$

or $$f_1(x)f_1'(x)-f_2(x)f_2'(x)=0$$

Integrating, we get $f_2^2(x)-f_1^2(x)=C_2$

or $$f_1(x)+f_2(x)=\frac{C_2}{f_1(x)-f_2(x)}$$

$$=\pm\frac{C_2}{C_1x}=\frac{\pm\lambda_2}{x} \qquad (2)$$

From equations (1) and (2), we get

$$2f_1(x)=\pm\left(\frac{\lambda_2}{x}+C_1x\right),$$

$$2f_2(x)=\pm\left(\frac{\lambda_2}{x}-C_1x\right)$$

We have $f_1(1)=1$ and $f(2)=3$

$\therefore$ $$f_1(x)=\frac{2}{x}-x,\ f_2(x)=\frac{2}{x}+x$$

EXAMPLE 10.15

Given two curves: $y=f(x)$ passing through the point (0, 1) and $g(x)=\int_{-\infty}^{x}f(t)dt$ passing through the point $\left(0,\frac{1}{n}\right)$. The tangents drawn to both the curves at the points with equal abscissas intersect on the x-axis. Find the curve $y=f(x)$.

Sol. Equation of tangent to the curve $y=f(x)$ is

$$Y-y=f'(x)(X-x)$$

Equation of tangents to the curve

$$g(x)=y_1=\int_{-\infty}^{x}f(t)dt$$

is $$Y-y_1=f(x)(X-x) \qquad \left(\frac{dy_1}{dx}=g'(x)=f(x)\right)$$

Since the tangent with equal abscissas intersect on the x-axis,

$$x-\frac{y}{f'(x)}=x-\frac{y_1}{f(x)}$$

or $$\frac{f(x)}{f'(x)}=\frac{y_1}{f(x)}$$

or $$\frac{g'(x)}{g''(x)}=\frac{g(x)}{g'(x)}$$

or $$\frac{g'(x)}{g(x)}=\frac{g''(x)}{g'(x)}$$

Integrating both sides,

$$\ln g(x)=\ln c\,g'(x)$$

or $$g(x)=c\,g'(x)$$

or $$\frac{g'(x)}{g(x)}=c$$

Integrating both sides,

$$g(x)=ke^{cx}$$

or $$f(x)=g'(x)=kce^{cx}$$

The curve $y=f(x)$ passes through (0, 1). Thus, $kc=1$.

The curve $y=g(x)$ passes through $\left(0,\frac{1}{n}\right)$. Thus,

$$k=\frac{1}{n}\text{ or }c=n\text{ or }f(x)=e^{nx}$$

EXAMPLE 10.16

A cyclist moving on a level road at 4 m/s stops pedalling and lets the wheels come to rest. The retardation of the cycle has two components: a constant 0.08 m/s^2 due to friction in the working parts and a resistance of 0.02 v^2/m, where v is speed in meters per second. What distance is traversed by the cycle before it comes to rest? (consider ln 5 = 1.61).

Sol. Let the cyclist starting to move from the point O and moving along OX, attain a velocity v at point P in time t such that $OP=x$. Let the acceleration of the moving cycle at P be a. Then we know that

$$v=\frac{dx}{dt}\text{ and }a=\frac{dv}{dt}=\frac{d^2x}{dt^2}=v\frac{dv}{dx} \qquad (1)$$

By hypothesis, retardation = 0.08 + 0.02 v^2 = 0.02 (4 + v^2)

or $$v\frac{dv}{dx}=-0.02(4+v^2)$$

or $$dx=-\frac{1}{0.02}\frac{v\,dv}{4+v^2} \qquad (2)$$

Integrating equation (2) between the limits $x=0$; $v=4$ m/s, and $x=x'$ meters, $v=0$, we get

$$\int_0^{x'}dx=-\frac{1}{0.04}\int_4^0\frac{2vdv}{4+v^2}$$

or $$x'=-\frac{1}{0.04}[\ln(4+v^2)]_4^0$$

$$=-\frac{1}{0.04}[\ln 4-\ln 20]$$

$$=\frac{\ln 5}{0.04}=\frac{1.61}{0.04}=\frac{161}{4}\text{ m}$$

EXAMPLE 10.17

The force of resistance encountered by water on a motor boat of mass m going in still water with velocity v is proportional to the velocity v. At $t=0$ when its velocity is v_0, the engine shuts off. Find an expression for the position of motor boat at time t and also the distance travelled by the boat before it comes to rest. Take the proportionality constant as $k>0$.

Sol. The resistance force opposing the

$$\text{motion}=m\times\text{acceleration}=m\frac{dv}{dt}$$

Hence, differential equation is $m\dfrac{dv}{dt}=-kv$

or $$\frac{dv}{v}=-\frac{k}{m}dt$$

Integrating, we get $\ln v=-\dfrac{k}{m}t+c$

At $t = 0$, $v = v_0$. Hence, $c = \ln v_0$

$$\therefore \quad \ln \frac{v}{v_0} = -\frac{k}{m} \cdot t \text{ or } v = v_0 \, e^{-\frac{k}{m}t} \qquad (1)$$

where v is the velocity at time t.

Now, $\frac{ds}{dt} = v_0 \, e^{-\frac{k}{m}t}$

or $ds = v_0 e^{-\frac{k}{m}t} \, dt$

Boat's position at time t is

$$s(t) = -\frac{v_0 m}{k} e^{-\frac{kt}{m}} + c$$

If $t = 0$, $s = 0$ or $c = \frac{v_0 m}{k}$

$$\therefore \quad s(t) = \frac{v_0 m}{k}\left[1 - e^{-\frac{kt}{m}}\right] \qquad (2)$$

To find how far the boat goes, we have to find $\lim_{t \to \infty} s(t)$ which is $\frac{mv_0}{k}$.

EXAMPLE 10.18

A and B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at the time. One hour after the water is released, the quantity of water is reservoir A is $1\frac{1}{2}$ times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water ?

Sol. Let at any instant t, x be the volume of water in the reserver A and y of that in B. Then

$$\frac{dx}{dt} \propto x$$

or $\frac{dx}{dt} = k_1 x$

or $\frac{dx}{x} = k_1 \, dt$

or $\log x = k_1 t + C_1$

or $x = e^{k_1 t} e^{C_1}$ (1)

Similarly for B, $\frac{dy}{dt} \propto y$

or $\log y = k_2 t + C_2$

or $y = e^{k_2 t} e^{C_2}$ (2)

Now, at $t = 0$, $x = 2y$, i.e., $\frac{x}{y} = 2$

Thus, from equations (1) and (2), we get $\frac{e^{C_1}}{e^{C_2}} = 2$ (3)

Also, at $t = 1$, $x = \frac{3}{2}y$, i.e., $\frac{x}{y} = \frac{3}{2}$

or $\frac{e^{k_1} e^{C_1}}{e^{k_2} e^{C_2}} = \frac{3}{2}$

or $e^{k_1 - k_2} = \frac{3}{4}$

Let at $t = T$, $x = y$, i.e., $\frac{x}{y} = 1$

Then $\frac{e^{k_1 T} e^{C_1}}{e^{k_2 T} e^{C_2}} = 1$

or $(e^{k_1 - k_2})^T \, 2 = 1$

or $\left(\frac{3}{4}\right)^T = \frac{1}{2}$

Taking log on both sides, we get

$$T \log (3/4) = \log (1/2)$$

or $T = \frac{-\log 2}{-\log 4/3} = \left(\frac{\log 2}{\log 4/3}\right)$

Exercises

Single Correct Answer Type

1. The degree of the differential equation satisfying $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$ is
(1) 1 (2) 2
(3) 3 (4) None of these

2. Number of values of $m \in N$ for which $y = e^{mx}$ is a solution of the differential equation $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}-4\frac{dy}{dx}+12y=0$
(1) 0 (2) 1
(3) 2 (4) More than 2

3. The order of differential equation of family of circles passing through intersection of $3x + 4y - 7 = 0$ and $S \equiv x^2 + y^2 - 2x - 2y + 1 = 0$ is
(1) 1 (2) 2 (3) 3 (4) 4

4. The differential equation of all non-horizontal lines in a plane is
(1) $\frac{d^2y}{dx^2}$ (2) $\frac{d^2x}{dy^2}=0$
(3) $\frac{dy}{dx}=0$ (4) $\frac{dx}{dy}=0$

5. The differential equation of the family of curves $y = e^x (A \cos x + B \sin x)$, where A and B are arbitrary constants is
(1) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$
(2) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-2y=0$
(3) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+y=0$
(4) $\frac{d^2y}{dx^2}-7\frac{dy}{dx}+2y=0$

6. Differential equation of the family of circles touching the line $y = 2$ at $(0, 2)$ is
(1) $x^2+(y-2)^2+\frac{dy}{dx}(y-2)=0$
(2) $x^2+(y-2)\left(2-2x\frac{dx}{dy}-y\right)=0$
(3) $x^2+(y-2)^2+\left(\frac{dx}{dy}+y-2\right)(y-2)=0$
(4) None of these

7. The order and degree of the differential equation of all tangent lines to the parabola $y = x^2$ is
(1) 1, 2 (2) 2, 3 (3) 2, 1 (4) 1, 1

8. The differential equation for the family of curve $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant, is
(1) $2(x^2-y^2)y' = xy$ (2) $2(x^2+y^2)y' = xy$
(3) $(x^2-y^2)y' = 2xy$ (4) $(x^2+y^2)y' = 2xy$

9. The differential equation whose general solution is given by $y = c_1\cos(x+c_2) - c_3e^{(-x+c_4)} + c_5 \sin x$, where c_1, c_2, c_3, c_4, c_5 are arbitrary constants, is
(1) $\frac{d^4y}{dx^4}-\frac{d^2y}{dx^2}+y=0$
(2) $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$
(3) $\frac{d^5y}{dx^5}+y=0$
(4) $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0$

10. If $y=\frac{x}{\log|cx|}$ (where c is an arbitrary constant) is the general solution of the differential equation $dy/dx = y/x + \phi(x/y)$, then the function $\phi(x/y)$ is
(1) x^2/y^2 (2) $-x^2/y^2$
(3) y^2/x^2 (4) $-y^2/x^2$

11. The differential equation of the curve $\frac{x}{c-1}+\frac{y}{c+1}=1$ is given by
(1) $\left(\frac{dy}{dx}-1\right)\left(y+x\frac{dy}{dx}\right)=2\frac{dy}{dx}$
(2) $\left(\frac{dy}{dx}+1\right)\left(y-x\frac{dy}{dx}\right)=\frac{dy}{dx}$
(3) $\left(\frac{dy}{dx}+1\right)\left(y-x\frac{dy}{dx}\right)=2\frac{dy}{dx}$
(4) None of these

12. If $y = y(x)$ and $\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right) = -\cos x$, $y(0) = 1$, then $y(\pi/2)$ equals
(1) 1/3 (2) 2/3 (3) – 1/3 (4) 1

13. The equation of the curves through the point (1, 0) and whose slope is $\frac{y-1}{x^2+x}$ is
(1) $(y-1)(x+1)+2x=0$
(2) $2x(y-1)+x+1=0$
(3) $x(y-1)(x+1)+2=0$
(4) None of these

14. The solution of the equation $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$ is
(1) $y\sin y = x^2\log x+\frac{x^2}{2}+c$
(2) $y\cos y = x^2(\log x+1)+c$

(3) $y\cos y = x^2\log x + \frac{x^2}{2} + c$

(4) $y\sin y = x^2\log x + c$

15. The solution of the equation $(x^2y + x^2)dx + y^2(x-1)\,dy = 0$ is given by

(1) $x^2 + y^2 + 2(x-y) + 2\ln\frac{(x-1)(y+1)}{c} = 0$

(2) $x^2 + y^2 + 2(x-y) + \ln\frac{(x-1)(y+1)}{c} = 0$

(3) $x^2 + y^2 + 2(x-y) - 2\ln\frac{(x-1)(y+1)}{c} = 0$

(4) None of these

16. Solution of differential equation $dy - \sin x\sin y\,dx = 0$ is

(1) $e^{\cos x}\tan\frac{y}{2} = c$ (2) $e^{\cos x}\tan y = c$

(3) $\cos x\tan y = c$ (4) $\cos x\sin y = c$

17. The solution of $\frac{d\upsilon}{dt} + \frac{k}{m}\upsilon = -g$ is

(1) $\upsilon = ce^{-\frac{k}{m}t} - \frac{mg}{k}$ (2) $\upsilon = c - \frac{mg}{k}e^{-\frac{k}{m}t}$

(3) $\upsilon e^{-\frac{k}{m}t} = c - \frac{mg}{k}$ (4) $\upsilon e^{\frac{k}{m}t} = c - \frac{mg}{k}$

18. The general solution of the differential equation $\frac{dy}{dx} + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$ is

(1) $\log\tan\left(\frac{y}{2}\right) = c - 2\sin x$

(2) $\log\tan\left(\frac{y}{4}\right) = c - 2\sin\left(\frac{x}{2}\right)$

(3) $\log\tan\left(\frac{y}{2} + \frac{\pi}{4}\right) = c - 2\sin x$

(4) $\log\tan\left(\frac{y}{4} + \frac{\pi}{4}\right) = c - 2\sin\left(\frac{x}{2}\right)$

19. If $y + x\frac{dy}{dx} = x\frac{\phi(xy)}{\phi'(xy)}$, then $\phi(xy)$ is equal to

(1) $ke^{x^2/2}$ (2) $ke^{y^2/2}$

(3) $ke^{xy/2}$ (4) ke^{xy}

20. The solution of differential equation $x^2 = 1 + \left(\frac{x}{y}\right)^{-1}\frac{dy}{dx} + \frac{\left(\frac{x}{y}\right)^{-2}\left(\frac{dy}{dx}\right)^2}{2!} + \frac{\left(\frac{x}{y}\right)^{-3}\left(\frac{dy}{dx}\right)^3}{3!} + \dots$ is

(1) $y^2 = x^2(\ln x^2 - 1) + c$

(2) $y = x^2(\ln x - 1) + c$

(3) $y^2 = x(\ln x - 1) + c$

(4) $y = x^2e^{x^1} + c$

21. The solution of the differential equation

$$\frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots} = \frac{dx - dy}{dx + dy} \text{ is}$$

(1) $2y\,e^{2x} = C\,e^{2x} + 1$ (2) $2y\,e^{2x} = C\,e^{2x} - 1$

(3) $y\,e^{2x} = C\,e^{2x} + 2$ (4) None of these

22. The solution of the differential equation

$$x = 1 + xy\frac{dy}{dx} + \frac{x^2y^2}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{x^3y^3}{3!}\left(\frac{dy}{dx}\right)^3 + \dots \text{ is}$$

(1) $y = \ln(x) + c$ (2) $y^2 = (\ln x)^2 + c$

(3) $y = \log x + xy$ (4) $xy = x^y + c$

23. A curve passing through (2, 3) and satisfying the differential equation $\int_0^x t\,y(t)\,dt = x^2y(x),\ (x > 0)$ is

(1) $x^2 + y^2 = 13$ (2) $y^2 = \frac{9}{2}x$

(3) $\frac{x^2}{8} + \frac{y^2}{18} = 1$ (4) $xy = 6$

24. The solution of the differential equation $\frac{d^2y}{dx^2} = \sin 3x + e^x + x^2$ when $y_1(0) = 1$ and $y(0) = 0$ is

(1) $y = \frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x - 1$

(2) $y = \frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x$

(3) $y = \frac{-\cos 3x}{3} + e^x + \frac{x^4}{12} + \frac{1}{3}x + 1$

(4) None of these

25. The solution of the differential equation $y'y''' = 3(y'')^2$ is

(1) $x = A_1y^2 + A_2y + A_3$ (2) $x = A_1y + A_2$

(3) $x = A_1y^2 + A_2y$ (4) None of these

26. The solution of the differential equation $y''' - 8y'' = 0$, where $y(0) = \frac{1}{8}, y'(0) = 0, y''(0) = 1$, is

(1) $y = \frac{1}{8}\left(\frac{e^{8x}}{8} + x - \frac{7}{9}\right)$ (2) $y = \frac{1}{8}\left(\frac{e^{8x}}{8} + x + \frac{7}{8}\right)$

(3) $y = \frac{1}{8}\left(\frac{e^{8x}}{8} - x + \frac{7}{8}\right)$ (4) None of these

27. The slope of the tangent at (x, y) to a curve passing through $\left(1, \frac{\pi}{4}\right)$ is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$, then the equation of the curve is

(1) $y = \tan^{-1}\left(\log\left(\frac{e}{x}\right)\right)$ (2) $y = x\tan^{-1}\left(\log\left(\frac{x}{e}\right)\right)$

(3) $y = x\tan^{-1}\left(\log\left(\frac{e}{x}\right)\right)$ (4) None of these

28. If $x\frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is

(1) $\log\frac{x}{y} = cy$ (2) $\log\frac{y}{x} = cy$

(3) $\log\frac{y}{x} = cx$ (4) None of these

29. The solution of differential equation

$yy' = x\left(\frac{y^2}{x^2} + \frac{f(y^2/x^2)}{f'(y^2/x^2)}\right)$ is

(1) $f(y^2/x^2) = cx^2$ (2) $x^2 f(y^2/x^2) = c^2y^2$

(3) $x^2 f(y^2/x^2) = c$ (4) $f(y^2/x^2) = cy/x$

30. The solution of $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$ is

(1) $\log x = \log(x - y) + \frac{y}{x} + c$

(2) $\log x = 2\log(x - y) + \frac{y}{x} + c$

(3) $\log x = \log(x - y) + \frac{x}{y} + c$

(4) None of these

31. The solution of $(y + x + 5)\,dy = (y - x + 1)\,dx$ is

(1) $\log((y + 3)^2 + (x + 2)^2) + \tan^{-1}\frac{y+3}{y+2} + C$

(2) $\log((y + 3)^2 + (x - 2)^2) + \tan^{-1}\frac{y-3}{x-2} = C$

(3) $\log((y + 3)^2 + (x + 2)^2) + 2\tan^{-1}\frac{y+3}{x+2} = C$

(4) $\log((y + 3)^2 + (x + 2)^2) - 2\tan^{-1}\frac{y+3}{x+2} = C$

32. The slope of the tangent at (x, y) to a curve passing through a point $(2, 1)$ is $\frac{x^2 + y^2}{2xy}$, then the equation of the curve is

(1) $2(x^2 - y^2) = 3x$ (2) $2(x^2 - y^2) = 6y$

(3) $x(x^2 - y^2) = 6$ (4) $x(x^2 + y^2) = 10$

33. Solution of the differential equation

$(y + x\sqrt{xy}(x + y))\,dx + (y\sqrt{xy}(x + y) - x)dy = 0$ is

(1) $\frac{x^2 + y^2}{2} + \tan^{-1}\sqrt{\frac{y}{x}} = c$

(2) $\frac{x^2 + y^2}{2} + 2\tan^{-1}\sqrt{\frac{x}{y}} = c$

(3) $\frac{x^2 + y^2}{2} + 2\cot^{-1}\sqrt{\frac{x}{y}} = c$

(4) None of these

34. The solution of $x^2\frac{dy}{dx} - xy = 1 + \cos\frac{y}{x}$ is

(1) $\tan\left(\frac{y}{2x}\right) = c - \frac{1}{2x^2}$ (2) $\tan\frac{y}{x} = c + \frac{1}{x}$

(3) $\cos\left(\frac{y}{x}\right) = 1 + \frac{c}{x}$ (4) $x^2 = (c + x^2)\tan\frac{y}{x}$

35. The solution of the differential equation

$2x^2y\frac{dy}{dx} = \tan(x^2y^2) - 2xy^2$, given $y(1) = \sqrt{\frac{\pi}{2}}$, is

(1) $\sin x^2y^2 = e^{x+1}$ (2) $\sin(x^2y^2) = x$

(3) $\cos x^2y^2 + x = 0$ (4) $\sin(x^2y^2) = e^{x-1}$

36. Solution of the differential equation

$\left\{\frac{1}{x} - \frac{y^2}{(x-y)^2}\right\}dx + \left\{\frac{x^2}{(x-y)^2} - \frac{1}{y}\right\}dy = 0$ is

(1) $\ln\left|\frac{x}{y}\right| + \frac{xy}{x-y} = c$ (2) $\frac{xy}{x-y} = ce^{x/y}$

(3) $\ln|xy| = c + \frac{xy}{x-y}$ (4) None of these

37. The solution of differential equation $(2y + xy^3)dx + (x + x^2y^2)\,dy = 0$ is

(1) $x^2y + \frac{x^3y^3}{3} = c$ (2) $xy^2 + \frac{x^3y^3}{3} = c$

(3) $x^2y + \frac{x^4y^4}{4} = c$ (4) None of these

38. The solution of $ye^{-x/y}dx - (xe^{(-x/y)} + y^3)dy = 0$ is

(1) $e^{-x/y} + y^2 = C$ (2) $xe^{-x/y} + y = C$

(3) $2e^{-x/y} + y^2 = C$ (4) $e^{-x/y} + 2y^2 = C$

39. The curve satisfying the equation $\frac{dy}{dx} = \frac{y(x + y^3)}{x(y^3 - x)}$ and passing through the point $(4, -2)$ is

(1) $y^2 = -2x$ (2) $y = -2x$

(3) $y^3 = -2x$ (4) None of these

40. The solution of differential equation

$\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = \frac{x\cos^2(x^2 + y^2)}{y^3}$ is

(1) $\tan(x^2 + y^2) = \frac{x^2}{y^2} + c$

(2) $\cot(x^2 + y^2) = \frac{x^2}{y^2} + c$

(3) $\tan(x^2 + y^2) = \frac{y^2}{x^2} + c$

(4) $\cot(x^2 + y^2) = \frac{y^2}{x^2} + c$

41. The solution of the differential equation

$\frac{dy}{dx} = \frac{3x^2 y^4 + 2xy}{x^2 - 2x^3 y^3}$ is

(1) $\frac{y^2}{x} - x^3 y^2 = c$ (2) $\frac{x^2}{y^2} + x^3 y^3 = c$

(3) $\frac{x^2}{y} + x^3 y^2 = c$ (4) $\frac{x^2}{3y} - 2x^3 y^2 = c$

42. The solution of the differential equation

$\{1 + x\sqrt{(x^2 + y^2)}\}dx + \{\sqrt{(x^2 + y^2)} - 1\} y\, dy = 0$ is equal to

(1) $x^2 + \frac{y^2}{2} + \frac{1}{3}(x^2 + y^2)^{3/2} = c$

(2) $x - \frac{y^3}{3} + \frac{1}{2}(x^2 + y^2)^{1/2} = c$

(3) $x - \frac{y^2}{2} + \frac{1}{3}(x^2 + y^2)^{3/2} = c$

(4) None of these

43. The solution of the differential equation

$y(2x^4 + y)\frac{dy}{dx} = (1 - 4xy^2)x^2$ is given by

(1) $3(x^2 y)^2 + y^3 - x^3 = c$

(2) $xy^2 + \frac{y^3}{3} - \frac{x^3}{3} + c = 0$

(3) $\frac{2}{5}yx^5 + \frac{y^3}{3} = \frac{x^3}{3} - \frac{4xy^3}{3} + c$

(4) None of these

44. The solution of the differential equation $(x \cot y + \log \cos x)\, dy + (\log \sin y - y \tan x)\, dx = 0$ is

(1) $(\sin x)^y (\cos y)^x = c$ (2) $(\sin y)^x (\cos x)^y = c$

(3) $(\sin x)^x (\cos y)^y = c$ (4) None of these

45. If $\frac{dy}{dx} = (e^y - x)^{-1}$, where $y(0) = 0$, then y is expressed explicitly as

(1) $\frac{1}{2}\log_e(1 + x^2)$ (2) $\log_e(1 + x^2)$

(3) $\log_e\left(x + \sqrt{1 + x^2}\right)$ (4) $\log_e\left(x + \sqrt{1 - x^2}\right)$

46. The general solution of the differential equation, $y' + y\phi'(x) - \phi(x) \cdot \phi'(x) = 0$, where $\phi(x)$ is a known function, is

(1) $y = ce^{-\phi(x)} + \phi(x) - 1$ (2) $y = ce^{+\phi(x)} + \phi(x) - 1$

(3) $y = ce^{-\phi(x)} - \phi(x) + 1$ (4) $y = ce^{-\phi(x)} + \phi(x) + 1$

where c is an arbitrary constant.

47. The integrating factor of the differential equation

$\frac{dy}{dx}(x \log_e x) + y = 2\log_e x$ is given by

(1) x (2) e^x

(3) $\log_e x$ (4) $\log_e(\log_e x)$

48. The solution of the differential equation $x(x^2 + 1)(dy/dx) = y(1 - x^2) + x^3 \log x$ is

(1) $y(x^2 + 1)/x = \frac{1}{4}x^2 \log x + \frac{1}{2}x^2 + c$

(2) $y^2(x^2 - 1)/x = \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + c$

(3) $y(x^2 + 1)/x = \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + c$

(4) None of these

49. Integrating factor of differential equation

$\cos x \frac{dy}{dx} + y \sin x = 1$ is

(1) $\cos x$ (2) $\tan x$ (3) $\sec x$ (4) $\sin x$

50. Solution of the equation $\cos^2 x \frac{dy}{dx} - (\tan 2x)y = \cos^4 x$, $|x| < \frac{\pi}{4}$, when $y\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8}$ is

(1) $y = \tan 2x \cos^2 x$ (2) $y = \cot 2x \cos^2 x$

(3) $y = \frac{1}{2}\tan 2x \cos^2 x$ (4) $y = \frac{1}{2}\cot 2x \cos^2 x$

51. If integrating factor of $x(1 - x^2)\, dy + (2x^2 y - y - ax^3)\, dx = 0$ is $e^{\int p dx}$, then P is equal to

(1) $\frac{2x^2 - ax^3}{x(1 - x^2)}$ (2) $2x^3 - 1$

(3) $\frac{2x^2 - a}{ax^3}$ (4) $\frac{2x^2 - 1}{x(1 - x^2)}$

52. A function $y = f(x)$ satisfies $(x + 1)f'(x) - 2(x^2 + x)f(x) = \frac{e^{x^2}}{(x+1)}$, $\forall x > -1$. If $f(0) = 5$, then $f(x)$ is

(1) $\left(\frac{3x+5}{x+1}\right)e^{x^2}$ (2) $\left(\frac{6x+5}{x+1}\right)e^{x^2}$

(3) $\left(\frac{6x+5}{(x+1)^2}\right)e^{x^2}$ (4) $\left(\frac{5-6x}{x+1}\right)e^{x^2}$

53. The general solution of the equation $\frac{dy}{dx} = 1 + xy$ is

(1) $y = ce^{-x^2/2}$ (2) $y = ce^{x^2/2}$

(3) $y = (x + c), e^{-x^2/2}$ (4) None of these

54. The solution of the differential equation $(x + 2y^3)\frac{dy}{dx} = y$ is

(1) $\frac{x}{y^2} = y + c$ (2) $\frac{x}{y} = y^2 + c$

(3) $\frac{x^2}{y} = y^2 + c$ (4) $\frac{y}{x} = x^2 + c$

55. The solution of the differential equation

$x^2 \frac{dy}{dx}\cos\frac{1}{x} - y \sin\frac{1}{x} = -1$, where $y \to -1$ as $x \to \infty$ is

(1) $y = \sin\frac{1}{x} - \cos\frac{1}{x}$ (2) $y = \frac{x+1}{x\sin\frac{1}{x}}$

(3) $y = \cos\frac{1}{x} + \sin\frac{1}{x}$ (4) $y = \frac{x+1}{x\cos 1/x}$

56. The solution of $\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$ satisfying $y(1) = 1$ is given by
(1) a system of parabolas
(2) a system of circles
(3) $y^2 = x(1+x) - 1$
(4) $(x-2)^2 + (y-3)^2 = 5$

57. The solution of the differential equation $\frac{dy}{dx} = \frac{1}{xy[x^2 \sin y^2 + 1]}$ is
(1) $x^2(\cos y^2 - \sin y^2 - 2Ce^{-y^2}) = 2$
(2) $y^2(\cos x^2 - \sin y^2 - 2Ce^{-y^2}) = 2$
(3) $x^2(\cos y^2 - \sin y^2 - e^{-y^2}) = 4C$
(4) None of these

58. The equation of a curve passing through (2, 7/2) and having gradient $1 - \frac{1}{x^2}$ at (x, y) is
(1) $y = x^2 + x + 1$ (2) $xy = x^2 + x + 1$
(3) $xy = x + 1$ (4) None of these

59. Which of the following is not the differential equation of family of curves whose tangent form an angle of $\pi/4$ with the hyperbola $xy = c^2$?
(1) $\frac{dy}{dx} = \frac{x-y}{x+y}$ (2) $\frac{dy}{dx} = \frac{x}{x-y}$
(3) $\frac{dy}{dx} = \frac{x+y}{y-x}$ (4) None of these

60. Tangent to a curve intercepts the y-axis at a point P. A line perpendicular to this tangent through P passes through another point (1, 0). The differential equation of the curve is
(1) $y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 1$ (2) $\frac{xd^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$
(3) $y\frac{dx}{dy} + x = 1$ (4) None of these

61. Orthogonal trajectories of family of the curve $x^{2/3} + y^{2/3} = a^{2/3}$, where 'a' is any arbitrary constant, is
(1) $x^{2/3} - y^{2/3} = c$ (2) $x^{4/3} - y^{4/3} = c$
(3) $x^{4/3} + y^{4/3} = c$ (4) $x^{1/3} - y^{1/3} = c$

62. The curve in the first quadrant for which the normal at any point (x, y) and the line joining the origin to that point form an isosceles triangle with the x-axis as base is
(1) an ellipse (2) a rectangular hyperbola
(3) a circle (4) None of these

63. The equation of the curve which is such that the portion of the axis of x cut off between the origin and tangent at any point is proportional to the ordinate of that point is
(1) $x = y(a - b \log y)$ (2) $\log x = by^2 + a$
(3) $x^2 = y(a - b \log y)$ (4) None of these
(b is a constant of proportionality)

64. The family of curves represented by $\frac{dy}{dx} = \frac{x^2+x+1}{y^2+y+1}$ and $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$
(1) Touch each other
(2) Are orthogonal
(3) Are one and the same
(4) None of these

65. A normal at $P(x, y)$ on a curve meets the x-axis at Q and N is the foot of the ordinate at P. If $NQ = \frac{x(1+y^2)}{1+x^2}$, then the equation of curve given that it passes through the point (3, 1) is
(1) $x^2 - y^2 = 8$ (2) $x^2 + 2y^2 = 11$
(3) $x^2 - 5y^2 = 4$ (4) None of these

66. A curve is such that the mid-point of the portion of the tangent intercepted between the point where the tangent is drawn and the point where the tangent meets the y-axis lies on the line $y = x$. If the curve passes through (1, 0), then the curve is
(1) $2y = x^2 - x$ (2) $y = x^2 - x$
(3) $y = x - x^2$ (4) $y = 2(x - x^2)$

67. The normal to a curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is a
(1) parabola (2) circle
(3) hyperbola (4) ellipse

68. The x-intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point (1, 1) is
(1) $ye^{x/y} = e$ (2) $xe^{x/y} = e$
(3) $xe^{y/x} = e$ (4) $ye^{y/x} = e$

69. The equation of a curve passing through (1, 0) for which the product of the abscissa of a point P and the intercept made by a normal at P on the x-axis equals twice the square of the radius vector of the point P is
(1) $x^2 + y^2 = x^4$ (2) $x^2 + y^2 = 2x^4$
(3) $x^2 + y^2 = 4x^4$ (4) None of these

70. The curve with the property that the projection of the ordinate on the normal is constant and has a length equal to a is
(1) $a\ln\left(\sqrt{y^2 - a^2} + y\right) = x + c$
(2) $x + \sqrt{a^2 - y^2} = c$
(3) $(y - a)^2 = cx$
(4) $ay = \tan^{-1}(x + c)$

71. Spherical rain drop evaporates at a rate proportional to its surface area. The differential equation corresponding to the rate of change of the radius of the rain drop if the constant of proportionality is $K > 0$ is

(1) $\frac{dr}{dt} + K = 0$ (2) $\frac{dr}{dt} - K = 0$

(3) $\frac{dr}{dt} = Kr$ (4) None of these

72. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = \frac{1}{15}$, then the time to drain the tank if the water is 4 m deep to start with is

(1) 30 min (2) 45 min

(3) 60 min (4) 80 min

73. The population of a country increases at a rate proportional to the number of inhabitants. f is the population which doubles in 30 years, then the population will triple in approximately

(1) 30 years (2) 45 years

(3) 48 years (4) 54 years

74. An object falling from rest in air is subject not only to the gravitational force but also to air resistance. Assume that the air resistance is proportional to the velocity with constant of proportionality as $k > 0$, and acts in a direction opposite to motion (g = 9.8 m/s^2). Then velocity cannot exceed

(1) 9.8/k m/s (2) 98/k m/s

(3) $\frac{k}{9.8}$ m/s (4) None of these

Multiple Correct Answers Type

1. Which one of the following function(s) is/are homogeneous?

(1) $f(x, y) = \frac{x - y}{x^2 + y^2}$

(2) $f(x, y) = x^{\frac{1}{3}} y^{-\frac{2}{3}} \tan^{-1}\frac{x}{y}$

(3) $f(x, y) = x\,(\ln\sqrt{x^2 + y^2} - \ln y) + ye^{x/y}$

(4) $f(x,y) = x\left[\ln\frac{2x^2 + y^2}{x} - \ln(x + y)\right] + y^2\tan\frac{x + 2y}{3x - y}$

2. For the differential equation whose solution is $(x - h)^2 + (y - k)^2 = a^2$ (a is a constant), its

(1) order is 2 (2) order is 3

(3) degree is 2 (4) degree is 3

3. The equation of the curve satisfying the differential equation $y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$ can be a

(1) Circle (2) Straight line

(3) Parabola (4) Ellipse

4. Which of the following equation(s) is/are linear?

(1) $\frac{dy}{dx} + \frac{y}{x} = \log x$ (2) $y\left(\frac{dy}{dx}\right) + 4x = 0$

(3) $(2x + y^3)\left(\frac{dy}{dx}\right) = 3y$ (4) None of these

5. The solution of $\frac{dy}{dx} = \frac{ax + h}{by + k}$ represents a parabola when

(1) $a = 0, b \neq 0$ (2) $a \neq 0, b \neq 0$

(3) $b = 0, a \neq 0$ (4) $a = 0, b \in R$

6. The equation of the curve satisfying the differential equation $y_2(x^2 + 1) = 2xy_1$ passing through the point (0, 1) and having slope of tangent at $x = 0$ as 3 (where y_2 and y_1 represent 2nd and 1st order derivative), then

(1) $y = f(x)$ is a strictly increasing function

(2) $y = f(x)$ is a non-monotonic function

(3) $y = f(x)$ has three distinct real roots

(4) $y = f(x)$ has only one negative root

7. Identify the statement(s) which is/are true.

(1) $f(x, y) = e^{y/x} + \tan\frac{y}{x}$ is a homogeneous of degree zero.

(2) $x\ln\frac{y}{x}dx + \frac{y^2}{x}\sin^{-1}\frac{y}{x}dy = 0$ is a homogeneous differential equation.

(3) $f(x, y) = x^2 + \sin x \cos y$ is a non-homogeneous.

(4) $(x^2 + y^2)dx - (xy^2 - y^3)dy = 0$ is a homegeneous differential equation.

8. The graph of the function $y = f(x)$ passing through the point (0, 1) and satisfying the differential equation $\frac{dy}{dx} + y\cos x = \cos x$ is such that

(1) it is a constant function

(2) it is periodic

(3) it is neither an even nor an odd function

(4) it is continuous and differentiable for all x

9. If $f(x)$, $g(x)$ is twice differential functions on [0, 2] satisfying $f''(x) = g''(x)$, $f'(1) = 2g'(1) = 4$ and $f(2) = 3g(2) = 9$, then

(1) $f(4) - g(4) = 10$

(2) $|f(x) - g(x)| < 2 \Rightarrow -2 < x < 0$

(3) $f(2) = g(2) \Rightarrow x = -1$

(4) $f(x) - g(x) = 2x$ has real root

10. The solution of the differential equation $(x^2y^2 - 1)\,dy + 2xy^3\,dx = 0$ is

(1) $1 + x^2y^2 = cx$ (2) $1 + x^2y^2 = cy$

(3) $y = 0$ (4) $y = -\frac{1}{x^2}$

11. $y = ae^{-1/x} + b$ is a solution of $\frac{dy}{dx} = \frac{y}{x^2}$, then
(1) $a \in R - \{0\}$
(2) $b = 0$
(3) $b = 1$
(4) a takes finite number of values

12. For equation of the conic whose subnormal is constant, then
(1) its eccentricity is 1
(2) its eccentricity is $\sqrt{2}$
(3) its axis is the x-axis
(4) its axis is the y-axis.

13. The solution of $\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}}$ is
(1) $\sqrt{x^2 + y^2} = \sin\{\tan^{-1}(y/x) + C\}$
(2) $\sqrt{x^2 + y^2} = \cos\{(\tan^{-1} y/x) + C\}$
(3) $\sqrt{x^2 + y^2} = (\tan(\sin^{-1} y/x) + C)$
(4) $y = x\tan\left(c + \sin^{-1}\sqrt{x^2 + y^2}\right)$

14. The curves for which the length of the normal is equal to the length of the radius vector is/are
(1) circles
(2) rectangular hyperbola
(3) ellipses
(4) straight lines

15. In which of the following differential equation degree is not defined?
(1) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x\log\frac{d^2y}{dx^2}$
(2) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$
(3) $x = \sin\left(\frac{dy}{dx} - 2y\right)$, $|x| < 1$
(4) $x - 2y = \log\left(\frac{dy}{dx}\right)$

16. If $y = f(x)$ is the solution of equation $ydx + dy = -e^x y^2 dy$, $f(0) = 1$ and area bounded by the curve $y = f(x)$, $y = e^x$ and $x = 1$ is A, then
(1) curve $y = f(x)$ is passing through $(-2, e)$
(2) curve $y = f(x)$ is passing through $(1, 1/e)$
(3) curve $y = f(x)$ is passing through $(1, 1/3)$
(4) $A = e + \frac{2}{\sqrt{e}} - 3$

17. A particle falls in a medium whose resistance is proportional to the square of the velocity of the particle. If the differential equation of the free fall is $\frac{dv}{dt} = g - kv^2$ (k is constant) then
(1) $v = 2\sqrt{\frac{g}{k}}\frac{e^{2t\sqrt{g/t}} + 1}{e^{2r\sqrt{g/k}} - 1}$
(2) $v = \sqrt{\frac{g}{k}}\frac{e^{2t\sqrt{gk}} - 1}{e^{2t\sqrt{gk}} + 1}$
(3) $v \to 0$ as $t \to \infty$
(4) $v \to \sqrt{\frac{g}{k}}$ as $t \to \infty$

Linked Comprehension Type

For Problems 1–3

For certain curve $y = f(x)$ satisfying $\frac{d^2y}{dx^2} = 6x - 4$, $f(x)$ has local minimum value 5 when $x = 1$.

1. The number of critical point for $y = f(x)$ for $x \in [0, 2]$ is
(1) 0 (2) 1 (3) 2 (4) 3

2. Global minimum value of $y = f(x)$ for $x \in [0, 2]$ is
(1) 5 (2) 7 (3) 8 (4) 9

3. Global maximum value of $y = f(x)$ for $x \in [0, 2]$ is
(1) 5 (2) 7 (3) 8 (4) 9

For Problems 4–6

The differential equation $y = px + f(p)$, (i)

where $p = \frac{dy}{dx}$, is known as Clairout's equation. To solve equation (i), differentiate it with respect to x, which gives either

$$\frac{dp}{dx} = 0 \Rightarrow p = c \quad \text{(ii)}$$

or $$x + f'(p) = 0 \quad \text{(iii)}$$

Note:
- If p is eliminated between equations (i) and (ii), the solution obtained is a general solution of equation (i).
- If p is eliminated between equations (i) and (iii), then solution obtained does not contain any arbitrary constant and is not particular solution of equation (i). This solution is called singular solution of equation (i).

4. Which of the following is true about solutions of differential equation $y = xy' + \sqrt{1 + y'^2}$?
(1) The general solution of equation is family of parabolas
(2) The general solution of equation is family of circles
(3) The singular solution of equation is circle
(4) The singular solution of equation is ellipse

5. The number of solution of the equation $f(x) = -1$ and the singular solution of the equation $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$ is
(1) 1 (2) 2 (3) 4 (4) 0

6. The singular solution of the differential equation $y = mx + m - m^3$, where $m = \frac{dy}{dx}$; passes through the point
(1) (0, 0) (2) (0, 1) (3) (1, 0) (4) (– 1, 0)

For Problems 7–9

Let $f(x)$ be a non-positive continuous function and $F(x) = \int_0^x f(t)dt$ $\forall\, x \geq 0$ and $f(x) \geq cF(x)$ where $c > 0$ and let $g : [0, \infty) \to R$ be a function such that $\frac{dg(x)}{dx} < g(x) \forall\, x > 0$ and $g(0) = 0$.

7. The total number of root(s) of the equation $f(x) = g(x)$ is/are

(1) ∞ (2) 1 (3) 2 (4) 0

8. The number of solution(s) of the equation $|x^2 + x - 6| = f(x) + g(x)$ is/are

(1) 2 (2) 1 (3) 0 (4) 3

9. The solution set of inequation $g(x)(\cos^{-1}x - \sin^{-1}x) \le 0$

(1) $\left[-1, \frac{1}{\sqrt{2}}\right]$ (2) $\left[\frac{1}{\sqrt{2}}, 1\right]$

(3) $\left[0, \frac{1}{\sqrt{2}}\right]$ (4) $\left(0, \frac{1}{\sqrt{2}}\right]$

For Problems to 10–12

A curve '*C*' with negative slope through the point (0, 1) lies in the *I* Quadrant. The tangent at any point '*P*' on it meets the *x*-axis at '*Q*'. Such that $PQ = 1$. Then

10. The curve in parametric form is

(1) $x = \cos\theta + \log_e \tan(\theta/2),\ y = \sin\theta$

(2) $x = -\cos\theta + \log_e \tan(\theta/2),\ y = \sin\theta$

(3) $x = -\cos\theta - \log_e \tan\theta/2, y = \sin\theta$

(4) None of these

11. The area bounded by '*C*' and the co-ordinate axes is

(1) 1 (2) $\log_e 2$ (3) $\pi/4$ (4) $\pi/2$

12. The orthogonal trajectories of '*C*' are

(1) Circles (2) Parabolas

(3) Ellipses (4) Hyperbolas

For Problems 13–15

Let $y = f(x)$ satisfies the equation

$$f(x) = (e^{-x} + e^{x}) \cos x - 2x - \int_0^x (x - t)\ f'(t)\, dt$$

13. y satisfies the differential equation

(1) $\frac{dy}{dx} + y = e^x(\cos x - \sin x) - e^{-x}(\cos x + \sin x)$

(2) $\frac{dy}{dx} - y = e^x(\cos x - \sin x) + e^{-x}(\cos x + \sin x)$

(3) $\frac{dy}{dx} + y = e^x(\cos x + \sin x) - e^{-x}(\cos x - \sin x)$

(4) $\frac{dy}{dx} - y = e^x(\cos x - \sin x) + e^{-x}(\cos x - \sin x)$

14. The value of $f(0) + f'(0)$ equal

(1) -1 (2) 0 (3) 1 (4) 2

15. $f(x) =$

(1) $e^{-x}(\cos x - \sin x) + \frac{e^x}{5}(3\cos x + \sin x) + \frac{2}{5}e^{-x}$

(2) $e^{-x}(\cos x + \sin x) + \frac{e^x}{5}(3\cos x - \sin x) - \frac{2}{5}e^{-x}$

(3) $e^{-x}(\cos x - \sin x) + \frac{e^x}{5}(3\cos x - \sin x) + \frac{2}{5}e^{-x}$

(4) $e^{-x}(\cos x + \sin x) + \frac{e^x}{5}(3\cos x - \sin x) - \frac{2}{5}e^{-x}$

For Problems 16–18

A certain radioactive material is known to decay at a rate proportional to the amount present. Initially there is 50 kg of the material present and after 2 h it is observed that the material has lost 10 percent of its original mass. Based on these data answer the following questions:

16. The expression for the mass of the material remaining at any time t is

(1) $N = 50e^{(1/2)(\ln 0.9)t}$ (2) $50e^{-(1/4)(\ln 9)t}$

(3) $N = 50e^{-(\ln 0.9)t}$ (4) None of these

17. The mass of the material after 4 h is

(1) $50^{-0.5\ln 9}$ (2) $50e^{-2\ln 9}$

(3) $50e^{-2\ln 0.9}$ (4) None of these

18. The time at which the material has decayed to one half of its initial mass is

(1) $\frac{\ln 0.25}{\ln 0.9}$ h (2) $\frac{\ln 0.5}{\ln 0.81}$ h

(3) $\frac{\ln 0.25}{\ln 0.81}$ h (4) None of these

For Problems 19–21

Consider a tank which initially holds V_0 liter of brine that contains a lb of salt. Another brine solution, containing b lb of salt per liter is poured into the tank at the rate of e L/min while, simultaneously, the well-stirred solution leaves the tank at the rate of f L/min. The problem is to find the amount of salt in the tank at any time t.

Let Q denote the amount of salt in the tank at any time. The time rate of change of Q, dQ/dt, equals the rate at which salt enters the tank minus the rate at which salt leaves the tank. Salt enters the tank at the rate of be lb/min. To determine the rate at which salt leaves the tank, we first calculate the volume of brine in the tank at any time t, which is the initial volume V_0 plus the volume of brine added et minus the volume of brine removed ft. Thus, the volume of brine at any time is

$$V_0 + et - ft \quad \text{(a)}$$

The concentration of salt in the tank at any time is $Q/(V_0 + et - ft)$, from which it follows that salt leaves the tank at the rate of $f\left(\frac{Q}{V_0 + et - ft}\right)$ lb/min. Thus,

$$\frac{dQ}{dt} = be - f\left(\frac{Q}{V_0 + et - ft}\right) \quad \text{(b)}$$

or

$$\frac{dQ}{dt} + \frac{f}{V_0 + et - ft}Q = be$$

19. A tank initially holds 100 L of a brine solution containing 20 lb of salt. At $t = 0$, fresh water is poured into the tank at the rate of 5 L/min, while the well-stirred mixture leaves

the tank at the same rate. Then the amount of salt in the tank after 20 min.

(1) $20/e$ (2) $10/e$ (3) $40/e^2$ (4) $5/e$ L

20. A 50 L tank initially contains 10 L of fresh water. At $t = 0$, a brine solution containing 1 lb of salt per gallon is poured into the tank at the rate of 4 L./min, while the well-stirred mixture leaves the tank at the rate of 2 L/min. Then the amount of time required for overflow to occur is

(1) 30 min (2) 20 min (3) 10 min (4) 40 min

21. In the above question, the amount of salt in the tank at the moment of overflow is

(1) 20 lb (2) 50 lb
(3) 30 lb (4) None of these

Matrix Match Type

1. Match the following lists:

List I	List II: Differential equation
a. order 1	p. of all parabolas whose axis is the x-axis
b. order 2	q. of family of curves $y = a(x + a)^2$, where a is an arbitrary constant
c. degree 1	r. $\left(1+3\frac{dy}{dx}\right)^{2/3} = \frac{4d^3y}{dx^3}$
d. degree 3	s. of family of curve $y^2 = 2c\,(x + \sqrt{c})$, where $c > 0$

2. Match the following lists:

List I	List II
a. If the function $y = e^{4x} + 2e^{-x}$ is a solution of the differential equation $\dfrac{\frac{d^3y}{dx^3} - 13\frac{dy}{dx}}{y} = K$, then the value of $K/3$ is	p. 3
b. Let a curve $y = f(x),\ f(x) \geq 0\ \forall x \in R$ have property that for every point P on the curve, length of subnormal is equal to abscissa of P. If $f(1) = 3$, then $\frac{f(4)}{\sqrt{6}} =$	q. 4
c. If real value of m for which the substitution, $y = u^m$ will transform the differential equation, $2x^4y\frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation, then the value of $2m$ is	r. 2
d. If the solution of differential equation $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 12y$ is $y = Ax^m + Bx^{-n}$, then possible value of $\lvert m+n \rvert$ is	s. 0

3. Match the differential equation in List I with its solution in List II and then choose the correct code.

List I	List II
a. $y = kx + k^2$	p. $1 - \left(\frac{dy}{dx}\right)^2 = 2x.\frac{dy}{dx}$
b. $y = ae^{2x} + be^{3x}$	q. $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 = 2$
c. $y^2 = 4a\,(x + a)$	r. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$
d. $xy = ae^x + be^{-x} + x^2$	s. $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$

Codes:

	a	b	c	d
(1)	p	s	q	r
(2)	s	r	p	q
(3)	r	p	s	q
(4)	q	p	r	s

Numerical Value Type

1. If $y = y(x)$ and it follows the relation $4xe^{xy} = y + 5\sin^2 x$, then $y'(0)$ is equal to_____

2. If $x\frac{dy}{dx} = x^2 + y - 2,\ y(1) = 1,$ then $y(2)$ equals_____

3. If the dependent variable y is changed to z by the substitution $y = \tan z$ and the differential equation $\frac{d^2y}{dx^2} = 1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2$ is changed to $\frac{d^2z}{dx^2} = \cos^2 z + k\left(\frac{dz}{dx}\right)^2$, then the value of k equals_____

4. Let $y = y\,(t)$ be a solution to the differential equation $y' + 2ty = t^2$, then $16 \lim_{t\to\infty} \frac{y}{t}$ is_____

5. If the solution of the differential equation $\frac{dy}{dx} = \frac{1}{x\cos y + \sin 2y}$ is $x = ce^{\sin y} - k\,(1 + \sin y)$, then the value of k is_____

6. If the independent variable x is changed to y, then the differential equation $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0$ is changed to $x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 = k$ where k equals_____

7. Let y_1 and y_2 be two different solutions of the equation $\frac{dy}{dx} + P(x)\cdot y = Q(x)$. Then $\alpha y_1 + \beta y_2$ will be a solution of the given equation if $\alpha + \beta =$ _____.

8. Tangent is drawn at the point (x_i, y_i) on the curve $y = f(x)$, which intersects the x-axis at $(x_{i+1}, 0)$. Now, again a tangent is drawn at (x_{i+1}, y_{i+1}) on the curve which intersects the x-axis at $(x_{i+2}, 0)$ and the process is repeated n times, i.e., $i = 1, 2, 3, \ldots, n$. If $x_1, x_2, x_3, \ldots, x_n$ form an arithmetic progression with common difference equal to $\log_2 e$ and curve passes through $(0, 2)$. Now if curve passes through the point $(-2, k)$, then the value of k

9. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Also curve passes through the point (1, 1). Then the length of intercept of the curve on the x-axis is______

10. If the eccentricity of the curve for which tangent at point P intersects the y-axis at M such that the point of tangency is equidistant from M and the origin is e, then the value of $5e^2$ is______

11. If the solution of the differential equation $\frac{dy}{dx} - y = 1 - e^{-x}$ and $y(0) = y_0$ has a finite value, when $x \to \infty$, then the value of $|2/y_0|$ is______

12. Let f be a function defined on the interval $[0, 2\pi]$ such that $\int_0^x (f'(t) - \sin 2t)dt = \int_x^0 f(t)\tan t\, dt$ and $f(0) = 1$. Then the maximum value of $f(x)$ is______.

13. Let $y(x)$ be a function satisfying $\frac{d^2y}{dx^2} - \frac{dy}{dx} + e^{2x} = 0$, $y(0) = 2$ and $y'(0) = 1$. If maximum value of $y(x)$ is $y(\alpha)$, then integral part of 2α is ______.

14. If the differential equation of a curve, passing through $\left(0, -\frac{\pi}{4}\right)$ and $(t, 0)$ is

$$\cos y\left(\frac{dy}{dx} + e^{-x}\right) + \sin y\left(e^{-x} - \frac{dy}{dx}\right) = e^{e^{-x}},$$

then the value of $\left|t \cdot e^{e^{-t}}\right|$ is ______.

15. Let f be a continuous function satisfying the equation $\int_0^x f(t)\, dt + \int_0^x t\, f(x-t)\, dt = e^{-x} - 1$, then find the value of $e^9 f(8)$ is equal to______.

Archives

JEE MAIN

Single Correct Answer Type

1. The differential equation which represents the family of curves $y = c_1 e^{c_2 x}$, where c_1 and c_2 are arbitrary constants, is

(1) $y' = y^2$ (2) $y'' = y'y$

(3) $yy'' = y'$ (4) $yy'' = (y')^2$

(AIEEE 2009)

2. The solution to the differential equation $\cos x\, dy = y(\sin x - y)\, dx$, $0 < x < \frac{\pi}{2}$, is

(1) $\tan x = (\sec x + c)\, y$

(2) $\sec x = (\tan x + c)\, y$

(3) $y \sec x = \tan x + c$

(4) $y \tan x = \sec x + c$ **(AIEEE 2010)**

3. Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then the scrap value $V(T)$ of the equipment is

(1) e^{-kT} (2) $T^2 - \frac{I}{k}$

(3) $I - \frac{kT^2}{2}$ (4) $I - \frac{k(T-t)^2}{2}$

(AIEEE 2011)

4. If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\ln 2)$ is equal to

(1) -2 (2) 7 (3) 5 (4) 13

(AIEEE 2011)

5. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is

(1) 6/7 (2) 4/9

(3) 2/9 (4) None of these

(AIEEE 2012)

6. The population $p(t)$ at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5\, p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is

(1) $2 \ln 18$ (2) $\ln 9$

(3) $\frac{1}{2} \ln 18$ (4) $\ln 18$ **(AIEEE 2012)**

7. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P w.r.t. additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is

(1) 2500 (2) 3000

(3) 3500 (4) 4500 **(JEE Main 2013)**

8. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ equals

(1) $400 - 300\, e^{t/2}$ (2) $300 - 200\, e^{-t/2}$

(3) $600 - 500\, e^{t/2}$ (4) $400 - 300\, e^{-t/2}$

(JEE Main 2014)

9. Let $y(x)$ be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x, (x \geq 1)$. Then $y(e)$ is equal to

(1) e (2) 0 (3) 2 (4) $2e$

(JEE Main 2015)

10. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)\,dx = x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to

(1) $-\frac{4}{5}$ (2) $\frac{2}{5}$ (3) $\frac{4}{5}$ (4) $-\frac{2}{5}$

(JEE Main 2016)

11. If $(2+\sin x)\frac{dy}{dx}+(y+1)\cos x = 0$ and $y(0)=1$ then $y\left(\frac{\pi}{2}\right)$ is equal to

(1) $\frac{4}{3}$ (2) $\frac{1}{3}$ (3) $-\frac{2}{3}$ (4) $-\frac{1}{3}$

(JEE Main 2017)

12. Let $y = y(x)$ be the solution of the differential equation $\sin x\frac{dy}{dx} + y\cos x = 4x$, $x \in (0, \pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to

(1) $-\frac{4}{9}\pi^2$ (2) $\frac{4}{9\sqrt{3}}\pi^2$ (3) $\frac{-8}{9\sqrt{3}}\pi^2$ (4) $-\frac{8}{9}\pi^2$

(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right), x > 0$. Then the equation of the curve is

(1) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$

(2) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(3) $\sec\left(\frac{2y}{x}\right) = \log x + 2$

(4) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

(JEE Advanced 2013)

2. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is

(1) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (2) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

(3) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (4) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

(JEE Advanced 2014)

3. If $y = y(x)$ satisfies the differential equation

$$8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx,$$

$x > 0$ and $y(0) = \sqrt{7}$, then $y(256) =$

(1) 3 (2) 9 (3) 16 (4) 80

(JEE Advanced 2017)

Multiple Correct Answers Type

1. If $y(x)$ satisfies the differential equation $y' - y\tan x = 2x\sec x$ and $y(0) = 0$, then

(1) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (2) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(3) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (4) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

(IIT-JEE, 2012)

2. Consider the family of all circles whose centers lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are functions of x, y and y' $\left(\text{here } y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}\right)$, then which of the following statements is (are) true?

(1) $P = y + x$

(2) $P = y - x$

(3) $P + Q = 1 - x + y + y' + (y')^2$

(4) $P - Q = x + y - y' - (y')^2$

(JEE Advanced 2015)

3. Let $y(x)$ be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statements is (are) true?

(1) $y(-4) = 0$

(2) $y(-2) = 0$

(3) $y(x)$ has a critical point in the interval $(-1, 0)$

(4) $y(x)$ has no critical point in the interval $(-1, 0)$

(JEE Advanced 2015)

4. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$, $x > 0$, passes through the point $(1, 3)$. Then the solution curve

(1) intersects $y = x + 2$ exactly at one point

(2) intersects $y = x + 2$ exactly at two points

(3) intersects $y = (x + 2)^2$

(4) does NOT intersect $y = (x + 3)^2$

(JEE Advanced 2016)

5. Let $f: R \to R$ and $g: R \to R$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x)-g(x))})\, g'(x)$ for all $x \in R$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE?
(1) $f(2) < 1 - \log_e 2$
(2) $f(2) > 1 - \log_e 2$
(3) $g(1) > 1 - \log_e 2$
(4) $g(1) < 1 - \log_e 2$
(JEE Advanced 2018)

Numerical Value Type

1. Let f be a real-valued differentiable function on R (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point $P(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscissa of P, then the value of $f(-3)$ is equal to _______ **(IIT-JEE 2010)**

2. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\dfrac{df(x)}{dx}$, and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then the value of $y(2)$ is _______. **(IIT-JEE 2011)**

3. Let $f: [1, \infty) \to [2, \infty)$ be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)\,dt = 3x\, f(x) - x^3$ for all $x \ge 1$, then the value of $f(2)$ is _______. **(IIT-JEE 2011)**

4. Let $f: R \to R$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation $\dfrac{dy}{dx} = (2 + 5y)(5y - 2)$, then the value of $\lim_{x \to -\infty} f(x)$ is ___________. **(JEE Advanced 2018)**

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	2. (3)	3. (1)	4. (2)	5. (1)
6. (4)	7. (1)	8. (3)	9. (2)	10. (4)
11. (3)	12. (1)	13. (1)	14. (4)	15. (1)
16. (1)	17. (1)	18. (2)	19. (1)	20. (1)
21. (2)	22. (2)	23. (4)	24. (1)	25. (1)
26. (3)	27. (3)	28. (3)	29. (1)	30. (2)
31. (3)	32. (1)	33. (2)	34. (1)	35. (4)
36. (1)	37. (1)	38. (3)	39. (3)	40. (1)
41. (3)	42. (3)	43. (1)	44. (2)	45. (3)
46. (1)	47. (3)	48. (3)	49. (3)	50. (3)
51. (4)	52. (2)	53. (4)	54. (2)	55. (1)
56. (3)	57. (1)	58. (2)	59. (2)	60. (1)
61. (2)	62. (2)	63. (1)	64. (2)	65. (3)
66. (3)	67. (3)	68. (1)	69. (1)	70. (1)
71. (1)	72. (3)	73. (3)	74. (1)	

Multiple Correct Answers Type

1. (1), (2), (3)	2. (1), (3)
3. (1), (2)	4. (1), (3)
5. (1), (3)	6. (1), (4)
7. (1), (2), (3)	8. (1), (2), (4)
9. (1), (2), (3)	10. (2)
11. (1), (2)	12. (2)
13. (1), (4)	14. (1), (2)
15. (1), (2)	16. (1), (4)
17. (2), (4)	

Linked Comprehension Type

1. (3)	2. (1)	3. (2)	4. (3)	5. (2)
6. (4)	7. (2)	8. (3)	9. (1)	10. (3)
11. (3)	12. (1)	13. (1)	14. (2)	15. (3)
16. (1)	17. (3)	18. (3)	19. (1)	20. (2)
21. (4)				

Matrix Match Type

1. a $\to$ q, s; b $\to$ p; c $\to$ p; d $\to$ q, r, s
2. a $\to$ q; b $\to$ r; c $\to$ p; d $\to$ s
3. (2)

Numerical Value Type

1. (4)	2. (2)	3. (2)	4. (8)	5. (2)
6. (1)	7. (1)	8. (8)	9. (2)	10. (5)
11. (4)	12. (1.125)	13. (1)	14. (1)	15. (8)

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (4)	2. (2)	3. (3)	4. (2)	5. (3)
6. (1)	7. (3)	8. (1)	9. (3)	10. (3)
11. (2)	12. (4)			

JEE ADVANCED

Single Correct Answer Type

1. (1) 2. (2) 3. (1)

Multiple Correct Answers Type

1. (1), (4)	2. (2), (3)
3. (1), (3)	4. (1), (4)
5. (2), (3)	

Numerical Value Type

1. (9) 2. (0) 3. (6) 4. (0.4)